Computable versions of basic theore
functional analysis II

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Main topic:

Riesz's representation theorem

of a bounded linear functional on a Hilbert space

Some hints of reviews —

1) what is a Hilbert space ?
2) some history of Riesz's theorem.
3) how Riesz's theorem is useful ?
4) the version in computability analysis.

1) what is a Hilbert space ?

Summary:

a) A linear space $\mathbf{H}$ (over the real number field $\mathbb{R}$ for

b) endowed with a scalar product $\langle \, , \rangle$

(= symmetric positive definite bilinear form)

c) complete with respect to the metric induced by $\langle \, ,$

N.B. These are terminologies in Mathematical Analys

Question : What kind of care is required to incorporat
space in the context of computability analysis ?
(Recall Professor Yasugi's lecture — Key : How to
the notion of recursiveness into Hilbert spaces.)

a) A linear space $\mathbf{H}$ (over the real number field $\mathbb{R}$ for

The elements $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\cdots$ of $\mathbf{H}$ are vectors.
The real numbers a, b, c, $\cdots$ are scalars.
Their linear combinations $a\,\mathbf{x} + b\,\mathbf{y}\,, \cdots$, are defined a
There is the zero vector $\mathbf{0} \in \mathbf{H}$.
Each $\mathbf{x}$ has its negative $-\mathbf{x}$: $\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0}$

The commutative, composition, ditributive laws hold.
Some consequences: $1 \cdot \mathbf{x} = \mathbf{x}$, $0 \cdot \mathbf{x} = \mathbf{0}$, $(-1) \cdot \mathbf{x} = -\mathbf{x}$

b) endowed with a scalar product $\langle\, , \rangle$
(= symmetric positive definite bilinear form)

$$\mathbf{H} \times \mathbf{H} \ni (\mathbf{x}, \mathbf{y}) \quad \mapsto \quad \langle \mathbf{x}, \mathbf{y} \rangle \in \mathbb{R}$$

symmetry : $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$
positivity: $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$
definiteness: $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \quad \Longleftrightarrow \quad \mathbf{x} = \mathbf{0}$
bilinearity: $\langle a\, \mathbf{x} + b\, \mathbf{y}, \mathbf{z} \rangle = a \langle \mathbf{x}, \mathbf{z} \rangle + b \langle \mathbf{y}, \mathbf{z} \rangle$.

N.B. Consult any standard textbook of the Hilbert sp
for the case of complex scalars. Symmetry and Biline
suitably be modified then.

A standard example : $\mathcal{L}^2(D)$

D : bounded open set $\subset \mathbb{R}^n$.
f : real-valued measurable function defined on D sati

$$\int_D |f(x)|^2 \, dx < +\infty$$

$\mathcal{L}^2(D)$: the set of such functions f
[actually functions f and f_1 are identified
if they differ only on the subset of measure zero
linear combination : $(a\, f + b\, g)(x) = a\, f(x) + b\, g(x)$, x
scalar product :

$$\langle f\,, g\rangle = \int_D f(x)\, g(x)\, dx.$$

c) complete with respect to the metric induced by $\langle \,,$

norm :

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}, \quad \mathbf{x} \in \mathbf{H}.$$

Property-1: $\quad \|\mathbf{x}\| \geq 0, \quad \|\mathbf{x}\| = 0 \iff \mathbf{x} = \mathbf{0}$

Property-2: $\quad \|a\,\mathbf{x}\| = |a|\,\|\mathbf{x}\|, \quad \mathbf{x} \in \mathbf{H}, \quad a \in \mathbf{R}.$

Property-3: $\quad \|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|, \quad \mathbf{x},\, \mathbf{y} \in \mathbf{H}.$

Property-4: $\quad |\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\|\,\|\mathbf{y}\|, \quad \mathbf{x},\, \mathbf{y} \in \mathbf{H}.$

$\mathbf{H}$ is a metric space : dist$(\mathbf{x}, \mathbf{y})$ =distance of $\mathbf{x}$ and $\mathbf{y}$=

$\{\mathbf{x}_n\} \subset \mathbf{H}$: Cauchy sequence $\iff$ $\lim_{n,m\to\infty} \|\mathbf{x}_n - \mathbf{x}_m|$

$\mathbf{H}$ complete $\quad\iff\quad$ Every Cauchy sequence converg

The example revisited $\mathcal{L}^2(D)$

norm:

$$\|f\| = \sqrt{\int_D |f(x)|^2 \, dx}, \quad f \in \mathcal{L}^2(D).$$

Completeness:

The Riesz-Fischer theorem (L^2-version):

Let $f_n \in \mathcal{L}^2(D)$ with $\lim_{n,m\to\infty} \|f_n - f_m\| = 0$.
Then, for a unique $f \in \mathcal{L}^2(D)$, $\lim_{n\to\infty} \|f_n - f\| = 0$
Actually, for an appropriate subsequence $f_{n'}$ of f_n,
$f_{n'}(x) \to f(x)$ as $n' \to \infty$ for almost all $x \in D$.

Other examples and related discussions will be
in due courses.

Still some patience for how the Hilbert space the
incorporated into the context of computability ar

Axiomatic Approach — Pour-El & Richards
Type-Two Turing-Machine Effectivity Approach — V

Their key ideas :
Pick up a class of countable sets (sequences) and
Concentrate considerations on analytical objects whic
— describable recursively through these countable
— and which turn out ample enough to be very in
— yet suggest full of philosophico-mathematical q
— . . .

2) some history of Riesz's theorem.

Summary:

a) bounded linear functional

b) Riesz's Theorem

c) some historical notes

d) Proofs

N.B. Here we follow the arguments of mathematical
You will see not all of them are valid in the contex
putability analysis, including the formulation of Riesz
itself.

a) bounded linear functional

A map F on a Hilbert space $\mathbf{H}$, i.e., $F : \mathbf{H} \ni \mathbf{x} \mapsto F(\mathbf{x}$
is linear if

$$F(a\,\mathbf{x} + b\,\mathbf{y}) = a\,F(\mathbf{x}) + b\,F(\mathbf{y}),\ \ \mathbf{x},\,\mathbf{y} \in \mathbf{H},\ \ a,\, b \in$$

and is bounded (or equivalently continuous) if

$$|F(\mathbf{x})| \leq M\,\|\mathbf{x}\|, \quad \mathbf{x} \in \mathbf{H} \quad (\text{for some} \quad M > 0$$

(or $\|\mathbf{x}_n - \mathbf{x}\| \to 0$ implies $F(\mathbf{x}_n) \to F(\mathbf{x})$).

b) Riesz's Theorem

The Riesz-Fréchet Theorem (Riesz's Theorem):

$F : \mathbf{H} \to \mathbb{R}$: bounded linear $\Leftrightarrow$ $F(\mathbf{x}) = \langle \mathbf{x}, \mathbf{v}_F \rangle$ for som

N.B. $\mathbf{v}_F \in \mathbf{H}$ uniquely determined.

c) Some historical literatures:

F. Riesz. Sur une espèce de Géométrie analytique de
de fonctions sommables. Comptes Rendus Acad.
144 (1907), 1409 – 1411.

$\cdots$ Pour fixer mes résultats, conséquences immédiates de mon th
enonce encore deux. Voici le premier, intimement lié à certaine
de MM. Hadamard et Fréchet. Pour l'ensemble des fonctions
de carré sommable, j'appelle *opération continue* chaque opér
correspondre à toute fonction f de l'ensemble un nombre $\mathsf{U}(f)$
quand f_n converge en moyenne vers f, $\mathsf{U}(f_n)$ converge vers $\mathsf{U}(f)$.
est dite linéaire si $\mathsf{U}(f_1 + f_2) = \mathsf{U}(f_1) + \mathsf{U}(f_2)$ et $\mathsf{U}(cf) = c\,\mathsf{U}(f)$.
**chaque opération linéaire continue il existe une fonction k
valeur de l'opération pour une fonction quelconque f est
l'intégrale du produit des fonctions f et k.** $\cdots$ (no explicit

M. Fréchet. Sur les ensembles de fonctions et les
linéaires. Comptes Rendus Acad. Sc. Paris. **144** (19
– 1416. — no explicit proof either —

Orthogonal Projection:

F. Riesz. Zur Theorie des Hilbertischen Raums. A
Math. Szeged. **7** (1934), 34 – 38.

F. Riesz et B. Sz. Nagy. *Leçons d'Analyse Fonctio*
ème éd.) Akadémiai Kiadó (1965) — Reproduction of the
—

d) Proofs

Two proofs:

d-1) separable case

d-2) orthogonal projection — valid for non-separable

N.B. Even the separable case is not immediately tra

computability context !

d-1) When $\mathbf{H}$ is separable:

$\{\mathbf{e}_n\}$ — complete orthonormal basis : $\langle \mathbf{e}_j, \mathbf{e}_k \rangle = \delta_{jk}$

$$\mathbf{x} = \sum_{k=1}^{\infty} x_n \, \mathbf{e}_k \in \mathbf{H}, \quad \|\mathbf{x}\|^2 = \sum_{k=1}^{\infty} |x_k|^2 < +\infty$$

$\mathbf{H}_N$ — linear span of $\mathbf{e}_1, \cdots, \mathbf{e}_N$ — closed subspace o

F_N — restriction of F on $\mathbf{H}_N$: $\mathbf{x}_N = \sum_{k=1}^{N} x_k \, \mathbf{e}_k \in \mathbf{H}$

$$F(\mathbf{x}_N) = F_N(\mathbf{x}_N) = \sum_{k=1}^{N} x_k \, F(\mathbf{e}_k)$$

representation of F_N : $\mathbf{v}_N = \sum_{k=1}^{N} F(\mathbf{e}_k) \, \mathbf{e}_k \in \mathbf{H}_N$.

$$F_N(\mathbf{x}_N) = \langle \mathbf{x}_N, \mathbf{v}_N \rangle = \langle \mathbf{x}, \mathbf{v}_N \rangle$$

d-1)[contd.]

Consequence of boundedness of F :

$$|F(\mathbf{x}_N)| = |\sum_{k=1}^{N} x_k\, F(\mathbf{e}_k)| \le M\,\|\mathbf{x}_N\| = M\sqrt{\sum_{k=1}^{N} |$$

$$\therefore \quad \sqrt{\sum_{k=1}^{N} |F(\mathbf{e}_k)|^2} \le M \quad \text{for any} \quad N$$

candidate of $\mathbf{v}_F$ which realizes $F(\mathbf{x}) = \langle \mathbf{x}, \mathbf{v}_F \rangle$.

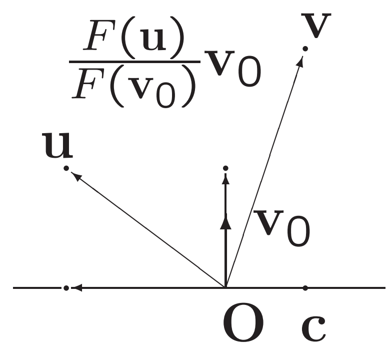
$$\mathbf{v}_F = \sum_{k=1}^{\infty} F(\mathbf{e}_k)\,\mathbf{e}_k \in \mathbf{H} \quad \text{since} \quad \sum_{k=1}^{\infty} |F(\mathbf{e}_k)|^2 <$$

d-1)[Remark]

There is the idea of friendly looking countability
— but no way of foreseeing the convergence rate of

This remark will be recalled when we discuss Riesz's T
the context of computability analysis.

d-2) Orthogonal Projection



$$C_F = \{F = 0\}$$

$$\mathbf{u}_1 = \mathbf{u} - \frac{F(\mathbf{u})}{F(\mathbf{v}_0)}\mathbf{v}_0$$

d-2) [Contd-1]

$C_F = \{\mathbf{x} \in \mathbf{H}\,;\ F(\mathbf{x}) = 0\,\}$ — null-space of F
$\mathbf{v} \notin C_F$ (i.e., $F(\mathbf{v}) \neq 0$)
$\mathbf{c} = \mathrm{pd}_{C_F}(\mathbf{v})$ — foot of $\mathbf{v}$ on C_F : $\mathrm{dist}(\mathbf{v}, C_F) = \|\mathbf{v} -$

$$\mathbf{v} - \mathbf{c} \perp C_F \quad \text{or} \quad \langle \mathbf{v} - \mathbf{c}, \mathbf{w} \rangle = 0,\ F(\mathbf{w}) = 0$$

$$\mathbf{v}_0 = \frac{1}{\|\mathbf{v} - \mathbf{c}\|}(\mathbf{v} - \mathbf{c}) \quad (\|\mathbf{v}_0\| = 1,\ \mathbf{v}_0 \perp C_F)$$

candidate of $\mathbf{v}_F$: $\mathbf{v}_F = F(\mathbf{v}_0)\,\mathbf{v}_0$

d-2) [Contd-2]

$$\mathbf{u} \in \mathbf{H} \quad \Longrightarrow \quad \mathbf{u}_1 = \mathbf{u} - \frac{F(\mathbf{u})}{F(\mathbf{v}_0)}\mathbf{v}_0 \in C_F \quad \text{or} \quad F(\mathbf{u}$$

Thus, $\langle \mathbf{v}_0, \mathbf{u}_1 \rangle = 0$, that is,

$$\langle \mathbf{u}, \mathbf{v}_0 \rangle - \frac{F(\mathbf{u})}{F(\mathbf{v}_0)} = 0 \quad (\langle \mathbf{v}_0, \mathbf{v}_0 \rangle = 1)$$

d-2) [foot]

C : closed convex subset in $\mathbf{H}$. $\quad \mathbf{v} \notin C$.

There is a unique $\mathbf{c} \in C$ such that

$$\text{dist}(\mathbf{v}, C) = \inf_{\mathbf{w} \in C} \|\mathbf{v} - \mathbf{w}\| = \|\mathbf{v} - \mathbf{c}\|$$

$\mathbf{c}$ is the foot of $\mathbf{v}$ on C: $\mathbf{c} = \text{pd}_C(\mathbf{v})$

For Verification:

i) The basic identity of a Hilbert space:

$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2, \quad \mathbf{x}, \mathbf{y} \in$$

ii) And completeness of $\mathbf{H}$

d-2) [Supplements]

I) Basic identity $\Longrightarrow$

$$\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{4}\Big(\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2 \Big), \quad \mathbf{x}, \mathbf{y} \in \mathbf{H}$$

defines an inner-product. Basic identity is characte

Hilbert space structure.

II) The idea of foot will be seen quite useful in co

discussion of the null-space of a bounded linear funct

A second example:

$I = [0,1]$. Let f be absolutely continuous on I with

summable derivative f' on I (Thus, f' is also absol

grable.) Suppose $f(0) = f(1) = 0$.

Let $\mathbf{H}_0^1$ be the totality of such functions f.

$\mathbf{H}_0^1$ is a Hilbert space with the inner-product:

$$\langle f, g\rangle = \int_0^1 f'(t)\, g'(t)\, dt, \quad f,\, g \in \mathbf{H}_0^1.$$

For $h \in \mathcal{L}^2(I)$, consider a bounded linear functional o

$$H\ :\ \mathbf{H}_0^1 \ni f \quad \mapsto \quad \int_0^1 f(t)\, h(t)\, dt \in \mathbb{R}.$$

Then

$$\mathbf{v}_H(t) = (1-t)\int_0^t s\, h(s)\, ds + t\int_1^t (1-s)\, h(s)\, d$$

3) how Riesz's theorem is useful ?

Summary:

a) Examples from simple linear elliptic variational pro

b) Existence and uniqueness theorems

c) Numerical Analysis

a) Examples from simple linear elliptic variational pro

D : bounded, open subset of $\mathbb{R}^n$

with a nice (e.g., smooth) boundary ∂D

Is D a good object of computability analysis ?

— Yes, surely ! You can so set.

Variational problem in the background:

Let $f(x)$ be (a good) function defined on $\overline{D} = D \cup \partial I$

Find (an appropriate) $u(x)$ on $\overline{D}$ which minimizes

$$\mathscr{V}(u) = \frac{1}{2} \int_D \sum_{k=1}^{n} \Big(\partial_k u(x)\Big)^2 dx + \int_D u(x)\, f(x)\, dx \quad (\partial$$

b) Existence and uniqueness theorems

appropriate class of functions —

a certain smoothness requirement

the imposed boundary condition

Example:

Find smooth $u(x)$ which vanish at ∂D:

Dirichlet condition : $\quad u(x) = 0 \quad x \in \partial D$

The Euler equation:

$$-\nabla^2 u(x) + f(x) = 0, \quad x \in D \quad (\nabla^2 = \sum_{k=1}^{n} \partial_k^2$$

$$\frac{d}{d\epsilon}\mathscr{V}(u + \epsilon\, v)|_{\epsilon=0} = \sum_{k=1}^{n} \int_D \partial_k u(x)\, \partial_k v(x)\, dx + \int_D v(x)$$

Choice of conditions will affect discussions below:

Choice of the Hilbert space:

$$\mathbf{H}_0^1(D) \ni v(x) \ \Leftrightarrow \ v \in \mathcal{L}^2(D),\ \partial_k v(x) \in \mathcal{L}^2(D),\ v(x)$$

Remarks:

1) Here $\partial_k v$ are generalized derivatives of $v(x)$:

$$\int_D \partial_k v(x)\,\varphi(x)\,dx = -\int_D v(x)\,\partial_k \varphi(x)\,dx \quad (\varphi(x) \in C$$

2) Legitimacy of $v(x)|_{\partial D} = 0$ requires some discussion

Inner Product:

$$\langle v, w\rangle = \sum_{k=1}^{n} \int_D \partial_k v(x)\,\partial_k w(x)\,dx, \quad v,\, w \in \mathbf{H}_0^1($$

bounded linear funcitonal on $\mathbf{H}_0^1(D)$:

$$F \,:\, \mathbf{H}_0^1(D) \ni v \quad \mapsto \quad -\int_D v(x)\, f(x)\, dx \quad \text{(for} \quad f \in$$

Why ?

$$|F(v)| \leq \|v\| \|f\| = \|f\| \sqrt{\int_D v(x)^2 \, dx}$$

together with Poincaré's inequality:

$$\sqrt{\int_D v(x)^2 \, dx} \leq \delta_D \sqrt{\int_D |\nabla v(x)|^2 \, dx}\,, \quad v \in \mathbf{H}_0^1($$

Apply Riesz's Theorem:

For a unique $u \in \mathbf{H}_0^1(D)$, $F(v) = \langle u, v \rangle$ in $\mathbf{H}_0^1(D)$.

This u solves the variational problem.

c) Numerical Analysis

Actual computation —— Done on computer
Basic philosophy —— Genuineness of discrete procedur
Main interest therein —— Efficient algorithm and Erro
Origin of credibility —— Mathematics

However, of course, we do not enter in philosophical a

Recall the variational elliptic problem in the above.
How do Riesz's theorem and discretization procedure

Principle of approximation :

$\mathscr{S}_h = \{s_{h,j}(x)\}$ — Linearly independent functions in $\mathbf{H}$

$\mathbf{S}_h$ — Linear span of $\mathscr{S}_h$ in $\mathbf{H}_0^1(D)$

$$N_h = \dim \mathbf{S}_h < +\infty \quad \Longleftrightarrow \quad \mathscr{S}_h \text{ finite}$$

Approximate $v \in \mathbf{H}_0^1(D)$ by $v_h \in \mathbf{S}_h$:

$$v_h(x) = \sum_{j=1}^{N_h} v^{h,j}\, s_{h,j}(x)$$

($s_{h,j}(x)$ are not orthogonal)

N.B. One might imagine $\mathscr{S}_h$ as shape functions relate finite elements. In fact, the presentation is too simpl Here are just the idea and principle to avoid technica variational equation in $\mathbf{S}_h$:

$\langle\, ,\, \rangle_h$: restriction to $\mathbf{S}_h$ of the inner product $\langle\, ,\, \rangle$ of $\mathbf{H}$

unknown — $u_h(x) = \sum_{j=1}^{N_h} u^{h,j}\, s_{h,j}(x)$

$$\langle u_h, s_{h,k}\rangle_h = -\int_D f(x)\, s_{h,k}(x)\, dx = -f_{h,k}, \quad k = 1,$$

$$LHS = \sum_{j=1}^{N_h} u^{h,j}\, a_{h,jk}, \quad a_{h,jk} = \langle s_{h,j},\, s_{h,k}\rangle_h = \langle s_{h,j}$$

Matrix equation:

$A_h = \left(a_{h,jk} \right)$ — $N_h \times N_h$-square matrix.

A_h is symmetric positive definite !

$U_h = {}^t(u^{h,1}, \cdots, u^{h,N_h})$ — Unknown vector
$F_h = {}^t(-f^{h,1}, \cdots, -f^{h,N_h})$ — Known vector

The matrix equation: $A_h U_h = F_h$, which is uniquely s

$u_h(x) = \sum_{j=1}^{N_h} u^{h,j} s_{h,j}(x) \in \mathbf{S}_h$: determined.

Important fact:

$u(x) \in \mathbf{H}_0^1(D)$ — The solution of the variational ellipt

$u_h(x) \in \mathbf{S}_h$ — Approximate solution

$u_h(x)$ is the foot of the orthogonal projection of $u(x)$

This fact, together with the construction of $s_{h,j}($
provides the estimate of the difference $u - u_h$.
h corresponds to the order of approximation. This pa
quite akin to the arguments in computability analysis

4) The version in computability analysis

Summary:

a) Our Main Theorem

b) Axiomatic approach of Pour-El & Richards

c) TTE approach of Weihrauch. Some flavor

a) **Our Main Theorem**

Let $\mathbf{H}$ be an effective separable real Hilbert space wit
product $\langle\, ,\, \rangle$. Let $\{\mathbf{e}_n\}$ be an effective generating set o
constitutes a basis of $\mathbf{H}$.
Suppose F is a bounded linear functional on $\mathbf{H}$.
Assume $\{F(\mathbf{e}_n)\}$ be a computable sequence of reals.
(1) Then there is a uniquely determined element $\mathbf{v}_F$
that $F(\mathbf{u}) = \langle \mathbf{u}, \mathbf{v}_F \rangle$ holds for any $\mathbf{u} \in \mathbf{H}$.
(2) $\mathbf{v}_F$ is not necessarily a computable element in $\mathbf{H}$
counter-examples show.
(3) The element $\mathbf{v}_F$ is computable if and only if the
$C_F = \{\, \mathbf{u}\,;\, F(\mathbf{u}) = 0\,\}$ is a recursive set.

Here are lots of jargons

b) Axiomatic approach of Pour-El & Richards

Marian B. Pour-El and J. Ian Richards.
Computability in Analysis and Physics.
Springer-Verlag (**1989**)

Computability structures in Banach spaces
Discussions about classical theorems of harmonic ana
Written in rather familiar language of **lay** mathemati
Very important result — **First Main Theorem**

Some reviews of their results follow:

Computable Structure $\mathscr{S}(\neq \emptyset)$ as the set of computable sequences in $\mathbf{H}$ specified by three axioms

Axiom I [Linear Forms]

$\{\mathbf{x}_{nk}\}, \{\mathbf{y}_{nk}\}$: computable sequences in $\mathbf{H}$.

$\{\alpha_{nk}\}, \{\beta_{nk}\}$ computable sequences of reals (scalars).

$d : \mathbb{N} \to \mathbb{N}$: a recursive function.

Then the sequence $\{\sum_{k=0}^{d(n)} (\alpha_{nk}\mathbf{x}_k + \beta_{nk}\mathbf{y}_k)\}$ is compu

Axiom II [Limits]

$\{\mathbf{x}_{nk}\}$: a computable double sequence in $\mathbf{H}$

$\{\mathbf{x}_{nk}\}$ converges to $\{\mathbf{x}_n\}$ in $\mathbf{H}$ effectively in k, n as k -

Then $\{\mathbf{x}_n\}$ is a computable sequence in $\mathbf{H}$.

Axiom III [Norms]

$\{\mathbf{x}_n\}$: computable sequence in $\mathbf{H}$.

Then $\{\|\mathbf{x}_n\|_{\mathbf{H}}\}$ is a computable sequence of reals.

Remarks:

1) $\mathscr{S} \ni (\mathbf{0}, \mathbf{0}, \cdots)$ is assumed (i.e., $\mathscr{S} \neq \emptyset$)

2) $\{\mathbf{x}_{nk}\}$ converges to $\{\mathbf{x}_n\}$ in $\mathbf{H}$ effectively in k, n as
if and only if

$$\|\mathbf{x}_{nk} - \mathbf{x}_n\| < 2^{-N}, \quad k > e(n, N).$$

for a recursive function $e : \mathbb{N}^2 \to \mathbb{N}$.

3) $\mathbf{x} \in \mathbf{H}$ is a computable element $\iff$ $(\mathbf{x}, \mathbf{x}, \cdots) \in$

4) Axioms I-III are formulated for Banach spaces.

A version with the inner-product can be proposed.

Effective separable:

$\{\mathbf{e}_n\}$ — effective generating set $\iff$

computable sequence

its linear span : dense in $\mathbf{H}$

$\mathcal{E} = \{\sum_{n=0}^{k} q_n \, \mathbf{e}_n \ : \ q_n \in \mathbb{Q}\,\}$: computable. dense in $\mathbf{H}$

N.B.

$\{\mathbf{e}_n\}$ can be made into a complete orthonormal basis

Effective Density Lemma (Pour-El & Richards)

$\{\mathbf{e}_n\}$: a complete orthonormal system,
$\{\mathbf{e}_n\}$ generates a computability structure $\mathscr{S}$ of $\mathbf{H}$.
$\{\mathbf{x}_n\}$: a sequence in $\mathbf{H}$.

$\{\mathbf{x}_n\}$: a computable sequence in $\mathscr{S}$

$$\Longleftrightarrow$$

$\{\sum_{j=0}^{d(n,k)} \alpha_{nkj}\,\mathbf{e}_j\}$ converges to $\{\mathbf{x}_n\}$ in $\mathbf{H}$ effectively
$k \to \infty$

Here $\{\alpha_{nkj}\}$ a computable triple sequence of rational
$d : \mathbb{N}^2 \to \mathbb{N}$ a recursive function

First Main Theorem. (Pour-El & Richards, p.101):

Let X and Y be Banach spaces with computability

Let $\{e_n\}$ be a computable sequence in X whose line

dense in X (i.e. an effective generating set). Let T

be a closed linear operator whose domain $\mathscr{D}(T)$ co

and such that the sequence $\{Te_n\}$ is computable in Y

maps every computable element of its domain onto a c

element of Y if and only if T is bounded.

Complement.

Under the same assumptions, if T is bounded then

be said. The domain of T coincides with X, and T m

computable sequence in X into a computable sequen

Back to Hilbert spaces:

$\{\mathbf{e}_n\}$: orthonormal basis. effective generating set

Fundamental fact:

$\mathbf{x} = \sum_{n=1}^{\infty} x_n \, \mathbf{e}_n \in \mathbf{H}$ computable element

$$\iff$$

$\{x_n\}$: computable sequence of reals
$\sum_{n=1}^{\infty} x_n^2 < +\infty$ **effectively**

Application to a computable version of Riesz's T

$F : \mathbf{H} \to \mathbb{R}$ bounded linear functional

Definition

F is called (PR)-computable* if, for any $\{\mathbf{b}_n\} \in \mathscr{S}$, $\{$
computable sequence of reals.

Proposition

F is (PR)-computable if and only if $\{F(\mathbf{e}_n)\}$ is a c
sequence of reals such that $\sum_{n=1}^{\infty} |F(\mathbf{e}_n)|^2 < +\infty$

N.B. The square sum needs not converge effectively

*not properly following Pour-El & Richards. See First Main The

Remark about the proof of Proposition:

Only if part:

F (PR)-computable $\Longrightarrow \{F(\mathbf{e}_n)\}$ computable sequenc

The convergence of the square sum: See Proof d-1)

If part:

Actually contained in Pour-El & Richards (p.137)

Need to verify that $\{F(\mathbf{x}_n)\}$ is a computable sequen

for any $\{\mathbf{x}_n\} \in \mathscr{S}$.

Counter-example 1:

ℓ_0^∞ : Banach space with norm $\|\xi\|_\infty = \max|\xi_n|$

$\ell_0^\infty \ni \xi = (\xi_0, \xi_1, \cdots) \quad \Longleftrightarrow \quad \lim_{n\to\infty}|\xi_n| = 0$

ξ : computable $\Leftrightarrow$ $\{\xi_n\}$ computable & $\lim_{n\to\infty}|\xi_n| = 0$

ℓ^2 : Banach space with norm $\|\eta\|_2 = \sqrt{\sum_{n=0}^\infty |\eta_n|^2}$

$\ell^2 \ni \eta = (\eta_0, \eta_1, \cdots) \quad \Longleftrightarrow \quad \sum_{n=0}^\infty |\eta_n|^2 < +\infty$

η : computable $\Leftrightarrow$ $\{\eta_n\}$ computable & $\sum_{n=1}^\infty |\eta_n|^2 <$
tively.

The closed linear operator

$I : \ell_0^\infty \ni (c_0, c_1, \cdots) \mapsto (c_0, c_1, \cdots) \in \ell^2$. dom$(I) = \ell_0^\infty \cap$

Apply First Main Theorem to find $\zeta = (\zeta_0, \zeta_1, \cdots) \in$ d
which is computable in ℓ_0^∞ but not in ℓ^2.

$\mathbf{v}_F = \sum_{n=0}^\infty \zeta_n \mathbf{e}_n \in \mathbf{H}$: not computable in $\mathbf{H}$

But $F(\mathbf{x}) = \langle \mathbf{x}, \mathbf{v}_F \rangle$ is (PR)-computable.

Counter-example 2: Perhaps friendlier looking to lo

$a : \mathbb{N} \to \mathbb{N}$ a one-to-one recursive function
generating a recursively enumerable non-recursive set
Let $\quad \zeta_k = 2^{-a(k)/2}, \quad k \in \mathbb{N}, \quad$ and $\quad \mathbf{v}_F = \sum_{k=0}^{\infty} \zeta_k \, \mathbf{e}_k$
(See Pour-El & Richards, p.16. pp.22–24)

The rest is as in the previous counter-example.

N.B.
Actually $\zeta = (\zeta_0, \zeta_1, \cdots) \in \ell_0^{\infty} \cap \ell^2$.

Final comment before proceeding to c):

— key point of Proof d-2) —

$\mathbf{z} \in \mathbf{H}$, $\mathbf{z} \neq \mathbf{0}$

Orthogonal complement of $\{\mathbf{z}\}$: $\quad \{\mathbf{z}\}^{\perp} = \{\, \mathbf{w} \in \mathbf{H} ;\ \langle$

foot of $\mathbf{v} \in \mathbf{H}$ on $\{\mathbf{z}\}^{\perp}$: $\quad \mathrm{pd}_{\{\mathbf{z}\}^{\perp}}(\mathbf{v}) = \mathbf{v} - \dfrac{\langle \mathbf{v}, \mathbf{z} \rangle}{\|\mathbf{z}\|^2}\mathbf{z}$

$\mathrm{pd}_{\{\mathbf{z}\}^{\perp}} : \mathbf{H} \rightarrow \{\mathbf{z}\}^{\perp}$ orthogonal projectio

N.B. $\quad \mathrm{dist}(\mathbf{v}, \{\mathbf{z}\}^{\perp}) = \dfrac{|\langle \mathbf{v}, \mathbf{z} \rangle|}{\|\mathbf{z}\|}.$

comment before c) [contd.]

$\{\mathbf{e}_n\}$ effective generating set
$\mathbf{z} \in \mathbf{H}$. $\|\mathbf{z}\| = 1$

$\{\mathrm{pd}_{\{\mathbf{z}\}^\perp}(\mathbf{e}_n)\}$ computable sequence in $\mathbf{H}$ $\Leftrightarrow$ $\mathbf{z}$ com

key ingredient:

$$\mathrm{pd}_{\{\mathbf{z}\}^\perp}(\mathbf{e}_n) = \mathbf{e}_n - \langle \mathbf{e}_n, \mathbf{z} \rangle\, \mathbf{z} \quad (\|\mathbf{z}\| = 1)$$

c) TTE approach of Weihrauch. Some flavor

K. Weihrauch.
Computable Analysis.
Springer (**2000**)

The key objective here:
discuss computability of the null-space C_F

Our preparation of these lectures is very much indebt
Professor Ning Zhong's series lectures
given at Kyushu University, last November.

Summary:

c-1) TTE approach: **representation, name, code,**

c-2) (ρ, δ)-computable $\Leftrightarrow$ (PR)-computable

c-3) coding of closed and open sets in $\mathbf{H}$

c-4) recursive set

c-5) recursive closed set $\{\mathbf{z}\}^{\perp}$

c-6) A computable version of Riesz's Theorem and re c-1) TTE approach: **representation, name, code,** -

Review:

Cauchy representation of $\mathbb{R}$

$\mathbb{Q}$ = the set of rational numbers. countable. dense in

$\alpha_{\mathbb{Q}} : \mathbb{N} \to \mathbb{Q}$ standard effective coding

$$\rho \ :\subset \mathbb{N}^{\mathbb{N}} \ni p = (k_0, k_1, \cdots) \quad \mapsto \quad x \in \mathbb{R}$$

by $|x - \alpha_{\mathbb{Q}}(k_m)| < 2^{-m}$ as $m \to \infty$.

$$\rho(p) = x \quad \Leftrightarrow \quad p \text{ is a } \quad \rho - \text{name of} \quad x \in \mathbb{R}$$

(ρ : surjective partial map)

N.B. There are other "representations" of $\mathbb{R}$.

In some case, the set of $\mathbb{D}$ of positive dyadic rationals

$$\mathbb{D} \ni d \quad \Longleftrightarrow \quad d = 2^m \sum_{l=0}^{L} \frac{k_l}{2^l} \quad (L \in \mathbb{N},\, m \in \mathbb{Z},\, k_l \in \{0, 1$$

($2^{m-L} \leq d < 2^{m+1}$ in the above)

$\mathbb{D}$ countable, encoded by $\alpha_{\mathbb{D}} : \mathbb{N} \to \mathbb{D}$.

Recall

$\mathcal{E} = \{\sum_{n=0}^{k} q_n \, \mathbf{e}_n \, : \, q_n \in \mathbb{Q}\}$: computable. dense in $\mathbf{H}$

notation or coding of $\mathcal{E}$

$$\alpha \; : \; \mathbb{N} \ni 2^{\ell_0} 3^{\ell_1} \cdots \pi_k^{\ell_k} \quad \mapsto \quad \sum_{n=0}^{k} \alpha_{\mathbb{Q}}(\ell_n) \in \mathcal{E} \quad \text{(bije}$$

($\pi_n = n+1$-st prime, $\pi_0 = 2$, $\pi_1 = 3$, $\pi_2 = 5, \cdots$)

Coding of $\mathbf{H}$

Recall $\{\alpha(k_m)\}$ is a sequence in $\mathcal{E}$ for $(k_0, k_1, \cdots) \in \mathbb{N}^{\mathbb{N}}$

Cauchy representation of $\mathbf{H}$

$$\delta :\subset \mathbb{N}^{\mathbb{N}} \ni p = (k_0, k_1, \cdots) \quad \mapsto \quad \mathbf{x} \in \mathbb{H}$$

by $\|\alpha(k_m) - \mathbf{x}\| \leq 2^{-m}$ as $m \to \infty$.

p δ-name or δ-code

c-2) (ρ, δ)- **computability**

string $\quad p = (k_0, k_1, \cdots) \in \mathbb{N}^{\mathbb{N}}$ computable

$$\Longleftrightarrow$$

$p : \mathbb{N} \ni m \quad \mapsto \quad k_m \in \mathbb{N}$ recursive

$\mathbf{x} \in \mathbf{H}$: computable $\Leftrightarrow$ δ-name of $\mathbf{x}$: computab

$x \in \mathbb{R}$: computable $\Longleftrightarrow$ ρ-name of x : computa

Computability of string function:

A string function

$$f_T : \underbrace{\mathbb{N}^{\mathbb{N}} \times \cdots \times \mathbb{N}^{\mathbb{N}}}_{k} \rightarrow \mathbb{N}^{\mathbb{N}}$$

is computable if there is a Type 2 Turing machine w
putes it†.

†There is a Type 2 Turing machine T which reads the input st
input tapes, the j-th string $p^j = (n_0^j, n_1^j, \cdots)$ on the j-th tape
symbol by symbol *from the left to the right*. T then compu
$(p^1, \cdots, p^k)$ and writes out the output string $q = (m_0, m_1, \cdots$
symbol *from the left to the right*. Thus, $f_T(p^1, \cdots, p^k) = q$.

Notes on computable sequences

$\{\mathbf{x}_n\}$: a sequence in $\mathbf{H}$.

$\{\mathbf{x}_n\}$ is a computable sequence in $\mathbf{H}$

if and only if

there is a computable string function $\hat{\Xi} :\subset \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$

such that $\mathbf{x}_n = \delta(\hat{\Xi}(q^n))$, $n \in \mathbb{N}$.

With these, discussions in Pour-El & Richards are t

into Weihrauch's approach

$F : \mathbf{H} \to \mathbb{R}$ bounded linear functional

F is computable[‡] if

$$F(\delta(p)) = \rho(\Psi(p)), \quad p \in \mathbb{N}^{\mathbb{N}}$$

where $\Psi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a computable string function.

Ψ is a (δ, ρ)-realization of F
F is (δ, ρ)-computable if it has a computable (δ, ρ)-rea

[‡]Forget (PR)-computable for some time. They turn out to be e

c-3) **Coding of closed and open sets in $\mathbf{H}$.**

Open ball with center $\mathbf{c} \in \mathbf{H}$ and radius $r > 0$

$$B(\mathbf{c}, r) = \{\mathbf{x}\,;\ \|\mathbf{x} - \mathbf{c}\| < r\,\}$$

Countable family of open balls

$$\mathfrak{B} = \{\,B(\mathbf{y}, d)\,;\ \mathbf{y} \in \mathcal{E},\ d \in \mathbb{D}\,\}$$

notation or coding of $\mathfrak{B}$

$$\beta\,:\,\mathbb{N} \ni k\ \rightarrow\ (k_1, k_2)\ \rightarrow B(\alpha(k_1), \alpha_{\mathbb{D}}(k_2)) \in$$

$(k \rightarrow (k_1, k_2)$ is the inverse of the standard bijection

Some auxiliary symbols :

$$\mathfrak{B}_k = \{\, B(\mathbf{y}, d)\,;\, \mathbf{y} \in \mathcal{E},\, d \in \mathbb{D},\, d \le 2^{-k} \},\qquad k \in \mathbb{Z}$$

For $X \subset \mathbf{H}$,

$$\mathfrak{B}^X = \{\, B\,;\, B \in \mathfrak{B},\ B \cap X \neq \emptyset \,\}.$$
$$^X\mathfrak{B} = \{\, B\,;\, B \in \mathfrak{B},\, B \cap X = \emptyset \,\}.$$

Also $\mathfrak{B}_n^X = \mathfrak{B}_n \cap \mathfrak{B}^X$ and $^X\mathfrak{B}_n = \mathfrak{B} \cap {}^X\mathfrak{B}$.

Basic Proposition

Take a closed set $A \subset \mathbf{H}$. Then

$$A = \bigcap_{n \in \mathbb{N}} \bigcup_{B \in \mathfrak{B}_n^A} B$$

and

$$\mathbf{H} \setminus A = \bigcup_{n \in \mathbb{N}} \bigcup_{B \in {}^A\mathfrak{B}_n} B = \bigcup_{B \in {}^A\mathfrak{B}} B$$

hold.

$\mathfrak{A}$ = the totality of the closed sets in $\mathbf{H}$.
Basic Proposition $\Longrightarrow$ Each $A \in \mathfrak{A}$ is specified by $\mathfrak{B}^A$.
Via the bijection[§] $\beta : \mathbb{N} \to \mathfrak{B}$, $\beta^{-1}(\mathfrak{B}_A) \subset \mathbb{N}$.
Its enumeration determines $p_A \in \mathbb{N}^{\mathbb{N}}$.
$A \in \mathfrak{A}$ is encoded by $p_A \in \mathbb{N}^{\mathbb{N}}$

Encoding of $\mathfrak{A}$:

$$\psi_< :\subset \mathbb{N}^{\mathbb{N}} \ni p \quad \mapsto \quad A \in \mathfrak{A}$$

defined by

$$\psi_<(p) = \begin{cases} \emptyset, & p = (0, 0, \cdots) \\ A, & p = p_A \end{cases} .$$

[§] $p_A = (n_0, n_1, \cdots)$ iff $\mathfrak{B}^A = \{\beta(n_k); k = 0, 1, \cdots\}$ unless $A = \emptyset$. infinite when $A \neq \emptyset$. Of course, $\mathfrak{B}^\emptyset = \emptyset$ and $\emptyset$ can be coded by

$\mathfrak{O}$ = the totality of the open sets in $\mathbf{H}$.
The compliments of the closed sets are open and th
open sets are closed.
To encode an open set $O = \mathbf{H} \setminus A$,
employ the enumeration $p = p_O \in \mathbb{N}^{\mathbb{N}}$ of $\beta^{-1}({}^{A}\mathfrak{B})$, A =

Encoding of $\mathfrak{O}$:

$$\theta_< : \mathbb{N}^{\mathbb{N}} \ni p \quad \mapsto \quad O \in \mathfrak{O}$$

defined by

$$\theta_<(p) = \begin{cases} \emptyset, & p = (0, 0, \cdots) \\ O, & p = p_O \end{cases} .$$

c-4) recursive set

More detailed encodings:

$$\psi : \mathbb{N}^{\mathbb{N}} \to \mathfrak{A} \quad \text{and} \quad \theta : \mathbb{N}^{\mathbb{N}} \to \mathfrak{O}.$$

$\langle\,,\,\rangle$ = the standard pairing $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ induced by J
For p_1, $p_2 \in \mathbb{N}^{\mathbb{N}}$: $\psi_<(p_1) = A$ and $\theta_<(p_2) = \mathbf{H} \setminus A$,

$$\psi : \mathbb{N}^{\mathbb{N}} \ni p = \langle p_1, p_2 \rangle \mapsto A \in \mathfrak{A}$$

For $O \in \mathfrak{O}$, use ψ-code of $\mathbf{H} \setminus O \in \mathfrak{A}$ to define θ. Thus

$$\theta(\langle q_1, q_2 \rangle) = O \quad \Longleftrightarrow \quad \psi(\langle q_2, q_1 \rangle) = \mathbf{H} \setminus O.$$

$A \in \mathfrak{A}$ is recursively enumerable or r.e.
if its $\psi_<$-code p_A is computable
A is co-recursively enumerable or co-r.e.
if its complement $O = \mathbf{H} \setminus A$ has a computable $\theta_<$-co

In our previous language :
A is $\psi_<$-computable iff A is r.e.,
A is co-r.e. iff $\mathbf{H} \setminus A$ is $\theta_<$-computable.

For closed sets in the Hilbert space $\mathbf{H}$, the above enco
that $A \in \mathfrak{A}$ is r.e. and co-r.e. iff the set $\beta^{-1}(\mathfrak{B}^A) \subset \mathbb{N}$ is
Thus, it is natural to call $A \in \mathfrak{A}$ recursive or ψ-compu
is r.e. and co-r.e..

Some examples:

$\mathbf{0} \in \mathbf{H} \quad \Longrightarrow \quad \{\mathbf{0}\}$ is a recursive closed set.

$\mathbf{x} \in \mathbf{H}$, $\{\mathbf{x}\}$ a recursive closed set $\quad \Longrightarrow \quad \mathbf{x}$ is computa

$\mathbf{z} \in \mathbf{H}, \mathbf{z} \neq \mathbf{0}$ computable

$\Longrightarrow$ $\mathrm{pd}_{\{\mathbf{z}\}^\perp}(\mathbf{y})$ computable for computable $\mathbf{y} \in \mathbf{H}$

$\Longrightarrow$ $\{\mathrm{pd}_{\{\mathbf{z}\}^\perp}(\mathbf{y}_n)\}$ computable sequence

for computable sequence $\{\mathbf{y}_n\}$ in $\mathbf{H}$

c-5) recursive closed set $\{\mathbf{z}\}^\perp$

$\{\mathbf{z}\}^\perp$ is a recursive closed set iff
its $\psi_<$-name is computable and
equality $|\langle \mathbf{y}, \mathbf{z}\rangle| = d\,\|\mathbf{z}\|$ can be effectively determined
for each $\mathbf{y} \in \mathcal{E}$ and $d \in \mathbb{D}$.

$\{\mathbf{z}\}^\perp$ recursive closed set
$\Longrightarrow$ dist$(\mathbf{y}, \{\mathbf{z}\}^\perp)$ is computable $(\mathbf{y} \in \mathcal{E}$ outside $\{\mathbf{z}$

$\{\mathbf{z}\}^\perp$ recursive closed set
$\Longrightarrow$ pd$_{\{\mathbf{z}\}^\perp}(\mathbf{y})$ has computable δ-name $(\mathbf{y} \in \mathcal{E}$ outsic

Further properties:

$\{\mathbf{z}\}^{\perp}$ recursive closed set $\quad \|z\| = 1$

$\Longrightarrow \quad \mathbf{z}$ computable $\quad$ (show $\{\mathrm{pd}_{\{\mathbf{z}\}^{\perp}}(\mathbf{e}_n)\}$ computal c-6) A computable version of Riesz's Theorem and re

$F : \mathbf{H} \to \mathbb{R}$ bounded linear functional

null-space $C_F = \{\mathbf{x}\,;\, F(\mathbf{x}) = 0\,\}$ recursive closed set
$\{F(\mathbf{e}_n)\}$ computable sequence of reals
$\Longrightarrow \quad F(\mathbf{x}) = \langle \mathbf{x}, \mathbf{v}_F \rangle$ with computable $\mathbf{v}_F$

null-space $C_F = \{\mathbf{x}\,;\, F(\mathbf{x}) = 0\,\}$ recursive closed set
$\{F(\mathbf{e})\}$ computable for some computable $\mathbf{e} \in \mathbf{H}$
$\Longrightarrow \quad F(\mathbf{x}) = \langle \mathbf{x}, \mathbf{v}_F \rangle$ with computable $\mathbf{v}_F$

Some further observations:

$\{F(\mathbf{e}_n)\}$ computable sequence of reals

$\|\mathbf{y} - \mathsf{pd}_{C_F}(\mathbf{y})\|$ computable (for some $\mathbf{y} \in \mathcal{E}$, $F(\mathbf{y}) \neq 0$
$\Longrightarrow \quad \mathbf{v}_F$ computable

$\mathbf{v}_F$ computable
$\Longrightarrow \quad \|\mathbf{y} - \mathsf{pd}_{C_F}(\mathbf{y})\|$ computable (for any $\mathbf{y} \in \mathcal{E}$, $F(\mathbf{y})$

Final comment:

$F : \mathbf{H} \to \mathbb{R}$ bounded linear functional. ((PR)-)comput

$$\mathbf{v}_F \quad \text{computable} \quad \Longleftrightarrow \quad C_F \quad \text{recursive closed}$$

N.B. **Interpretation in <u>explicit</u> examples**: immer c

Invitation: In addition to the above notices
Related results according to various differentiate of computability are of course still to be exploite

N.B. Vasco Brattka called me attention that, in view of the f of F coincides with that of $\|\mathbf{v}_F\|$, the above statement may well b standard arguments of recursive closed sets, whence much of c could be simplified.

THANK YOU VERY MUCH FOR YOUR PATIE

There are still incomplete manuscripts in a pdf form
technical details (of various levels, though) about t
discussions. — I show only its references part here
you some ideas.