Effective sequence of uniformities and its effective limit

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Abstract

The *effective sequence of unifomities* on a set and its *effective limit* are defined. We can then express some notion of computability of a *sequence of functions* which have *different jump points*. As an example, a *sequence of uniformities* on the set $[0,1)$ is discussed.

1 Introduction

The standard notion of *computability* of a *real number* or of a sequence of real numbers as well as that of the computability of a *continuous function* or of a sequence of continuous functions is now generally agreed. There are many references on this subject. We refer the reader to [9]; there is also [20] for a quick read.

Very often, however, we compute values of a discontinuous function. We thus expect that some class of discontinuous functions can be attributed a certain kind of computability. In an attempt of computing a discontinuous function, a problem arises in the computation of the value at a jump point (a point of discontinuity). This is because it is not in general decidable if a real number is a jump point, that is, the question "$x = a$?" is not decidable even for computable x and a.

One method of dissolving this problem was proposed in [9] by Pour-El and Richards and was succeeded by Washihara (cf. [13],[14],[15]) and others. It was a functional analysis approach. In their treatment, a function is regarded as computable as a point in some function space. This is sufficient in order to draw a rough graph of the function, but does not supply us with information when computing individual values.

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There are many ways of characterizing *computation of a discontinuous function*. We have proposed some approaches to this problem. One is to express the value of a function at a jump point in terms of a recursive rational sequence approximating it with a "limiting recursive" modulus of convergence instead of a recursive one (Yasugi, Brattka, Washihara:[16]). (A *limiting recursive* function is obtained from a recursive function by taking the limit of values with respect to a parameter. The notion of limiting recursive functions is due to Gold [2], and has been utilized also in some works related to ours: cf. [8] and [3].) Another is to change the topology of the domain of a function (Tsujii, Mori, Yasugi:[11]). Under certain conditions, these two approaches are equivalent ([18]).

As for a *sequence of functions with different jump points*, we proposed an approach, guided by an example of the system of Rademacher functions (Yasugi, Washihara:[21]), in terms of a *limiting recursive modulus of convergence*. Let $\{\phi_l(x)\}$ be the sequence of Rademacher functions (cf., for example, [10]). In [21], it was shown that $\{\phi_l(x)\}$ has a "weak sequential computability," that is, there is a program which does the following job: input a computable sequence of real numbers $\{x_m\}$, a recursive sequence of rational numbers can be constructed so that it converges to $\{\phi_l(x_m)\}$ with a limiting recursive modulus of convergence.

There is an alternative way of expressing a notion of computability of a sequence of real functions which have different jump points, using the notion of an "effective sequence of uniformities." Here we *propose* the theory of "effective sequence of uniformities" and its "effective limit" (Section 3). Under a certain condition, the "limit" of such an effective sequence is again an effective uniformity (Section 3: Theorem 1).

Our theory of the effective uniform space can be seen in [11], [19], [18]. Some examples of the effective sequence of uniformities are given in Section 3.

We will first give a brief account of some fundamental notions such as computable real numbers and computable (continuous) functions (Section 2) for the reader's convenience. In Section 4, we present a special case of the theory proposed in Section 3 (Section 4). There we confine ourselves to the real numbers in the interval $I = [0, 1)$ and functions on it. The theory of an effective sequence of uniformities on I and its limit is developed. For the real sequences from I, **R**-computability (computability in the Euclidean topology), ν-computability (computability with respect to the ν-th uniformity in the sequence, $\nu = 1, 2, 3, \cdots$) and ω-computability (computability with respect to the limit of the sequence) will be defined and their mutual relationship will be worked out (Theorems 2 and 3).

In Section 5, an example of a function sequence whose uniform computability property can be stated in the limit space (Theorem 4). This limit space is topologically equivalent to the Fine-metric space.

In Section 6, other examples of the effective sequence of uniformities and function sequences which are computable in the limit spaces will be taken up. Some additional comments are attached at the end.

As a way of remark, we append a related subject, the diagonal-uniformity (Section 7: Appendix).

We only list some references which have close relationship with the present work.

We hope to modify our approach to a function sequence with points of discontinuity defined on a metric space which has a computability structure (cf. [7] and [17]). We would also like to investigate more applications of our theory.

2 Preliminaries

In the following, **N**, **Z** and **R** will denote respectively the set of natural numbers, the set of integers and the set of real numbers. In general, p, l, m, n, k will be used to denote elements of **N** or positive integers.

A real number x is called *computable* (**R**-*computable*) if

$$\forall m \geq \alpha(p). |x - r_m| < \frac{1}{2^p}$$

for a recursive function α and a recursive sequence of rational numbers $\{r_m\}$. This property will be expressed as $x \simeq \langle r_m, \alpha(p) \rangle$, or, for short, $x \simeq \langle r_m, \alpha \rangle$.

These definitions can be extended to a computable sequence of real numbers.

A real (continuous) function f is *computable* (**R**-*computable*) if the following (i) and (ii) hold.

(i) f preserves *sequential computability*, that is, for a computable $\{x_n\}$, $\{f(x_n)\}$ is computable.

(ii) f is *effectively continuous* with a recursive β:

$$\forall p \forall n \forall k \geq \beta(n,p) \forall x, y \in [-n, n]. |x - y| < \frac{1}{2^k} \Rightarrow |f(x) - f(y)| < \frac{1}{2^p}.$$

This definition can be extended to a sequence of functions.

In computing values of a piecewise continuous function, for example, it is a common practice to first compute the value at a jump point, and then compute values on the open interval where the function is continuous. Such an action corresponds to the mathematical notion of *isolating the jump points* with respect to a given function. We were thus led to the uniform topology of the real line induced from the Euclidean topology by isolating the jump points. (We have employed the definition of uniformity as defined in [4].) For details of three definitions below, see [11] and [18].

Let X be a non-empty set. A sequence $\{V_n\}_{n \in \mathbf{N}}$ of maps from X to the powerset of X, that is, $V_n : X \to P(X)$, is called a (countable) *uniformity* if it satisfies some axioms, Axioms $A_1 \sim A_5$ to be stated below. Let us note that, in fact, A_1 and A_2 in [4] can be unified to $A_1 \& A_2$: $\cap_n V_n(x) = \{x\}$. We will state Axioms $A_3 \sim A_5$ in the form of effective uniformity.

Definition 2.1 (Effective uniformity:[11]) Let $\{V_n\}$ be as above. It is called an *effective* uniformity if there are recursive functions α_1, α_2 and α_3 such that

$$(A_1 \& A_2) \quad \cap_n V_n(x) = \{x\};$$

$$(\text{Effective } A_3) \quad \forall n, m \forall x \in X. V_{\alpha_1(n,m)}(x) \subseteq V_n(x) \cap V_m(x);$$

$$\text{(Effective } A_4\text{)} \quad \forall n \forall x, y \in X . x \in V_{\alpha_2(n)}(y) \Rightarrow y \in V_n(x);$$

$$\text{(Effective } A_5\text{)} \quad \forall n \forall x, y, z \in X . x \in V_{\alpha_3(n)}(y) \wedge y \in V_{\alpha_3(n)}(z) \Rightarrow x \in V_n(z)$$

$\mathcal{T} := \langle X, \{V_n\}, \alpha_1, \alpha_2, \alpha_3 \rangle$ is called an *effective uniform topological space*.

It is known that $\{V_n(x)\}$ forms a system of fundamental neighborhoods of x for $x \in X$.

Definition 2.2 (Effective convergence:[11]) A sequence $\{x_k\}$ from X is said to *effectively converge* to x in X if there is a recursive function γ satisfying

$$\forall n \forall k \geq \gamma(n) . x_k \in V_n(x).$$

This can be extended to effective convergence of a multiple sequence.

Definition 2.3 (Computability structure:[11]) Let $\mathcal{S}$ be a family of sequences from X (multiple sequences included). $\mathcal{S}$ is called a *computability structure* if the following C1$\sim$C3 hold.

C1: (Non-emptiness) $\mathcal{S}$ is nonempty.

C2: (Re-enumeration) If $\{x_k\} \in \mathcal{S}$ and α is a recursive function, then $\{x_{\alpha(i)}\}_i \in \mathcal{S}$.

C3: (Limit) If $\{x_{lk}\}$ belongs to $\mathcal{S}$, $\{x_l\}$ is a sequence from X, and $\{x_{lk}\}$ converges to $\{x_l\}$ effectively as k tends to the infinity, then $\{x_l\} \in \mathcal{S}$. ($\mathcal{S}$ is closed with respect to effective convergence.)

In fact, C2 and C3 should be extended to multiple sequences.

A sequence belonging to $\mathcal{S}$ is called *computable*, and $x \in X$ is called *computable* if $\{x, x, \cdots\}$ is computable.

Remark Applications of the effective uniform space to the computability problems of some discontinuous (in the Euclidean topology) real functions are seen in [11], [12] and [18]. In the present article, we are interested in the computability property of a *function sequence* whose functions have *different jump points*, and, for that purpose, we need to work on a *sequence of uniformities*.

3 A sequence of uniformities

We will assume that $\nu, n \in \mathbf{N}$.

In our previous work [11], an effective uniformity, say $\{U_n\}$, was associated with a piecewise continuous (in the Euclidean topology) real function so that its computability problem could be reduced to that of a continuous function with respect to $\{U_n\}$. Often, however, we have to deal with a sequence of functions $\{f_\nu\}$ whose jump points vary according to ν. In such a case, one might associate a (an effective) uniformity $\{V_n^\nu\}_n$ to f_ν for each ν. Then it may not be possible to consider the function sequence $\{f_\nu\}$ with respect to one uniformity. In analysis, it is important that a sequence of functions can be dealt with in one space. Thus we would like to define the "limit" of such an effective sequence of uniformities $\{V_n^\nu\}_{\nu,n}$, so that we can talk about the computability of the function sequence $\{f_\nu\}$ in the "limit space." In fact, the sequecne $\{W_{\langle\nu,n\rangle}\}$ defined by $W_{\langle\nu,n\rangle} = V_n^\nu$ does the work under a certain condition, where $\langle\nu,n\rangle$ denotes a standard recursive paring function. (Details will be presented later.) We can then define the notion of computability of $\{f_\nu\}$ with respect to the "limit uniformity" $\{W_{\langle\nu,n\rangle}\}$. What is important is that, in this way, we can discuss the computability of $\{f_\nu\}$ as a function sequence while *preserving the information* that f_ν is computable with respect to $\{V_n^\nu\}_n$. We can make this idea clear by considering the sequence of pairs $\{(f_\nu, \{V_n^\nu\}_n)\}_\nu$.

Definition 3.1 (Effective sequence of uniformities) 1) Let $\{V_n^\nu\}$ be a sequence of uniformities on a set X satisfying the following conditions (cf. Definition 2.1).

For each ν, $A_1\&A_2$ holds, that is, $\cap_n V_n^\nu(x) = \{x\}$.

In A_3, A_4 and A_5 in Definition 2.1, the recursive functions α_1, α_2 and α_3 depend also on ν. Thus, for example, effective A_4 stands as follows.

$$\forall\nu, n\forall x, y \in X.x \in V^\nu_{\alpha_2(\nu;n)}(y) \Rightarrow y \in V_n^\nu(x).$$

Then $\{V_n^\nu\}_{\nu,n}$ will be called an *effective sequence of uniformities* (on X).

(An expression such as $\alpha_2(\nu;n)$ is used here to emphasize that $\alpha_2(\nu;n)$ concerns the νth uniformity in a sequence, although mathematically $\alpha_2(\nu,n)$ will do just as well.)

2) The *limit sequence* of $\{V_n^\nu\}$, denoted by $\{W_{\langle\nu,n\rangle}\}$, is defined by $W_{\langle\nu,n\rangle} = V_n^\nu$. More precisely, define $\{W_l\}$ by $W_l = V^{\pi_1(l)}_{\pi_2(l)}$, where π_1 and π_2 are recursive inverse functions of a recursive pairing function $\langle p,q\rangle$ so that $l = \langle\pi_1(l), \pi_2(l)\rangle$.

3) The limit sequence $\{W_l\}(=\{W_{\langle\nu,n\rangle}\})$ as defined above will be called the *effective limit* of $\{V_n^\nu\}_{\nu,n}$ if, for A_3, further holds that there are recursive functions β and γ satisfying the following:

$$\forall\nu_1,\nu_2;n,m\forall x \in X.V^{\gamma(\nu_1,\nu_2;n,m)}_{\beta(\nu_1,\nu_2;n,m)}(x) \subseteq V_n^{\nu_1}(x) \cap V_m^{\nu_2}(x).$$

We will call this condition the *sequential intersection property* (of $\{V_n^\nu\}$).

Theorem 1 (Limit uniformity) The effective limit of the sequence $\{V_n^\nu\}$, denoted by $\{W_{\langle\nu,n\rangle}\}$ (cf. 2) and 3) of Definition 3.1), is an effective uniformity on X (cf. Definition 2.1). We can thus call $\{W_{\langle\nu,n\rangle}\}$ the *effective limit uniformity* of $\{V_n^\nu\}$.

Proof $A_1\&A_2$ is obvious. A_4 and A_5 hold due to the conditions in 1) of Definition 3.1. For example, A_4 can be shown to hold as below.

$$\forall\nu, n\forall x, y \in X.x \in W_{\langle\nu,\alpha_2(\nu;n)\rangle}(y) \Rightarrow y \in W_{\langle\nu,n\rangle}(x).$$

A_3 holds due to the sequential intersection property:

$$W_{\langle\gamma(\nu_1,\nu_2;n,m),\beta(\nu_1,\nu_2;n,m)\rangle}(x) \subseteq W_{\langle\nu_1,n_1\rangle}(x) \cap W_{\langle\nu_2,n_2\rangle}(x).$$

We will denote the space with the effective limit uniformity by

$$\mathcal{X} = \left\langle X, \{W_{\langle \nu,n\rangle}\}, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \right\rangle.$$

Theorem 1 guarantees that, under the sequential intersection property, there is the effective limit uniformity of an effective sequence of uniformities.

We will list some examples of the effective sequence of uniformities on the real line **R** with the sequential intersection property (cf. 3) of Definition 3.1). We will later consider some function sequences defined on these uniform spaces (Section 6).

Example 1 Let i represent an integer. Define $\{V_n^\nu\}$ on **R** as follows. $V_n^\nu(x) = \{i\}$ for all n if $x = i$ for an i where $-\nu \leq i \leq \nu$. Suppose $x \neq i$ for any such i. Let J^ν denote one of the open intervals $(-\infty, -\nu), (-\nu, -\nu+1), \cdots, (\nu-1, \nu), (\nu, \infty)$.

$$V_n^\nu(x) = J^\nu \cap (x - \frac{1}{2^n}, x + \frac{1}{2^n}) \quad \text{if } x \in J^\nu \quad .$$

Corollary 1 The effective limit of $\{V_n^\nu\}$ in **Example** 1 turns out to be topologically *equivalent to the amalgamated space* in [11]. The amalgamated space is the uniform space on **R** such that all the integers are isolated and each interval between adjacent integers is assigned the usual interval topology.

Example 2 For each $\nu = 1, 2, 3, \cdots$, consider the computable sequence of real numbers $\{\frac{2i+1}{2^{\nu+1}}\pi\}$, where $i \in \mathbf{Z}$. If $x = \frac{2i+1}{2^{\nu+1}}\pi$ for some i, then define a sequence of uniformities on **R** by $V_n^\nu(x) = \{\frac{2i+1}{2^{\nu+1}}\pi\}$ for all n. If $x \in J_i^\nu = (\frac{2i+1}{2^{\nu+1}}\pi, \frac{2i+3}{2^{\nu+1}}\pi)$ for some i, then put

$$V_n^\nu(x) = J_i^\nu \cap (x - \frac{1}{2^n}, x + \frac{1}{2^n}).$$

Example 3 Consider the interval $I_0 = [0, 1]$. Define first a "ν-decomposition" of I_0, denoted by $K^\nu_{\langle i_1, i_2, \cdots, i_\nu\rangle}$, where $i_j = 0, 1, 2$, as follows.

$$K^\nu_{\langle i_1, i_2, \cdots, i_\nu\rangle} = [\Sigma_{k=1}^\nu \frac{i_k}{3^k}, \Sigma_{k=1}^\nu \frac{i_k}{3^k} + \frac{1}{3^\nu}]$$

if $i_1, \cdots, i_{\nu-1}, i_\nu \neq 1$;

$$K^\nu_{\langle i_1, i_2, \cdots, i_\nu\rangle} = (\Sigma_{k=1}^\nu \frac{i_k}{3^k}, \Sigma_{k=1}^\nu \frac{i_k}{3^k} + \frac{1}{3^\nu})$$

if $i_1, \cdots, i_{\nu-1} \neq 1$ and $i_\nu = 1$;

$$K^\nu_{\langle i_1, i_2, \cdots, i_\nu\rangle} = K^\mu_{\langle i_1, i_2, \cdots, i_\mu\rangle}$$

if there exists a $\mu \leq \nu - 1$ such that $i_1, \cdots, i_{\mu-1} \neq 1$ and $i_\mu = 1$.

It is obvious that, for each ν, the collection of $K^\nu_{\langle i_1, i_2, \cdots, i_\nu\rangle}$'s forms a decomposition of I_0, hence, for every $x \in I_0$, there is a unique tuple $\langle i_1, i_2, \cdots, i_\nu\rangle$ such that $x \in K^\nu_{\langle i_1, i_2, \cdots, i_\nu\rangle}$. Now define a sequence of uniformities by

$$V_n^\nu(x) = K^\nu_{\langle i_1, i_2, \cdots, i_\nu\rangle} \cap (x - \frac{1}{2^n}, x + \frac{1}{2^n})$$

presuming that $x \in K^{\nu}_{\langle i_1, i_2, \cdots, i_\nu \rangle}$.

Example 4 Consider the interval $I = [0, 1)$. Put $p_\nu = \nu + 1$.

$$V^{\nu}_n(x) = [\frac{k}{p_\nu}, \frac{k+1}{p_\nu}) \cap (x - \frac{1}{2^n}, x + \frac{1}{2^n})$$

if $x \in [\frac{k}{p_\nu}, \frac{k+1}{p_\nu})$, for $k = 0, 1, \cdots, \nu$ and $n \in \mathbf{N}$.

It is a straightforward practice to show that the sequence of uniformities in each of **Examples** 1∼4 is an effective one.

In **Examples** 1∼3, define β and γ in 3) of Definition 3.1 by

$$\beta(\nu_1, \nu_2; n, m) = \max(n, m);\ \gamma(\nu_1, \nu_2; n, m) = \max(\nu_1, \nu_2).$$

In **Example** 4, define

$$\beta(\nu_1, \nu_2; n, m) = \max(n, m);\ \gamma(\nu_1, \nu_2; n, m) = p_{\nu_1} \times p_{\nu_2}.$$

Then $\{V^{\nu}_n\}$ satisfies the sequential intersection property.

For **Examples** 1 and 2, for each ν, the computable sequences with respect to $\{V^{\nu}_n\}$ (ν-computable sequences) can be defined following Section 6 in [11]; for **Examples** 3 and 4, they can be defined similarly to those in Section 4 below. We will return to some of these examples in Section 6.

In the next section, we will present a model example of an effective sequence of uniformities.

4 A sequence of uniformities on $I = [0, 1)$

We will subsequently present a special example of an effective sequence of uniformities, denoted by $\{U^{\nu}_n\}$, with which one can prove some interesting interrelations between the computability problem with respect to $\{U^{\nu}_n\}_n$ for each ν and the computability problem with respect to the effective limit uniformity. In Section 5, we will also give an example which indicates the following fact. The index ν represents the "size of the mesh" of a function f which is continuous with respect to $\{U^{\nu}_n\}_n$, that is, ν plays the role of a parameter expressing the distance of two adjacent points of discontinuity of f (in the Euclidean topology).

Put $I = [0, 1)$. This is the domain of our discourse in Sections 4 and 5. We assume that $\nu, k \in \mathbf{N}$ and $0 \le k \le 2^\nu - 1$. We introduce an example of the effective sequence of uniformities $\{U^{\nu}_n\}$ and its effective limit $\{Z_{\langle \nu, n \rangle}\}$ on I.

Definition 4.1 (Neighborhoods and uniformity) Define subintervals of I, denoted by I^{ν}_k, and a sequence of maps $U^{\nu}_n : I \to P(I)$ as follows.

$$I^{\nu}_k = [\frac{k}{2^\nu}, \frac{k+1}{2^\nu}); U^{\nu}_n(x) = I^{\nu}_k \cap (x - \frac{1}{2^n}, x + \frac{1}{2^n}) \quad \text{if} \quad x \in I^{\nu}_k.$$

Lemma 1 1) $\{U_n^\nu\}$ is a decreasing sequence with respect to ν, that is, $U_n^{\nu+1}(x) \subseteq U_n^\nu(x)$.

2) $\{U_n^\nu\}$ is a decreasing sequence with respect to n, that is, $U_{n+1}^\nu(x) \subseteq U_n^\nu(x)$.

3) $x \in U_n^\nu(y)$ if and only if $y \in U_n^\nu(x)$ (symmetry).

4) If $x \in U_{n+1}^\nu(y), y \in U_{n+1}^\nu(z)$, then $x \in U_n^\nu(z)$ (transitivity).

Proposition 4.1 (Effective sequence) 1) $\{U_n^\nu\}$ forms an effective sequence of uniformities on I (cf. 1) of Definition 3.1).

2) Put $Z_l = U_{\pi_2(l)}^{\pi_1(l)}$. $\{Z_l\}$ is the effective limit of $\{U_n^\nu\}$ (cf. 3) of Definition 3.1). We will write $\{Z_l\}$ as $\{Z_{\langle\nu,n\rangle}\}$.

Proof 1) $A_1 \& A_2$: $\cap_n U_n^\nu(x) = \{x\}$ is obvious.

A_3, A_4, A_5 follow from 2), 3) and 4) of Lemma 1, that is, for each ν, take

$$\alpha_1(\nu; n, m) = \max(n, m);\ \ \alpha_2(\nu; n) = n;\ \ \alpha_3(\nu; n) = n + 1.$$

2) For β and γ in 3) of Definition 3.1, put

$$\beta(\nu_1, \nu_2; n, m) = \max(\nu_1, \nu_2);\ \ \gamma(\nu_1, \nu_2; n, m) = \max(n, m).$$

Then $U_{\gamma(\nu_1,\nu_2;n,m)}^{\beta(\nu_1,\nu_2;n,m)}(x) \subseteq U_n^{\nu_1}(x) \cap U_m^{\nu_2}(x)$.

Definition 4.2 (ν-computability) Let ν be an arbitrary (but fixed) natural number. In the following, μ will denote a finite sequence of indices, which is possibly empty. μi will denote a finite sequence (of natural numbers) μ followed by i.

1) A (multiple) sequence $\{a_{\mu i}\}$ is called a ν-sequence if, for each μ, there exists a $k = k_\mu \leq 2^\nu - 1$ such that $\{a_{\mu i}\}_i \subseteq I_k^\nu$.

2) A multiple sequence of rational numbers $\{r_{\mu i}\}$ is called a *recursive ν-sequence* if it is recursive and is a ν-sequence.

3) $\{a_{\mu m i}\}$ *converges ν-effectively* to $\{x_{\mu m}\}$ with respect to i if there is a recursive function α so that $i \geq \alpha(\mu, m, p)$ implies $a_{\mu m i} \in U_p^\nu(x_{\mu m})$. We write this property as

$$x_{\mu m} \simeq_\nu \langle a_{\mu m i}, \alpha(\mu, m, p)\rangle$$

or, for short, $x_{\mu m} \simeq_\nu \langle a_{\mu m i}, \alpha\rangle$.

4) A sequence of real numbers $\{x_{\mu m}\}$ is called *ν-computable* if there are a recursive ν-sequence $\{r_{\mu m i}\}$ and a recursive function α as in 3) such that

$$x_{\mu m} \simeq_\nu \langle r_{\mu m i}, \alpha(\mu, m, p)\rangle\,.$$

5) A real number x is called ν-computable if the sequence $\{x, x, \cdots\}$ is ν-computable.

Corollary 2 For each ν, the family of ν-computable sequences from I is a computability structure with respect to $\{U_n^\nu\}_n$ (cf. Definition 2.3). (Let us denote this structure by $\mathcal{C}_\nu$.)

Definition 4.3 (ω-computability) 1) A sequence $\{a^\nu_{\mu i}\}_{\nu\mu i} \subseteq I$ is called a $\{\nu\}$-*sequence* if $\{a^\nu_{\mu i}\}_{\mu i}$ is a ν-sequence for each ν.

2) A sequence $\{r^\nu_{\mu i}\}_{\nu\mu i}$ of rational numbers from I is called a *recursive* $\{\nu\}$-*sequence* if it is recursive and is a $\{\nu\}$-sequence.

(In this case, $k = k^\nu_\mu$ in 1) of Definition 4.2 can be effectively found by cheking "$r^\nu_{\mu 1} \in I^\nu_k$?", and hence the sequence $\{k^\nu_\mu\}$ is recursive.)

3) A $\{\nu\}$-sequence $\{a^\nu_{\mu m i}\}$ is said to *converge* ω-*effectively* to $\{x_{\mu m}\}$ (with respect to i) if there is a recursive function α such that

$$\forall\nu\forall\mu\forall m\forall p\forall i \geq \alpha(\nu;\mu,m,p).a^\nu_{\mu m i} \in U^\nu_p(x_{\mu m}).$$

This fact will be expressed by

$$x_{\mu m} \simeq_\omega \left\langle a^\nu_{\mu m i}, \alpha(\nu;\mu,m,p)\right\rangle$$

or simply $x_{\mu m} \simeq_\omega \left\langle a^\nu_{\mu m i}, \alpha\right\rangle$.

4) A sequence $\{x_{\mu m}\}$ is called ω-*computable* if there are a recursive $\{\nu\}$-sequence $\{r^\nu_{\mu m i}\}$ and a recursive function α as in 3) above, that is,

$$x_{\mu m} \simeq_\omega \left\langle r^\nu_{\mu m i}, \alpha(\nu;\mu,m,p)\right\rangle.$$

5) A real number x is called ω-computable if the sequence $\{x, x, \cdots\}$ is ω-computable.

Lemma 2 A recursive sequence of rational numbers, say $\{q_j\}$, regarded as a sequence of real numbers, is ω-computable, since we can obtain an approximating sequence $\{r^\nu_{ji}\}$ by $r^\nu_{ji} = q_j$ for all ν and i.

Theorem 2 (ω-computability structure) ω-computable sequences form a computability structure with respect to $\{Z_l\}$ (cf. Definition 2.3).

Proof C1 is guaranteed by Lemma 2. C2 is obvious.

C3: Suppose $\{x_{lm}\}$ is ω-computable, that is, there is a recursive $\{\nu\}$-sequence $\{r^\nu_{lmi}\}$ and a recursive α satisfying $x_{lm} \simeq_\omega \langle r^\nu_{lmi}, \alpha\rangle$. Suppose also that $\{x_l\} \subseteq I$ and $\{x_{lm}\}$ converges ω-effectively to it, that is, there is a recursive γ satisfying $x_l \simeq_\omega \langle x_{lm}, \gamma\rangle$. Define

$$q^\nu_{lp} = r^\nu_{l\gamma(\nu;l,p+1)\alpha(\nu;l,\gamma(\nu;l,p+1),p+1)}.$$

$\{q^\nu_{lp}\}_p$ is a recursive $\{\nu\}$-sequence, and the following hold:

$$q^\nu_{lp} \in U^\nu_{p+1}(x_{l\gamma(\nu;l,p+1)}), x_{l\gamma(\nu;l,p+1)} \in U^\nu_{p+1}(x_l).$$

From this, by virtue of 4) and 2) of Lemma 1, follows $\forall s \geq p.q^\nu_{ls} \in U^\nu_p(x_l)$, that is, $\{x_l\}$ is ω-computable with a recursive $\{\nu\}$-sequence $\{q^\nu_{lp}\}$ and the identity function as a modulus of convergence.

Theorem 3 (**R**-computability, ν-computability and ω-computability) 1) If $\{x_{\mu m}\}$ is ω-computable, then it is ν-computable for all ν.

2) For each ν, if $\{x_{\mu m}\}$ is ν-computable, then it is **R**-computable, hence by 1), an ω-computable sequence is **R**-computable.

3) For one real number $x \in I$, **R**-computability, ω-computability and ν-computability (for each fixed ν) are mutually equivalent.

4) For each ν, there is a sequence $\{x_m\}$, which is **R**-computable but is not ν-computable.

5) For each ν_1, ν_2 where $\nu_1 < \nu_2$, there is a sequence $\{x_m\}$ which is ν_1-computable but not ν_2- computable.

6) If $\nu_1 < \nu_2$ and $\{x_m\}$ is ν_2-computable, then it is ν_1-computable.

Proof 1) is obvious from the definitions (cf. Definitions 4.2 and 4.3).

2) is obvious from the definition (cf. 4) of Definition 4.2).

3) The facts that an ω-computable real number is ν-computable for each ν and a ν computable real number is **R**-computable follow from 1) and 2) as special cases.

To finish up the equivalences, let us note that each rational number is both **R**- and ω-computable (cf. Lemma 2). Suppose then that x is irrational and **R**-computable with $x \simeq \langle q_m, \alpha \rangle$ (cf. Section 2). We will show that x is ω-computable. Then the equivalences follow.

Since x is irrational, for each ν, there is a unique $k = k_\nu \leq 2^\nu - 1$ such that $x \in I_k^\nu$, or $\frac{k}{2^\nu} < x < \frac{k+1}{2^\nu}$. $\{k_\nu\}$ can be effectively determined. With this $\{k_\nu\}$, we will construct a recursive $\{\nu\}$-sequence $\{t_p^\nu\}$ and a recursive δ such that $x \simeq_\omega \left\langle t_p^\nu, \delta \right\rangle$. (We will omit the ν in k_ν for the simplicity of notations.)

Construction of t_p^ν is given below:

$$
\begin{aligned}
t_p^\nu = q_{\alpha(p+\nu)} \quad &\text{if} \quad \frac{k}{2^\nu} \leq q_{\alpha(p+\nu)} < \frac{k+1}{2^\nu}; \\
= q_{\alpha(p+\nu)} + \frac{1}{2^{p+\nu}} \quad &\text{if} \quad q_{\alpha(p+\nu)} < \frac{k}{2^\nu}; \\
= \frac{k+1}{2^\nu} - \frac{1}{2^{p+\nu}} \quad &\text{if} \quad \frac{k+1}{2^\nu} \leq q_{\alpha(p+\nu)}.
\end{aligned}
$$

$\{t_p^\nu\}$ is recursive. We will show that it is a $\{\nu\}$-sequence, and that it converges ω-effectively to x, that is,

1. $t_p^\nu > \frac{k}{2^\nu}$;
2. $t_p^\nu < \frac{k+1}{2^\nu}$;
3. $|x - t_p^\nu| < \frac{1}{2^p}$.

We then have that x is ω-computable (cf. 4) of Definition 4.3).

Proof of 1$\sim$3.

Case where $q_{\alpha(p+\nu)} < \frac{k}{2^\nu} (< x)$.

1. Since $0 < x - q_{\alpha(p+\nu)} < \frac{1}{2^{p+\nu}}$, it follows that

$$
\frac{k}{2^\nu} < x < q_{\alpha(p+\nu)} + \frac{1}{2^{p+\nu}} = t_p^\nu.
$$

2. Since $q_{\alpha(p+\nu)} < \frac{k}{2^\nu}$, it follows that

$$t_p^\nu = q_{\alpha(p+\nu)} + \frac{1}{2^{p+\nu}} < \frac{k}{2^\nu} + \frac{1}{2^{p+\nu}} < \frac{k+1}{2^\nu}.$$

3. $|x - t_p^\nu| = |x - (q_{\alpha(p+\nu)} + \frac{1}{2^{p+\nu}})|$

$$\leq |x - q_{\alpha(p+\nu)}| + \frac{1}{2^{p+\nu}} < \frac{1}{2^{p+\nu}} + \frac{1}{2^{p+\nu}} \leq \frac{1}{2^p}.$$

The last inequality holds presuming that $\nu \geq 1$.

Other cases are similarly treated.

4) Let a be a recursive injection whose image is not recursive.

Define $y_m = \frac{1}{2^l}$ if $m = a(l)$; $= 0$ if there is no l such that $m = a(l)$. The sequence $\{y_m\}$ is known to be **R**-computable (cf. [9]). Next define x_m by

$$x_m = \frac{1}{2^\nu} - \frac{y_m}{2^\nu}.$$

Then $\{x_m\}$ is **R**-computable. $x_m = \frac{1}{2^\nu} - \frac{1}{2^{l+\nu}} < \frac{1}{2^\nu}$ if there is an l such that $m = a(l)$, and $= \frac{1}{2^\nu}$ otherwise. In the former case, $x_m \in I_0^\nu$ and in the latter case $x_m \in I_1^\nu$.

Suppose there were a recursive ν-sequence $\{q_{mi}\}$ which ν-effectively converges to $\{x_m\}$ (cf. Definition 4.2). Then the following equivalences hold.

$$\{q_{mi}\}_i \subseteq I_0^\nu \leftrightarrow x_m < \frac{1}{2^\nu} \leftrightarrow y_m = \frac{1}{2^l} \quad \text{for some} \quad l \leftrightarrow m \in a(\mathbf{N});$$

$$\{q_{mi}\}_i \subseteq I_1^\nu \leftrightarrow x_m = \frac{1}{2^\nu} \leftrightarrow y_m = 0 \leftrightarrow m \notin a(\mathbf{N}).$$

Since $\{q_{mi}\}$ is a recursive ν-sequence, it can be determined in which of I_0^ν or I_1^ν it is contained by checking q_{m1}, hence it would be recursively determined whether $m \in a(\mathbf{N})$ or not for every m, contradicting non-recursiveness of $a(\mathbf{N})$. So, $\{x_m\}$ cannot be ν-computable.

5) Define $\{y_m\}$ as above for ν_2, and then $\{x_m\}$ defined below will do:

$$x_m = \frac{1}{2^{\nu_2}} - \frac{y_m}{2^{\nu_2}}.$$

6) Immediate from the definition.

5 ω-computability of a function sequence

As for the computability problem of a function sequence with respect to an effective limit uniformity, the Rademacher function sequence is a typical example. We will thus explain how the uniform computability of this function system can be treated in $\{Z_l\}$ (cf. Proposition 4.1). The Rademacher function system (among various piecewise continuous functions) has mathematical significance, since it is a subsystem of the Walsh function system, which plays an important role in analysis.

Other examples will be given later.

Definition 5.1 (ω-uniform computability of a function sequence) Let $\{f_n\}$ be a sequence of real functions on I.

1) (ω-sequential computability) $\{f_n\}$ is called ω-*sequentially computable* if, for every ω-computable sequence from I (cf. 4) of Definition 4.3), say $\{x_m\}$, $\{f_n(x_m)\}$ is an $\mathbf{R}$-computable (double) sequence.

2) (Effective ω-uniform continuity) $\{f_n\}$ is called ω-*uniformly continuous* if there are recursive functions δ and ε such that

$$\forall n \forall p \forall x \in I \forall y \in Z_{\langle \delta(n,p), \varepsilon(n,p) \rangle}(x). |f_n(x) - f_n(y)| < \frac{1}{2^p}.$$

3) (ω-uniform computability) $\{f_n\}$ is called ω-*uniformly computable* (with respect to $\{Z_l\}$) if it is ω-sequentially computable and is effectively ω-uniformly continuous.

The νth *Rademacher function* $\phi_\nu(x)$, $\nu = 0, 1, 2, 3, \cdots$, is defined as follows.

$$\phi_0(x) = 1, \quad x \in [0, 1)$$

$$\phi_\nu(x) = \begin{cases} 1, & x \in [\frac{2i}{2^\nu}, \frac{2i+1}{2^\nu}) \\ -1, & x \in [\frac{2i+1}{2^\nu}, \frac{2i+2}{2^\nu}) \end{cases}$$

where $\nu \geq 1$ and $i = 0, 1, 2, \cdots, 2^{\nu-1} - 1$.

The sequence $\{\phi_\nu(x)\}$ is called the *Rademacher function system*.

Note To each ϕ_ν, the effective uniform space $\{U_n^\nu\}_n$ is associated, and the size of the mesh is $\frac{1}{2^\nu}$.

As has been explained in Introduction, Yasugi and Washihara [21] have shown that the function system $\{\phi_\nu\}$ is endowed with some kind of computational attributes. Here in this article, we will show that the computability problem of the function sequence $\{\phi_\nu\}$ can be treated in the effective limit $\{Z_l\}$ (cf. Proposition 4.1).

Proposition 5.1 (ω-sequential computability of $\{\phi_\nu\}$) The Rademacher functions system $\{\phi_\nu\}$ is ω-sequentially computable with respect to $\{Z_l\}$.

Proof Let $\{x_m\}$ be an ω-computable sequence from I: $x_m \simeq_\omega \langle q_{mi}^\nu, \alpha \rangle$ (cf. 4) of Definition 4.3). We show that $\{\phi_\nu(x_m)\}$ is an $\mathbf{R}$-computable double sequence of real numbers. From the assumption follows that, for all $i \geq \alpha(\nu; m, p)$, $q_{mi}^\nu \in U_p^\nu(x_m)$. There is (classically) a $k = k_{\nu mp}$ such that $U_p^\nu(x_m) = I_k^\nu \cap (x_m - \frac{1}{2^p}, x_m + \frac{1}{2^p})$, hence, in particular, $q_{m\alpha(\nu;m,p)}^\nu \in I_k^\nu$. Recall that $\{q_{ni}^\nu\}$ is a ν-sequence, and hence the $k = k_{\nu mp}$ in fact does not depend on p. Since $\{q_{m\alpha(\nu;m,1)}^\nu\}$ is a recursive sequence of rational numbers, $k = k_{\nu m1}$ above can in fact be computed by examining the inequalities

$$\frac{k}{2^\nu} \leq q_{m\alpha(\nu;m,1)}^\nu < \frac{k+1}{2^\nu},$$

and hence $\{k_{\nu m1}\}$ is a recursive sequence of integers. By definition, $\phi_\nu(x_m) = \phi_\nu(q^\nu_{m\alpha(\nu;m,1)})$, and hence, to evaluate $\phi_\nu(x_m)$, it suffices to compute $\phi_\nu(q^\nu_{m\alpha(\nu;m,1)})$ ($= 1$ or $= -1$). This value can be determined according as $k_{\nu m1}$ is even or odd. It is then a trivial matter to show that $\{\phi_\nu(x_m)\}$ is a computable sequence of real numbers.

Theorem 4 (Uniform ω-computability of $\{\phi_\nu\}$) The function system $\{\phi_\nu\}$ is ω-uniformly computable with respect to $\{Z_l\}$.

Proof ω-sequential computability has been proved in Proposition 5.1. The effective ω-uniform continuity follows from the fact that $\phi_\nu(x) = \phi_\nu(y)$ holds when $y \in U^\nu_p(x)$.

Note The effective limit uniform space $\langle I, \{Z_l\}\rangle$ is topologically equivalent with the space $\langle I, \{U^\nu_\nu\}\rangle$ (called diagonal-space: cf. Section 7 below), which is in turn topologically equivalent with the Fine-metric space. The effective uniform computability of $\{\phi_\nu\}$ holds in the Fine-metric space (cf. [5],[6]). The virtue of the treatment in $\langle I, \{Z_l\}\rangle$ is that it preserves the information on the mesh of each function in the system.

Example (Brattka's example: [1]) $\mu(x)$ will denote the binary representation of x with infinitely many 0's.

Let $f : I \to \mathbf{R}$ be defined as follows.

$$f(x) = \begin{cases} \sum_{i=0}^{\infty}(\ell_i \bmod 2)2^{-n_i-\sum_{j=0}^{i-1}(n_j+\ell_j)} \\ \qquad \text{if } \ \mu(x) = 0^{n_0}1^{\ell_0}0^{n_1}1^{\ell_1}0^{n_2}\cdots \\ \sum_{i=0}^{m}(\ell_i \bmod 2)2^{-n_i-\sum_{j=0}^{i-1}(n_j+\ell_j)} \\ \qquad \text{if } \ \mu(x) = 0^{n_0}1^{\ell_0}0^{n_1}1^{\ell_1}0^{n_2}\cdots 1^{\ell_m}0^{\omega} \end{cases},$$

where $n_0 \geq 0, n_i > 0$ for $i > 0$ and $l_i > 0$ for all $i \geq 0$.

Brattka showed that the function f is "Fine-computable" but is "not locally uniformly Fine-computable." f can be approximated by the function sequence $\{\varphi_\nu\}$ defined below. Each φ_ν is continuous in I^ν_k, and jumps at $\frac{k}{2^\nu}$, $k \leq 2^\nu - 1$. So, φ_ν is continuous in the uniform space $\{U^\nu_n\}$. In our language, $\{\phi_\nu\}$ is an "ω-computable" sequence of functions and f is an "ω-computable" function.

$$\varphi_1(x) = \begin{cases} 0 & \text{if } \ 0 \leq x < \frac{1}{2} \\ 1 & \text{if } \ \frac{1}{2} \leq x < 1 \end{cases},$$

$$\varphi_\nu(x) = \begin{cases} \frac{1+(-1)^i}{2} + \frac{\varphi_{\nu-i}(2^i(x-1+\frac{1}{2^{i-1}}))}{2^i} & \text{if } \ 1-\frac{1}{2^{i-1}} \leq x < 1-\frac{1}{2^i} \\ (i = 1, 2, \ldots, \nu-1) & \\ \frac{1+(-1)^\nu}{2} & \text{if } \ 1-\frac{1}{2^{\nu-1}} \leq x < 1-\frac{1}{2^\nu} \\ \frac{1+(-1)^{\nu+1}}{2} & \text{if } \ 1-\frac{1}{2^\nu} \leq x < 1 \end{cases}.$$

For each ν, the "mesh" of φ_ν is $\frac{1}{2^\nu}$.

6 Other examples

Here we will take up **Examples** 3 and 2 in Section 3.

Example 3 It is easy to show that the sequence of uniformities in Example 3 is an effective sequence of uniformities. The notions of the ν-computability and the ω-computability of sequences from I_0 with respect to $\{V_n^\nu\}$ can be defined similarly to those in Definitions 4.2 and 4.3, replacing I_k^ν by $K^\nu_{\langle i_1,\cdots,i_\nu\rangle}$.

Define next a function sequence $\{\gamma_\nu\}$ as follows. Suppose $x \in K^\nu_{\langle i_1,\cdots,i_\nu\rangle}$.

$$\gamma_\nu(x) = 0 \quad \text{if} \quad i_1, \cdots, i_{\nu-1} \neq 1, i_\nu = 1;$$

$$\gamma_\nu(x) = 1 \quad \text{if} \quad i_1, \cdots, i_\nu \neq 1;$$

$$\gamma_\nu(x) = \gamma_\mu(x)(=0) \quad \text{if} \quad \exists \mu \leq \nu - 1. i_1, \cdots, i_{\mu-1} \neq 1, i_\mu = 1.$$

Each γ_ν is uniformly continuous with respect to $\{V_n^\nu\}_n$. It is also sequentially computable and is effectively uniformly continuous with respect to $\{V_n^\nu\}_n$. In fact, the function sequence $\{\gamma_\nu\}$ is uniformly ω-computable with respect to $\{W_l\} = \{V_{\pi_2(l)}^{\pi_1(l)}\}$. These facts can be shown similarly to the proofs of Proposition 5.1 and Theorem 4.

The pointwise limit, that is, $\lim_\nu \gamma_\nu(x)$ for each x, exists. Let us denote this limit by $c(x)$. It is obvious that $c(x)$ is the characteristic function of the Cantor set. Since $\{\gamma_\nu(x)\}$ pointwise approximates $c(x)$, in a way, we can identify the Cantor set with the function sequence $\{\gamma_\nu\}$.

We can sum up the result above as follows.

Proposition 6.1 (Computability of $\{\gamma_\nu\}$) The sequence of functions $\{\gamma_\nu\}$, which pointwise approximates (in the Euclidean topology, classically) the Cantor ternary set, is uniformly computable in the effective limit $\langle I_0, \{W_l\}\rangle$.

Note The function $c(x)$ is not continuous with respect to $\{W_l\}$, and hence the convergence is not uniform.

Remark In the space $L^2[0,1]$, a set such as the Cantor set is negligible, and hence there is no way of discussing its approximation (or the approximation of its characteristic function) with a computable sequence of functions in the functional space approach to computability.

Example 2 Consider the function sequence $\tau_\nu(x)$ defined on $\mathbf{R}$ as follows. $\tau_\nu(x) = \tan(2^\nu x)$ if $x \in J_i^\nu$; $= 0$ if $x = \frac{2i+1}{2^{\nu+1}}\pi$. $\{\tau_\nu(x)\}$ is not uniformly computable in the effective limit, but is "locally uniformly computable." (The notions of locally uniform computability and computability in an effective uniform space and related examples are also mentioned in [12].) The "mesh" of $\tau_\nu(x)$ is $\frac{\pi}{2^\nu}$.

7 Appendix: Diagonal uniformity

As was notified in Note of Section 5, we will show that $\{Z_l\}$ in Section 4 is equivalent to "diagonal-uniformity" $\{U_n^n\}$.

Definition 7.1 (Diagonal sequence) The sequence $\{U_n^n\}(=\{Z_{\langle n,n\rangle}\})$ will be called the *diagonal sequence* of $\{U_n^\nu\}$, and will be denoted by $\{U_n\}$. A sequence of real numbers $\{x_m\} \subseteq I$ is called diagonal-computable if it is effectively approximated by a recursive sequence of rational numbers with respect to $\{U_n\}$.

Lemma 3 1) For any n and any $x \in I$, $U_n(x) = I_k^n$ for some k. So,

$$y \in U_n(x) \leftrightarrow U_n(y) = I_k^n = U_n(x).$$

2) $U_{n+1}(x) \subset U_n(x)$.

Proposition 7.1 (Diagonal sequence and limit) The sequence $\{U_n\}$ forms an effective uniformity which is topologically effectively equivalent to the effective limit $\{Z_l\}$ (cf. Proposition 4.1).

Proof We use Lemma 3.

$A_1\&A_2$ is obvious. For A_3, $m = \max(n_1, n_2)$ will do. For A_4 and A_5, put $m = n$. $\{U_n\}$ thus forms an effective uniformity.

As for the effective equivalence, notice that $\{U_n\}$ is a subsequence of $\{U_n^\nu\}$. For the converse, put $n_0 = \max(\nu, n)$. Then $U_n^\nu(x) \supset U_{n_0}(x)$.

Definition 7.2 (Diagonal uniformity) The sequence of diagonals $\{U_n\}$ as in Definition 7.1 will be called the *diagonal-uniformity* determined by $\{U_n^\nu\}$ or $\{Z_{\langle \nu,n\rangle}\}$, and the space $\langle I, \{U_n\}\rangle$ will be called the *diagonal space* determined by $\{U_n^\nu\}$.

Definition 7.3 (Diagonal-computability) A sequence of real numbers $\{x_m\} \subset I$ is *diagonal-computable* if there is a recursive sequence $\{q_{mp}\}$ of rational numbers which converges to $\{x_m\}$ effectively with respect to $\{U_n\}$ in a manner that, for a recursive function γ and for $k \geq \gamma(m,p)$, $q_{mk} \in U_p(x_m)$. We will write this property as

$$x_m \simeq_{\mathcal{D}} \langle q_{mk}, \gamma\rangle .$$

The definition can be extended to a multiple sequence.

Proposition 7.2 (Diagonal computability structure) The family of diagonal-computable sequences of real numbers, say $\mathcal{R}$, forms a computability structure for $\langle I, \{U_n\}\rangle$ (cf. C1∼C3 of Definition 2.3).

Proof C1 and C2 are obvious. For C3, suppose that $\{x_{lm}\} \in \mathcal{R}$ is represented by $x_{lm} \simeq_{\mathcal{D}} \langle q_{lmp}, \gamma\rangle$, and that $\{x_{lm}\}$ effectively converges to $\{x_l\}$ with a recursive modulus of convergence ε. In particular, $q_{lm\gamma(l,m,p)} \in U_p(x_{lm})$ and $x_{l\varepsilon(l,p)} \in U_p(x_l)$.

If we put $r_{lp} = q_{l\varepsilon(l,p)\gamma(l,\varepsilon(l,p),p)}$, then $\{r_{lp}\}$ is a recursive sequence of rational numbers satisfying $r_{lp} \in U_p(x_{l\varepsilon(l,p)})$. On the other hand, since $x_{l\varepsilon(l,p)} \in U_p(x_l)$, $U_p(x_{l\varepsilon(l,p)}) = U_p(x_l)$ by 1) of Lemma 3. So, $r_{lp} \in U_p(x_{l\varepsilon(l,p)}) = U_p(x_l)$.

Put next $\eta(l,p) = p$. If $q \geq p$, then by 2) of Lemma 3, $r_{lq} \in U_q(x_l) \subset U_p(x_l)$. It thus follows that $x_l \simeq_{\mathcal{D}} \langle r_{lp}, \eta \rangle$, or $\{x_l\} \in \mathcal{R}$.

Theorem 5 (Diagonal-computability and ω-computability) A sequence $\{x_m\} \subset I$ is diagonal-computable if and only if it is ω-computable (cf. Definitions 7.3 and 4.3).

Proof Suppose $\{x_m\}$ is diagonal-computable. Then there are recursive $\{q_{mp}\}$ and γ such that $x_m \simeq_{\mathcal{D}} \langle q_{mk}, \gamma \rangle$ (cf. Definition 7.3). Define $r^{\nu}_{mi} = q_{m\gamma(m,\nu)+i}$, and put $\alpha(\nu; m, p) = \gamma(m, \nu + p)$. $\{r^{\nu}_{mi}\}$ is a recursive $\{\nu\}$-sequence and α is a recursive function. Suppose $j = \alpha(\nu; m, p) + l$. Then

$$r^{\nu}_{mj} = r^{\nu}_{m\alpha(\nu;m,p)+l} = r^{\nu}_{m\gamma(m,\nu+p)+l} = q_{m\gamma(m,\nu)+\gamma(m,\nu+p)+l}$$

$$= q_{m\gamma(m,\nu+p)+\gamma(m,\nu)+l} \in U^{\nu+p}_{\nu+p}(x_m) \subset U^{\nu}_{p}(x_m)$$

by Lemma 1, and so $x_m \simeq_{\omega} \langle r^{\nu}_{mj}, \alpha \rangle$.

Suppose conversely, that $\{x_m\}$ is ω-computable: $x_m \simeq_{\omega} \langle r^{\nu}_{mi}, \alpha \rangle$, where $\{r^{\nu}_{mp}\}$ is a recursive $\{\nu\}$-sequence. Put $q_{mp} = r^{p}_{mp}$. Then $q_{mp} \in U_p(x_m) = I^p_k$ for some appropriate k. Put $\beta(m,p) = \alpha(p; m, p) + p$, and suppose $i \geq \beta(m,p)$. Then $i = \alpha(p; m, p) + j + p$ for some j. So, $q_{mi} = r^{i}_{m\alpha(p;m,p)+j+p} \in U^{i}_{p}(x_m) \subset U^{p}_{p}(x_m)$ (cf. 1) of Lemma 1), that is, $x_m \simeq_{\mathcal{D}} \langle q_{mi}, \beta \rangle$.

Note Yasugi, Tsujii and Mori [19] showed that the metric induced from an effective uniformity by a *general construction* preserves effective convergence, while left it open if it preserves sequential computability. The metric obtained from the uniformity $\{U_n\}$ by the general construction method indeed preserves sequential computability. The resulting metric induces the same system of fundamental neighborhoods as the one that is induced from the Fine metric.

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