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Effective sequence of uniformities and the Rademacher function system

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Abstract. An *effective sequence of unifomities* on a set and its *limit* are defined. By taking the *diagonal* of the limit space, we can express the *uniform computability* of the Rademacher function system, which is the basis of Walsh-Fourier analysis.

1. Introduction

It is important and mathematically significant to review some theories of mathematics from an algorithmic standpoint.

The standard notion of computability of a real number or of a sequence of real numbers as well as that of computability of a continuous or of a sequence of continuous functions is now generally agreed. There are many references on this subject. We have referred to [8]; there is also [18] for a quick read.

The reason why one can define a reasonable notion of computability for a continuous function is that for a continuous function there is a way to nicely approximate the values for computable inputs, and this notion depends on the continuity.

Very often, however, we compute values and draw a graph of a discontinuous function. We thus expect that some class of discontinuous functions can be attributed a certain kind of computability. In an attempt of computing a discontinuous function, a problem arises in the computation of the value at a jump point (a point of discontinuity). This is because it is not

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in general decidable if a real number is a jump point, that is, the question "$x = a$?" is not decidable even for computable x and a.

(For the subsequent discussion, let us here note the following: $=, \leq, <$ on natural numbers and fractional numbers are decidable. $a < b$ is decidable for computable real numbers a and b, while $a = b$ and $a \leq b$ are not necessarily decidable even for computable real numbers.)

One method of dissolving this problem was proposed in [8] by Pour-El and Richards and was succeeded by Washihara (cf. [11],[12],[13]). It was a functional analysis approach. In their treatment, a function is regarded as computable as a point in a space. This is sufficient in order to draw a rough graph of the function, but does not supply us with information when computing individual values.

There are many ways of characterizing computation of a discontinuous function. We have proposed some approaches to this problem. One is to express the value of a function at a jump point in terms of a "limiting recursive" modulus of convergence instead of a recursive one (Yasugi, Brattka, Washihara:[14]). Another is to change the topology of the domain of a function (Tsujii, Mori, Yasugi:[10]). In a way, these two approaches are equivalent ([16]).

As for a sequence of functions with different jump points, we proposed an approach with an example of the system of Rademacher functions (Yasugi, Washihara:[19]) in terms of a "limiting recursive" modulus of convergence. (The notion of limiting recursive functions is due to [1], and has been utilized also in [7] and [2] along a similar line as ours.)

Let $\{\phi_n(x)\}$ be the sequence of Rademacher functions, that is, for each n, $\phi_n(x)$ is defined on $[0, 1)$, is discontinuous at the dyadic rational numbers of the form $\frac{k}{2^n}$, and assumes the values 1 and -1 alternatingly. The mathematical significance of the Rademacher function system among various discontinuous functions is that it is a subsystem of the Walsh function system, and the latter plays an important role in Walsh-Fourier analysis. (As for Rademacher and Walsh functions, one can refer to [9].)

In [19], it was shown that $\{\phi_l\}$ has a "weak computation" in the following sense: there is a program which acts in a manner that, input a sequence of information $\{\langle r_{mn}, \alpha\rangle\}$ of $\{x_m\}$, a recursive sequence of rational numbers $\{s_{lmn}\}$ which converges to $\{\phi_l(x_m)\}$ constructed with a modulus of convergence which is "limiting recursive" can be utilized.

Here we present an alternative way of expressing a notion of computability of the Rademacher function system by changing the topology of the real interval $[0, 1)$ by decomposing it to $\{[\frac{k}{2^\nu}, \frac{k+1}{2^\nu})\}$ for $k \leq 2^\nu - 1$, and then taking a kind of the limit with respect to ν. Our theory of the effective uniform space (cf. [10], [17], [16]).

We subsequently present a brief account of some basics such as the definitions of computable reals and computable (continuous) functions (Section 2), the definition of the Rademacher function system (Section 3), and the theory of effective uniformity (Section 4). Next a theory of a sequence of uniformities and its limit is introduced (Section 5). We then confine ourselves to the real numbers in the interval $I = [0, 1)$ and functions on it. The theory of an effective sequence of uniformities on I and its limit is developed in Section 6. The "limit uniformity" is proven to be effectively equivalent to the "diagonal uniformity" and two notions of computability, "diagonal computability" and "ω-computability," are shown to be equivalent (Theorem 1: Section 7). With all this preparation, we propose the notion of "uniform $\mathcal{D}$-computability" of a piecewise continuous function in the space of the diagonal uniform space and then prove that the Rademacher function system is "uniformly $\mathcal{D}$-computable" (Theorem 2: Section 8). As an additional topic, we show that the metric induced from the diagonal uniformity by a general construction "preserves comutability" (Theorem 3: Section 9).

In the references we only list those which have close relationship with the present work. We may add our approach to computability problems in the metric space, [6], [15].

2. Preliminaries

In the following, $\mathbf{N}$ will denote the set of natural numbers.

The basic definitions below are taken from [8]. A sequence of ratinal numbers $\{r_n\}$ is called *recursive* if

$$r_m = (-1)^{\beta(m)} \frac{\gamma(m)}{\delta(m)}$$

with recursive β, γ and δ.

A real number x is called *computable* ($\mathbf{R}$-computable) if

$$\forall m \geq \alpha(p). |x - r_m| < \frac{1}{2^p}$$

for recursive α and $\{r_m\}$. We will express such a circumstance as $x \simeq \langle r_m, \alpha(p) \rangle$, or for short $x \simeq \langle r_m, \alpha \rangle$.

These definitions can be extended to a *computable sequence* of real numbers.

A real (continuous) function f is *computable* ($\mathbf{R}$-computable) if the following hold.

(i) f preserves *sequential computability*, that is, for a computable $\{x_n\}$, $\{f(x_n)\}$ is computable.

(ii) f is continuous with *recursive modulus of continuity*, say β:

$$\forall p \forall n \in \mathbf{N} \forall k \geq \beta(n, p) \forall x, y \in [n, n+1].$$

$$|x - y| < \frac{1}{2^k} \Rightarrow |f(x) - f(y)| < \frac{1}{2^p}.$$

This can be extended to a computable sequence of functions.

3. Rademacher functions

Rademacher functions are step functions from $I = [0, 1)$ to $\{-1, 1\}$ defined below.

Definition 31 (Rademacher functions) Let n denote $0, 1, 2, 3, \cdots$. Then the nth *Rademacher function* $\phi_n(x)$ is defined as follows.

$$\phi_0(x) = 1, \quad x \in [0, 1)$$

$$\phi_n(x) = \begin{cases} 1, & x \in [\frac{2i}{2^n}, \frac{2i+1}{2^n}) \\ -1, & x \in [\frac{2i+1}{2^n}, \frac{2i+2}{2^n}) \end{cases}$$

where $n \geq 1$ and $i = 0, 1, 2, \cdots, 2^n - 1$.

The sequence $\{\phi_n(x)\}$ will be called the *system of Rademacher functions*, or the *Rademacher function system.*

A Rademacher function $\phi_n(x)$ is a step function which takes a value 1 or -1, and jumps at dyadic fractions $\frac{k}{2^n}$ for $k = 1, 2, \cdots, 2^n - 1$. It is right continuous with left limit.

As a sequence of functions, $\{\phi_n\}$ is eventually constant at each dyadic point. Namely, let x be a dyadic point $\frac{k}{2^n}$, where n is the first number with respect to which x can be expressed as such. Then k is an odd number, and $\phi_n(x) = -1$. For any $m > n$, $x = \frac{2l}{2^m}$ for an l, and this implies that $\phi_m(x) = 1$.

Yasugi and Washihara [19] have shown that the function system $\{\phi_n\}$ is endowed with some kind of computational attributes. The double sequence of values $\{\phi_n(x_m)\}$ for a computable sequence of real numbers $\{x_m\} \subset [0, 1)$ is "weak computable" in the sense that $\{\phi_n(x_m)\}$ is approximated by a recursive triple sequence of rational numbers with a "limiting recursive" modulus of convergence (Remark 1 in Section 4 of [19]). The result cannot be sharpened to a recursive modulus of convergence.

4. Topological computability

In computing the values or drawing the graph of a piecewise continuous function, it is a usual practice to first compute the value or plot a dot at a jump point, and then compute values or draw a curve on the open interval where the function is continuous. Such an action corresponds to the mathematical notion of isolating the jump points. We were thus led to the uniform topology of the real line induced from the Euclidean topology by isolating the jump points. (We have employed the definition of uniformity in [3].)

Let X be a non-empty set. A sequence $\{V_n\}_{n\in\mathbf{N}}$ of maps such that $V_n : X \to P(X)$ is called a *uniformity* if it satisfies some axioms, Axioms $A_1 \sim A_5$ to be stated below. In particular, A_1 and A_2 in [3] can be unified to $A_1 \& A_2$:

$$\cap_n V_n(x) = \{x\}.$$

We will state Axioms $A_3 \sim A_5$ in the form of effective uniformity.

Subsequent definitions are due to [10].

Definition 41 (Effective uniformity) A *uniformity* $\{V_n\}$ on X is *effective* if there are recursive functions $\alpha_1, \alpha_2, \alpha_3$ which satisfy the following.

$$\forall n, m \in \mathbf{N} \forall x \in X. V_{\alpha_1(n,m)}(x) \subset V_n(x) \cap V_m(x) \quad \text{(effective } A_3\text{)};$$

$$\forall n \in \mathbf{N} \forall x, y \in X. x \in V_{\alpha_2(n)}(y) \to y \in V_n(x) \quad \text{(effective } A_4\text{)};$$

$$\forall n \in \mathbf{N} \forall x, y, z \in X. x \in V_{\alpha_3(n)}(y), y \in V_{\alpha_3(n)}(z) \to x \in V_n(z) \quad \text{(effective } A_5\text{)}.$$

$\mathcal{T} = \langle X, \{V_n\}, \alpha_1, \alpha_2, \alpha_3 \rangle$ forms an *effective uniform topological space*.

Definition 42 (Effective convergence) $\{x_k\} \subset X$ *effectively converges* to x in X if there is a recursive function γ satisfying $\forall n \forall k \geq \gamma(n). x_k \in V_n(x)$.

This can be extended to effective convergence of a multiple sequence.

Definition 43 (Computability structure) Let $\mathcal{S}$ be a family of sequences from X (multiple sequences included). $\mathcal{S}$ is called a *computability structure* if the following hold.

C1: (Non-emptiness) $\mathcal{S}$ is nonempty.

C2: (Re-enumeration) If $\{x_k\} \in \mathcal{S}$ and α is a recursive function, then $\{x_{\alpha(i)}\}_i \in \mathcal{S}$.

This can be extended to multiple sequences.

C3: (Limit) If $\{x_{lk}\}$ belongs to $\mathcal{S}$, $\{x_l\}$ is a sequence from X, and $\{x_{lk}\}$ converges to $\{x_l\}$ effectively, then $\{x_l\} \in \mathcal{S}$. ($\mathcal{S}$ is closed with respect to effective convergence.)

This can be extended to multiple sequences.

A sequence belonging to $\mathcal{S}$ is called *computable*, and x is *computable* if $\{x, x, \cdots\}$ is computable.

We will henceforth consider the space

$$\mathcal{C}_{\mathcal{T}} = \langle X, \{V_n\}, \alpha_1, \alpha_2, \alpha_3, \mathcal{S} \rangle .$$

For each fixed n, an effective uniform space, say $\mathcal{U}_n$, will be associated to the Rademacher function ϕ_n and it will be claimed that ϕ_n is "uniformly computable" in it. Our present interest is in defining an effective uniform space which is a kind of the limit of $\{\mathcal{U}_n\}$ and in which the computability of the function sequence $\{\phi_n\}$ can be stated.

5. A sequence of uniformities

We assume that $\nu, n \in \mathbf{N}$.

Definition 51 (Effective sequence of uniformities) 1) Let $\{V_n^\nu\}$ be a sequence of uniformities on a set X with the following modifications (cf. Definition 41).

For each ν, $A_1 \& A_2$ holds, that is, $\cap_n V_n^\nu(x) = \{x\}$.

In A_3, A_4, A_5, the recursive functions $\alpha_1, \alpha_2, \alpha_3$ depend also on ν. Thus, for example, effective A_4 stands as follows.

$$\forall \nu, n \in \mathbf{N} \forall x, y \in X. x \in V^{\nu}_{\alpha_2(\nu;n)}(y) \to y \in V^{\nu}_n(x).$$

2) The *limit sequence* of $\{V^{\nu}_n\}$, denoted by $\{W_{\langle \nu,n\rangle}\}$, is defined as follows, where $\langle \nu, n\rangle$ represents a code for the pair of ν and n effectively enumerated:

$$W_{\langle \nu,n\rangle} = V^{\nu}_n.$$

3) The limit sequence $W_{\langle \nu,n\rangle}$ will be called the *effective limit* of $\{V^{\nu}_n\}$ if for A_3 further holds: there are recursive β and γ satisfying the following.

$$\forall \nu_1, \nu_2; n, m \in \mathbf{N} \forall x \in X. V^{\gamma(\nu_1,\nu_2;n,m)}_{\beta(\nu_1,\nu_2;n,m)}(x) \subset V^{\nu_1}_n(x) \cap V^{\nu_2}_m(x).$$

Proposition 51 (Limit uniformity) The sequence $\{W_{\langle \nu,n\rangle}\}$ defined in 2) of Definition 51 satisfying 3) forms an effective uniformity on X. We will then call $\{W_{\langle \nu,n\rangle}\}$ the limit uniformity of $\{V^{\nu}_n\}$.

Proof $A_1 \& A_2$ is obvious. A_4 and A_5 hold due to the conditions in 1) of Definition 51. For example, A_4 can be stated as follows.

$$\forall \nu, n \in \mathbf{N} \forall x, y \in X. x \in W_{\langle \nu, \alpha_2(\nu;n)\rangle}(y) \to x \in W_{\langle \nu,n\rangle}(y).$$

A_3 holds due to the condition of 3): taking the β, γ, it holds

$$W_{\langle \gamma(\nu_1,\nu_2;n,m), \beta(\nu_1,\nu_2;n,m)\rangle}(x) \subset W_{\langle \nu_1,n_1\rangle (x)} \cap W_{\langle \nu_2,n_2\rangle}(x).$$

Proposition 51 suggests that one can utilize the effective limit of an effective sequence of uniformities in an attempt to describe the computability of a sequence of functions $\{f_\nu\}$ whose jump points vary according to ν in a certain way.

We will henceforth confine ourselves to our original interest, that is, to the Rademacher function system. The sequence of uniformities associated with this system is not only an effective sequence satisfying the condition in 3) of Definition 51 but is endowed with some more favorable properties. In the subsequent sections we will investigate these properties.

6. A sequence of uniformities on I

Put $I = [0, 1)$. We assume that $\nu, k \in \mathbf{N}$ and $0 \le k \le 2^\nu - 1$. We will consider real numbers and sequences of real numbers in I.

Definition 61 (Intervals and uniformity) Define subintervals of I, I^{ν}_k, and a sequence of maps $U^{\nu}_n : I \to P(I)$ as follows.

$$I^{\nu}_k = [\frac{k}{2^\nu}, \frac{k+1}{2^\nu})$$

$$U^{\nu}_n(x) = I^{\nu}_k \cap (x - \frac{1}{2^n}, x + \frac{1}{2^n}) \quad \text{if} \quad x \in I^{\nu}_k.$$

Notice that

$$\forall x \forall \nu \exists ! k . x \in I_k^\nu$$

and

$$\forall n . x \in (x - \frac{1}{2^n}, x + \frac{1}{2^n}).$$

Lemma 1 1) $\{U_n^\nu\}$ is a decreasing sequence with respect to ν, that is, $U_n^{\nu+1}(x) \subset U_n^\nu(x)$.

2) $\{U_n^\nu\}$ is a decreasing sequence with respect to n, that is, $U_n^\nu(x) \subset U_m^\nu(x)$ if $n \geq m$.

3) $x \in U_n^\nu(y) \leftrightarrow y \in U_n^\nu(x)$ (symmety).

4) $x \in U_{n+1}^\nu(y), y \in U_{n+1}^\nu(z) \rightarrow x \in U_n^\nu(z)$ (transitivity).

The proofs are straightforward.

Proposition 61 (Effective sequence) 1) $\{U_n^\nu\}$ forms an effective sequence of uniformities on I (cf. 1) of Definition 51).

2) Put $Z_{\langle \nu, n \rangle} = U_n^\nu$. $\{Z_{\langle \nu, n \rangle}\}$ is the effective limit of $\{U_n^\nu\}$ (cf. 3) of Definition 51).

Proof 1) $A_1 \& A_2$: $\cap_n U_n^\nu(x) = \{x\}$ is obvious.

A_3, A_4, A_5 follow from 2), 3) and 4) of Lemma 1, that is, take $\max(n, m)$ for $\alpha_1(\nu; n, m)$, n for $\alpha_2(\nu; n)$ and $n+1$ for $\alpha_3(\nu; n)$.

2) We will show that there are β and γ as in 3) of Definition 51. Put

$$\beta(\nu_1, \nu_2; n, m) = \max(\nu_1, \nu_2); \ \gamma(\nu_1, \nu_2; n, m) = \max(n, m).$$

Then $U_{\gamma(\nu_1, \nu_2; n, m)}^{\beta(\nu_1, \nu_2; n, m)} \subset U_n^{\nu_1}(x) \cap U_m^{\nu_2}(x)$.

Definition 62 (ν-computability) Let ν be an arbitrary (but fixed) natural number.

1) A sequence $\{a_{\mu i}\}$ with multiple index μi is called a ν-sequence if, for a $k = k_\mu \leq 2^\nu - 1$, $\{a_{\mu i}\}_i \subset I_k^\nu$. ($\mu$ may be empty.)

2) A multiple sequence of rational numbers $\{r_{\mu i}\}$ is called a *recursive ν-sequence* if it is recursive and is a ν-sequence.

3) $\{a_{\mu m i}\}$ *converges ν-effectively* to $\{x_{\mu m}\}$ with respect to i if there is a recursive α so that $i \geq \alpha(\mu, m, p)$ implies $a_{\mu m i} \in U_p^\nu(x_{\mu m})$. We write this property as

$$x_{\mu m} \simeq_\nu \langle a_{\mu m i}, \alpha(\mu, m, p) \rangle$$

or, for short, $x_{\mu m} \simeq_\nu \langle a_{\mu m i}, \alpha \rangle$.

4) A sequence of real numbers $\{x_{\mu m}\}$ is called *ν-computable* if there are recursive ν-sequence $\{r_{\mu m i}\}$ and α as in 3), that is,

$$x_{\mu m} \simeq_\nu \langle r_{\mu m i}, \alpha(\mu, m, p) \rangle .$$

Note For each ν, $\langle I, \{U_n^\nu\}_n \rangle$ is a space in which one can express the uniform computability of the ν-th Rademacher function ϕ_ν in a natural manner. Our objective is then to set up a uniform space in which the computability problem of the function sequence $\{\phi_\nu\}$ can be expressed.

Definition 63 (ω-computability) 1) A sequence $\{a^{\nu}_{\mu i}\} \subset I$ is called a $\{\nu\}$-*sequence* if, for each ν, for a $k^{\nu}_{\mu} \leq 2^{\nu} - 1$, $\{a^{\nu}_{\mu i}\}_i \subset I^{\nu}_{k^{\nu}_{\mu}}$.

2) A multiple sequence of rational numbers from I, say $\{r^{\nu}_{\mu i}\}$, is called a *recursive* $\{\nu\}$-*sequence* if it is recursive and is a $\{\nu\}$-sequence.

In this case, $\{k^{\nu}_{\mu}\}$ can be recursive, for we can find k^{ν}_{μ} by cheking "$r^{\nu}_{\mu 1} \in I^{\nu}_{k}$?".

3) A multiple $\{\nu\}$-sequence $\{a^{\nu}_{\mu m i}\}$ *converges* $\{\nu\}$-*effectively* to $x_{\mu m}$ (with respect to i) if there is a recursive α such that

$$\forall \nu \forall \mu \forall m \forall p \forall i \geq \alpha(\nu; \mu, m, p). a^{\nu}_{\mu m i} \in U^{\nu}_{p}(x_{\mu m}).$$

This fact will be expressed by

$$x_{\mu m} \simeq_{\omega} \left\langle a^{\nu}_{\mu m i}, \alpha(\nu; \mu, m, p) \right\rangle$$

or simply

$$x_{\mu m} \simeq_{\omega} \left\langle a^{\nu}_{\mu m i}, \alpha \right\rangle.$$

4) A sequence $\{x_{\mu m}\}$ is ω-*computable* if there are a recursive $\{\nu\}$-sequence $\{r^{\nu}_{\mu m i}\}$ and a recursive α as in 3), that is,

$$x_{\mu m} \simeq_{\omega} \left\langle r^{\nu}_{\mu m p}, \alpha(\nu; \mu, m, i) \right\rangle.$$

Proposition 62 (**R**-computability, ω-computability and ν-computability) 1) For a single real number $x \in I$, **R**-computability (i.e., computable in the Euclidean topology), ω-computability and ν-computability (for each fixed ν) are equivalent.

2) If $\{x_{\mu m}\}$ is ω-computable, then it is ν-computable for all ν.

3) For each ν, if $\{x_{\mu m}\}$ is ν-computable, then it is **R**-computable, hence by 2), an ω-computable sequence is **R**-computable.

4) For each ν, there is a sequence $\{x_m\}$, which is **R**-computable but is not ν-computable.

5) For each ν_1, ν_2 where $\nu_2 > \nu_1$, there is a sequence $\{x_m\}$ which is ν_1-computable but not ν_2- computable.

6) If $\nu_1 < \nu_2$ and $\{x_{\mu m}\}$ is ν_2-computable, then it is ν_1-computable.

Proof 1) An ω-computable real number is ν-computable for each ν and a ν computable real number is **R**-computable as special cases of 2) below.

¿From the definition, each rational number is both **R**- and ν-computable.

Suppose x is **R**-computable with $x \simeq \langle q_m, \alpha \rangle$. We will show that x is ω-computable. Then the equivalences follow.

Given x and ν, $x \in I^{\nu}_{k}$ for some $k \leq 2^{\nu} - 1$. Note that for a single x, we need not attempt to compute k from x; the subsequent construction is carried out assuming that a k is fixed.

We will construct a recursive $\{\nu\}$-sequence $\{t^{\nu}_{p}\}$ and a recursive δ such that $x \simeq_{\omega} \left\langle t^{\nu}_{p}, \delta \right\rangle$.

Construction of t^{ν}_{p}:

$$t^{\nu}_{p} = q_{\alpha(p+\nu)} \quad \text{if} \quad \frac{k}{2^{\nu}} \leq q_{\alpha(p+\nu)} < \frac{k+1}{2^{\nu}};$$

$$= q_{\alpha(p+\nu)} + \frac{1}{2^{p+\nu}} \quad \text{if} \quad q_{\alpha(p+\nu)} < \frac{k}{2^\nu};$$
$$= \frac{k+1}{2^\nu} - \frac{1}{2^{p+\nu}} \quad \text{if} \quad \frac{k+1}{2^\nu} \leq q_{\alpha(p+\nu)}.$$

Given k, the double sequence $\{t^\nu_p\}$ is recursive. We will show that it is a $\{\nu\}$-sequence, and that it converges to x effectively, that is,

1. $t^\nu_p \geq \frac{k}{2^\nu}$;
2. $t^\nu_p < \frac{k+1}{2^\nu}$;
3. $|x - t^\nu_p| < \frac{1}{2^p}$.

We then have that x is ω-computable.

Proof of 1$\sim$3.

Case where $q_{\alpha(p+\nu)} < \frac{k}{2^\nu} (\leq x)$.

1. Since $0 < x - q_{\alpha(p+\nu)} < \frac{1}{2^{p+\nu}}$, it follows that

$$\frac{k}{2^\nu} \leq x < q_{\alpha(p+\nu)} + \frac{1}{2^{p+\nu}} = t^\nu_p.$$

2. Since $q_{\alpha(p+\nu)} < \frac{k}{2^\nu}$, it follows that

$$t^\nu_p = q_{\alpha(p+\nu)} + \frac{1}{2^{p+\nu}} < \frac{k}{2^\nu} + \frac{1}{2^{p+\nu}} < \frac{k+1}{2^\nu}.$$

3. $|x - t^\nu_p| = |x - (q_{\alpha(p+\nu)} + \frac{1}{2^{p+\nu}})|$

$$\leq |x - q_{\alpha(p+\nu)}| + \frac{1}{2^{p+\nu}} < \frac{1}{2^{p+\nu}} + \frac{1}{2^{p+\nu}} \leq \frac{1}{2^p}.$$

The last inequality holds presuming that $\nu \geq 1$.

Case where $q_{\alpha(p+\nu)} \geq \frac{k+1}{2^\nu}$ can be dealt with similarly.

Case where $\frac{k}{2^\nu} \leq q_{\alpha(p+\nu)} < \frac{k+1}{2^p}$.

Since $t^\nu_p = q_{\alpha(p+\nu)}$, $1 \sim 3$ for this case trivially hold.

2) and 3) Obvious from definitions.

4) Let a be a recursive injection whose image is not recursive.

Define $y_m = \frac{1}{2^l}$ if $m = a(l)$; $= 0$ if there is no l such that $m = a(l)$. $\{y_m\}$ is known to be $\mathbf{R}$-computable (cf. [8]). Next define x_m by

$$x_m = \frac{1}{2^\nu} - \frac{y_m}{2^\nu}.$$

Then $x_m = \frac{1}{2^\nu} - \frac{1}{2^{l+\nu}} < \frac{1}{2^\nu}$ if there is an l such that $m = a(l)$ and $= \frac{1}{2^\nu}$ otherwise. In the former case, $x_m \in I^\nu_0$ and in the latter case $x_m \in I^\nu_1$. $\{x_m\}$ is $\mathbf{R}$-computable.

Suppose there were a recursive ν-sequence $\{q_{mi}\}$ which ν-effectively converges to $\{x_m\}$ (cf. Definition 62). Then the following equivalences hold.

$$\{q_{mi}\}_i \subset I^\nu_0 \leftrightarrow x_m < \frac{1}{2^\nu} \leftrightarrow y_m = \frac{1}{2^l} \quad \text{for some} \quad l \leftrightarrow m \in a(\mathbf{N});$$

$$\{q_{mi}\}_i \subset I^\nu_1 \leftrightarrow x_m = \frac{1}{2^\nu} \leftrightarrow y_m = 0 \leftrightarrow m \notin a(\mathbf{N}).$$

Since $\{q_{mi}\}$ is a ν-sequence, it can be determined in which of I_0^ν or I_1^ν it is contained by checking q_{m1}, hence it can be recursively determined whether $m \in a(\mathbf{N})$ or not for every m, contradicting non-recursiveness of $a(\mathbf{N})$. So, $\{x_m\}$ cannot be ν-computable.

5) Define $\{y_m\}$ as above for ν_2, and then define $\{x_m\}$ as follows.

$$x_m = \frac{1}{2^{\nu_2}} - \frac{y_m}{2^{\nu_2}}.$$

With the same reason as in 4), $\{x_m\}$ is ν_1-computable, but not ν_2-computable.

6) Immediate from the definition.

Lemma 2 A recursive sequence of rational numbers, say $\{q_j\}$, regarded as a sequence of real numbers, is ω-computable, since we can obtain an approximating sequence by $\{r^\nu_{ji}\}$, $r^\nu_{ji} = q_j$.

Proposition 63 (ω-computability structure) ω-computable sequences form a computability structure with respect to $\{Z_{\langle \nu,n\rangle}\}$ (cf. Definition 43).

Proof C1 is guaranteed by Lemma 2. C2 is obvious.

C3: Suppose $\{x_{lm}\}$ is ω-computable, that is there is a recursive $\{\nu\}$-sequence $\{r^\nu_{lmi}\}$ and a recursive α satisfying

$$\forall l \forall m \forall \nu \forall p \forall i \geq \alpha(\nu; l, m, p). r^\nu_{lmi} \in U^\nu_p(x_{lm}).$$

Suppose also that $\{x_l\} \subset I$ and $\{x_{lm}\}$ converges effectively to it, that is, there is a recursive γ satisfying

$$\forall l \forall \nu \forall m \geq \gamma(\nu; p). x_{lm} \in U^\nu_p(x_l).$$

Define

$$q^\nu_{lp} = r^\nu_{l\gamma(\nu;p+1)\alpha(\nu;l,\gamma(\nu;p+1),p+1)}.$$

$\{q^\nu_{lp}\}_p$ is a recursive $\{\nu\}$-sequence, and the following hold.

$$q^\nu_{lp} \in U^\nu_{p+1}(x_{l\gamma(\nu;p+1)}),\ x_{l\gamma(\nu;p+1)} \in U^\nu_{p+1}(x_l),$$

from which follows

$$\forall s \geq p. q^\nu_{ls} \in U^\nu_p(x_l),$$

that is, $\{x_l\}$ is ω-computable with $\{q^\nu_{lp}\}$ and the identity function as a modulus of convergence.

7. Diagonal computability

Definition 71 (Diagonal sequence) The sequence $\{U_n^n\}$ will be called the *diagonal sequence* of $\{U_n^\nu\}$, and will be denoted by $\{U_n\}$.

Lemma 3 1) For any n and any $x \in I$, $U_n(x) = I_k^n$ for some k. So,

$$y \in U_n(x) \leftrightarrow U_n(y) = I_k^n = U_n(x).$$

2) $U_{n+1}(x) \subset U_n(x)$.

Proposition 71 (Diagonal sequence and limit) The sequence $\{U_n\}$ forms an effective uniformity which is topologically effectively equivalent to the effective limit $\{Z\langle\nu,n\rangle\}$ (cf. Proposition 61).

Proof We use Lemma 3.

A_1&A_2 is obvious. For A_3, $m = \max(n_1, n_2)$ will do. For A_4 and A_5, put $m = n$. $\{U_n\}$ thus forms an effective uniformity.

As for the effective equivalence, notice that $\{U_n\}$ is a subsequence of $\{U_n^\nu\}$. For the converse, put $n_0 = \max(\nu, n)$. Then $U_n^\nu(x) \supset U_{n_0}(x)$.

Definition 72 (Diagonal uniformity) The sequence of diagonals $\{U_n\}$ as in Definition 71 will be called the *diagonal uniformity* determined by $\{U_n^\nu\}$ or $\{Z_{\langle\nu,n\rangle}\}$, and the space $\langle I, \{U_n\}\rangle$ will be called the *diagonal space* determined by $\{U_n^\nu\}$.

Definition 73 (Diagonal computability) A sequence of real numbers $\{x_m\} \subset I$ is *diagonal computable* if there is a recursive sequence $\{q_{mp}\}$ of rational numbers which converges to $\{x_m\}$ effectively with respect to $\{U_n\}$ in a manner that, for a recursive γ and for $k \geq \gamma(m,p)$, $q_{mk} \in U_p(x_m)$. We will write this property as

$$x_m \simeq_{\mathcal{D}} \langle q_{mk}, \gamma\rangle .$$

More generally $\{x_{\mu m}\} \subset I$ is diagonal computable if there is a recursive sequence $\{q_{\mu mp}\}$ of rationals and a recursive γ such that for $k \geq \gamma(\mu, m, p)$, $q_{\mu mk} \in U_p(x_{\mu m})$, or

$$x_{\mu m} \simeq_{\mathcal{D}} \langle q_{\mu mp}, \gamma\rangle .$$

Proposition 72 The family of diagonal computable sequences of real numbers, say $\mathcal{R}$, forms a computability structure for $\langle I, \{U_n\}\rangle$ (cf. C1$\sim$C3 of Definition 43).

Proof It can easily been demonstrated that $\mathcal{R}$ is non-empty (A recursive sequence of rationals, for example, belongs to $\mathcal{R}$.) and is closed with respect to a recursive re-enumeration, hence C1 and C2. Suppose $\{x_{lm}\} \in \mathcal{R}$ represented by $x_{lm} \simeq_{\mathcal{D}} \langle q_{lmp}, \gamma\rangle$, and the sequence effectively converges to $\{x_l\}$ with a recursive modulus of convergence ε. In particular, $q_{lm\gamma(l,m,p)} \in U_p(x_{lm})$ and $x_{l\varepsilon(l,p)} \in U_p(x_l)$.

If we put $r_{lp} = q_{l\varepsilon(l,p)\gamma(l,\varepsilon(l,p),p)}$, then $\{r_{lp}\}$ is a recursive sequence of rational numbers satisfying $r_{lp} \in U_p(x_{l\varepsilon(l,p)})$. On the other hand, $x_{l\varepsilon(l,p)} \in$

$U_p(x_l)$ implies $U_p(x_{l\varepsilon(l,p)}) = U_p(x_l)$ by 1) of Lemma 3. So, $r_{lp} \in U_p(x_{l\varepsilon(l,p)}) = U_p(x_l)$.

Put next $\eta(l,p) = p$. If $q \geq p$, then by 2) of Lemma 3, $r_{lq} \in U_q(x_l) \subset U_p(x_l)$. It thus follows that $x_l \simeq_{\mathcal{D}} \langle r_{lp}, \eta \rangle$, or $\{x_l\} \in \mathcal{R}$.

Theorem 1 (Diagonal computability and ω-computability) A sequence $\{x_m\} \subset [0,1)$ is diagonal computable if and only if it is ω-computable (cf. Definitions 73 and 63).

Proof Suppose $\{x_m\}$ is diagonal computable. Then there are recursive $\{q_{mp}\}$ and γ such that

$$\forall m \forall p \forall k \geq \gamma(m,p). q_{mk} \in U_p(x_m).$$

Define $r^{\nu}_{mi} = q_{m\gamma(m,\nu)+i}$. If we put $\alpha(\nu; m, p) = \gamma(m, \nu + p)$, then $\{r^{\nu}_{mi}\}$ is a recursive $\{\nu\}$-sequence and converges to $\{x_m\}$ effecitvely with recursive modulus of convergence α shown as follows.

$$r^{\nu}_{m\alpha(\nu;m,p)} = r^{\nu}_{m\gamma(m,\nu+p)} = q_{m\gamma(m,\nu)+\gamma(m,\nu+p)}$$

$$= q_{m\gamma(m,\nu+p)+\gamma(m,\nu)} \in U^{\nu+p}_{\nu+p}(x_m) \subset U^{\nu}_{p}(x_m).$$

If $j = \alpha(\nu; m, p) + l$, then

$$r^{\nu}_{mj} = r^{\nu}_{m\alpha(\nu;m,p)+l} = r^{\nu}_{m\gamma(m,\nu+p)+l} = q_{m\gamma(m,\nu)+\gamma(m,\nu+p)+l}$$

$$= q_{m\gamma(m,\nu+p)+(\gamma(m,\nu)+l)} \in U^{\nu+p}_{\nu+p}(x_m) \subset U^{\nu}_{p}(x_m).$$

We have thus obtained ω-computability of $\{x_m\}$.

Suppose conversely, that $\{x_m\}$ is ω-computable: $x_m \simeq_\omega \langle r^{\nu}_{mi}, \alpha \rangle$, where $\{r^{\nu}_{mp}\}$ is a recursive $\{\nu\}$-sequence. Define $q_{mp} = r^{p}_{mp}$. Then $q_{mp} \in U_p(x_m) = I^p_k$ for some appropriate k. $\{q_{mp}\}$ converges to $\{x_m\}$ with modulus of convergence $\beta(m,p) = \alpha(p; m, p) + p$ as follows. Suppose $i \geq \beta(m,p) = \alpha(p; m, p) + p$. Then $i = \alpha(p; m, p) + j + p$. So, $q_{mi} = r^{i}_{m\alpha(p;m,p)+j+p} \in U^{i}_{p}(x_m) \subset U^{p}_{p}(x_m)$ (cf. 1) of Lemma 1), that is, $x_m \simeq_{\mathcal{D}} \langle q_{mi}, \beta \rangle$.

For a later use, we prove the following lemma.

Lemma 4 We can re-define the diagonal computability as follows. $\{x_m\}$ is diagonal computable if there is a recursive sequence of rational numbers $\{r_{mp}\}$ such that

$$\forall p \forall k \geq p. r_{mk} \in U_p(x_m).$$

Proof From the original definition, $\forall k \geq \gamma(m,p). q_{mk} \in U_p(x_m)$. Define $\{r_{mp}\}$ by $r_{mp} = q_{m\gamma(m,p)}$. Then $r_{mp} = q_{m\gamma(m,p)} \in U_p(x_m)$. For $k \geq p$, put $k = p + i$. Then

$$r_{mk} = r_{m(p+i)} = q_{m\gamma(m,p+i)} \subset U_{p+i}(x_m) \subset U_p(x_m)$$

by 2) of Lemma 3.

8. $\mathcal{D}$-sequential computability of a function sequence

Let $\mathbf{U} = \langle X, \{V_n\}, \mathcal{S} \rangle$ be an effective uniform space with a computability structure $\mathcal{S}$. We will consider functions from X to $\mathbf{R}$.

Definition 81 (Sequential computability) A function sequence $\{f_l\}$ is called sequentially computable in $\mathbf{U}$ if, for every sequence $\{x_m\}$ in $\mathcal{S}$, $\{f_l(x_m)\}$ is an $\mathbf{R}$-computable double sequence of real numbers.

Proposition 81 ($\mathcal{D}$-sequential computability of the Rademacher function system) Let $\{\phi_j(x)\}$ be the Rademacher function system (cf. Definition 31). Then it is sequentially computable in the diagonal uniform space. (We will call such a sequence $\mathcal{D}$-sequentially computable.)

Proof Let $\{x_m\}$ be a diagonal computable sequence of real numbers. We show that $\{\phi_l(x_m)\}$ is an $\mathbf{R}$-computable double sequence of real numbers. Suppose $\{x_m\}$ is represented by recursive $\{q_{mp}\}$ such that for all $k \geq p$, $q_{mk} \in U_p(x_m)$ (cf. Lemma 4). There is (classically) a $k = k_{mp}$ such that $U_p(x_m) = I^p_{k_{mp}}$, hence, in particular, $q_{mp} \in I^p_{k_{mp}}$. Since q_{mp} is a rational number, the k_{mp} can in fact be computed, by checking

$$\frac{k_{mp}}{2^p} \leq q_{mp} < \frac{k_{mp}+1}{2^p}.$$

By definition, $\phi_p(x_m) = \phi_p(q_{mp})$, hence to evaluate $\phi_p(x_m)$, it suffices to compute $\phi_p(q_{mp})$. $\phi_p(q_{mp})$ $(= 1$ or $= -1)$ can be determined according to the parity of k_{mp} (cf. Definition 31). It is then a trivial fact that $\{\phi_p(x_m)\}(= \{\phi_p(q_{mp})\})$ is a computable sequence of real numbers.

Definition 82 (Effective uniform continuity: Definition 4.5, [10]) A sequence of functions $\{f_n\}$ is called uniformly computable in $\mathbf{U}$ if it is sequentially computable and there is a recursive function α such that

$$y \in V_{\alpha(n,p)}(x) \Rightarrow |f_n(x) - f_n(y)| \leq \frac{1}{2^p}$$

(cf. Definition 4.5 of [10]).

Theorem 2 (Uniform computability of the Rademacher function system) The function sequence $\{\phi_l\}$ is uniformly computable with respect to $\{U_n\}$ (cf. Definition 82). We will call this computability "uniformly $\mathcal{D}$-computable."

The same conclusion holds for the Walsh function system.

Proof $\mathcal{D}$-sequential computability has been proved in Proposition 82.

As for effective uniform continuity, use $\phi_p(x) = \phi_p(y)$ if $y \in U_p(x)$.

Remark One might view that one could have defined the diagonal uniformity from the outset, which would have simplified the entire argument. Technically that may be so. However, that would not represent our natural thinking process as was explained in Introduction.

9. Metrization

Yasugi, Tsujii and Mori [17] showed that the metric induced from a uniformity by a general construction preserves effective convergence, and left it open if it preserves sequential computability. Here we show that such a metric induced from the uniformity $\{U_n\}$ preserves sequential computability.

We will follow [17] and prepare a lemma. Instead of quoting general definitions of relevant notions from Section 3 of [17], we will give definitions and equations here just necessary for the purpose.

Lemma 5 $\{U_n\}$ is an "effective sequence of uniform coverings" and is "normal".

Proof An effective uniformity is an effective sequence of uniform coverings (Theorem 3.5 [17]).

We prove that $\{U_n\}$ is normal, that is, $U^*_{n+1} \succ U_n$, where U^* denotes the "star refinement" of U, and $V \succ U$ expresses that $X \in V$ implies $X \subset Y$ for some $Y \in U$.

$$S(x, U_n) = \cup\{U_n(y) | y \in U_n(x)\} = U_n(x).$$

$$S(U_n(x), U_n) = \cup\{U_n(y) | U_n(x) \cap U_n(y) \neq \emptyset\} = U_n(x).$$

$$\begin{aligned} U^*_{n+1} &= \cup\{S(U_{n+1}(x), U_{n+1}) | x \in I\} \\ &= \{U_{n+1}(x) | x \in I\} = U_{n+1}. \end{aligned}$$

$$U^*_{n+1} = U_{n+1} \succ U_n$$

for

$$\forall x \in I. U_{n+1}(x) \subset U_n(x).$$

In order to attain our major purpose, we first define and evaluate some quantities.

$$V(x, \frac{1}{2^n}) = S(x, U_n) = U_n(x);$$

Suppose

$$r = \frac{1}{2^{i_1}} + \frac{1}{2^{i_2}} + \cdots + \frac{1}{2^{i_k}}, i_1 < i_2 < \cdots < i_k.$$

Then

$$\begin{aligned} V(x, r) &= S(\cdots S(S(x, U_{i_1}), U_{i_2}), \cdots, U_{i_n}) \\ &= S(\cdots S(U_{i_1}(x), U_{i_2}), \cdots, U_{i_n}). \end{aligned}$$

$$\begin{aligned} S(U_{i_1}(x), U_{i_2}) &= \cup\{U_{i_2}(y) | U_{i_1}(x) \cap U_{i_2}(y) \neq \emptyset\} \\ &= \cup\{U_{i_2}(y) | y \in U_{i_1}(x)\} = U_{i_1}(x) \end{aligned}$$

From this follows that in fact

$$V(x,r) = U_{i_1}(x),$$

that is,$V(x,r)$ is determined by i_1 alone.

Using this fact, we can define successively the following $f_x(y), d^*_x(y,z)$ and $d(y,z)$.

$$f_x(y) := \sup\{r | y \in V(x,r)^c\} = \sup\{r | y \in U^c_{i_1}(x)\}.$$

If $x = y$, then $f_x(y) = 0$, and otherwise $f_x(y) = \frac{1}{2^{i_0}}$, where i_0 is the last i such that $y \in U_i(x)$. For, suppose $i_1 = i_0 + 1$. Then $y \notin U_{i_1}(x)$, and for $r = \frac{1}{2^{i_1}} + \frac{1}{2^{i_2}} + \cdots, i_1 < i_2 < \cdots, \sup r = \frac{1}{2^{i_0}}$.

$$d^*_x(y,z) := |f_x(y) - f_x(z)| = |(\frac{1}{2^{i_0}})^* - (\frac{1}{2^{j_0}})^*|,$$

where $(a)^*$ represents either a or 0 as the case may be.

$$d(y,z) := \sup_x d^*_x(y,z) = \frac{1}{2^{i^*}}, \qquad (*)$$

where i^* is the last i such that for a $k \le 2^i - 1$, $y, z \in I^i_k$, for, $d^*_x(y,z)$ attains its maximum when $y = x$ or $z = x$. This is shown as follows. We may assume for a fixed x that $(\frac{1}{2^{i_0}})^* \ge (\frac{1}{2^{j_0}})^*$ holds. Now to maximize $d^*_x(y,z)$, it suffices to put $(\frac{1}{2^{j_0}})^* = 0$, or $z = x$.

Remark It is known that two notions of effective convergence, one with respect to an effective uniformity (here $\{U_n\}$) and one with respect to the derived metric (here d), are equivalent (cf. [17]).

Corollary 1 For **R**-computable y and z, $d(y,z)$ is **R**-computable.

Theorem 3 (Preservation of sequential computability) d preserves computability, that is, for diagonal computable sequences of real numbers in I, say $\{x_i\}$ and $\{y_j\}$, $\{d(x_i, y_j)\}$ is an **R**-computable double sequence of real numbers.

Proof Let $\{x_i\}$ and $\{y_j\}$ be diagonal computable sequences, respectively represented by $\{r_{ip}\}$ and $\{s_{jp}\}$. Recall that they satisfy the following.

$$\forall k \ge p. r_{ik} \in U_p(x_i), \quad \forall l \ge p. s_{jl} \in U_p(y_j)$$

(cf. Lemma 4). Notice that $U_p(x_i) = I^p_u$ for some $u = u_{ip} \le 2^p - 1$ with u_{ip} recursively determined by the relation $r_{ip} \in I^p_{u_{ip}}$. Similarly $U_p(y_j) = I^p_{v_{jp}}$ with v_{jp} recursively determined by $\{s_{jp}\}$.

Now define a triple sequence of rational numbers in I, $\{t_{ijq}\}$, as follows.

Let $R(i,j,p)$ denote the relation $u_{ip} = v_{jp} \wedge u_{i(p+1)} \ne v_{j(p+1)}$, and put

$$p_0 = \quad \text{the least} \quad p \quad \text{such that} \quad p < q \quad \text{and} \quad R(i,j,p)$$

depending on i, j, q. Now put $t_{ijq} = \frac{1}{2^{p_0}}$ if there is a (least) p as above. If $u_{ip} = v_{jp}$ for all $p \leq q$, then put $t_{ijq} = \frac{1}{2^q}$. $\{t_{ijq}\}$ is recursive, and $w_{ij} = \lim_q t_{ijq} = \frac{1}{2^p}$ if there is a (least) p such that $R(i, j, p)$. $w_{ij} = 0$ otherwise. The convergence to the limit is effective since $|w_{ip} - t_{ijq}| \leq \frac{1}{2^q}$ holds, and hence $\{w_{ij}\}$ is a computable double sequence of real numbers. Furthermore, $d(x_i, y_j) = w_{ij}$. So, $\{d(x_i, y_j)\}$ is a computable double sequence.

Remark 1) The metric d as above induced from the uniformity $\{U_n\}$ by a general metric construction is closely related to Fine metric. We plan to relate various notions of computability of a function or a function sequence with respect to Fine metric (cf. [4],[5]) to our theory of $\{U_n\}$.

2) The general theory of an effective sequence of uniformities is yet to be worked out. It is expected to have wide applications to the computability problems of piecewise continuous real functions.

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