Fine computable functions and effective Fine convergence

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Abstract

In this article, we discuss the Fine computability and the effective Fine convergence for functions on $[0, 1)$ with respect to the Fine metric as the beginning of the effective Walsh-Fourier analysis. First we treat classically the Fine continuity and the Fine convergence. Next, we prove that Fine computability does not depend on the choice of an effective separating set. Subsequently, we propose a notion of effective Fine convergence for a sequence of functions. We prove that the limit of an effectively Fine continuous sequence of functions and the limit of a Fine computable sequence of functions under this effective Fine convergence is effectively Fine continuous and Fine computable respectively. We also investigate some properties of Fine computable functions through examples. Especially, we extend the result of Brattka, which asserts the existence of a Fine computable but not locally uniformly Fine continuous function. Finally, we treat other examples of Fine computable functions.

Key Words: Fine metric, dyadic interval, Fine continuous function, Fine computable function, effective Fine convergence, continuous convergence

1 Introduction

Piecewise constant functions on the real line have become interesting and important objects in applications of mathematics to information technology. For example, the Walsh-Fourier series as well as the Haar wavelet plays an important role in digital processing. These applications are based on approximation of given data by a finite linear combination of Walsh functions or of Haar wavelets up to some order.

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Piecewise constant functions have inevitably discontinuity of the first kind with respect to the Euclidean topology. For such functions, the values at discontinuous points are often replaced by the corresponding right limits. Let $\mathcal{D}$ be the class consisting of all functions on $[0, 1]$ which are right-continuous and have left limits. It is well known that $\mathcal{D}$ is not separable with respect to the supremum norm, that $\mathcal{D}$ is separable with respect to the Skorohod metric and that $\mathcal{D}$ is complete and separable with respect to a certain metric which is equivalent to the Skorohod metric ([1, 14]). Unfortunately, the definitions of the above two metrices are complicated, and it does not seem feasible to treat them effectively.

On the other hand, in some applications, it is sufficient to deal with those functions which have discontinuities only at dyadic rationals. This is related to the fact that we can treat only dyadic rationals in the course of calculations using digital computers.

Fine introduced the *Fine metric* on the unit interval or on the nonnegative real line and initiated the theory of Walsh-Fourier analysis by proving some fundamental theorems ([6, 7, 8, 17]). He defined the Fine metric between two real numbers as the weighted sum of differences of corresponding bits in their binary representations with infinitely many 0's. Many topological properties concerning the Fine metric are derived from the property that a dyadic interval, that is, an interval $[a, b)$ with dyadic rationals a and b, is open and closed (clopen) with respect to the Fine metric. The topology generated by the set of all dyadic intervals is equivalent to that induced by the Fine metric. The point 1 is an isolated point with this topology in the unit interval $[0, 1]$. Therefore, we treat $[0, 1)$ with the above topology and call this space the *Fine space*.

In this article, we first consider various notions of continuity and the corresponding notions of convergence for functions on the Fine space. We use the term "function" as a mapping from some space to the real line $\mathbb{R}$ with the ordinary Euclidean topology. Subsequently, we treat the corresponding notions of computability. In order to specify the topological properties with respect to the Fine metric, we prefix "Fine" in front of the relevant terms. For example, the convergence of a sequence in $[0, 1)$ with respect to the Fine metric is called *Fine convergence*. Topological notions with no prefix or with the prefix "$\mathbb{E}$-" will mean the notions with respect to the Euclidean metric.

In classical analysis, we usually define some suitable notion of convergence for a function sequence in a space. It is a fundamental problem if the space is closed under the convergence. It is well known that, if a sequence of uniformly continuous functions converges uniformly to f, then f is also uniformly continuous. On the bounded closed interval, it also holds a function is uniformly continuous if and only if it can be approximated uniformly by a sequence of polynomials. In the measure theory, a function is measurable if and only if it can be approximated in measure by a sequence of step functions.

Since the Fine space is not complete, uniform Fine continuity and locally uniform Fine continuity are different. Let $\mathcal{C}_F$ be the set of all Fine continuous functions. It is well known that a function belongs to $\mathcal{C}_F$ if and only if it is $\mathbb{E}$-continuous at every dyadic irrational and right $\mathbb{E}$-continuous at every dyadic rational ([6, 17]). Moreover, a function in $\mathcal{C}_F$ is uniformly Fine continuous if and only if it has a left limit at every dyadic rational. So, a Fine continuous function may diverge. For example, $f(x) = \frac{1}{1-2x}\chi_{[0,\frac{1}{2})}(x)$ is locally uniformly Fine continuous and diverges at $\frac{1}{2}$, where χ_A denotes the indicator (characteristic) function of the set A. Brattka proved the existence of a Fine computable function, hence Fine continuous, which is not locally uniformly Fine continuous.

In the theory of Walsh-Fourier series, it is well known that Walsh functions are uniformly Fine continuous and form a complete orthogonal system in $L^2([0, 1))$. In the applications, we need to calculate Walsh-Fourier coefficients. Many attempts have been made to obtain an algorithm to calculate them fast. *Computability theory* of Walsh-Fourier series has thus become a significant subject. For this purpose, we must first formulate the notion of *computability of a function* on the Fine space and that of *effective convergence* for a sequence of functions.

As for the *effective theory* of the Walsh-Fourier analysis, we have proved in [11] that the Walsh-Fourier coefficients of a *uniformly Fine computable function* form an $\mathbb{E}$-*computable sequence* of reals, and have extended this result to the case of locally uniformly Fine computable functions in [10]. We have also proved in [12] the effective Riemann Lebesgue theorem, which asserts that the Walsh-Fourier coefficients of a locally uniformly Fine computable function $\mathbb{E}$-converges effectively to zero.

In order to extend the above theorems to a more general class of functions, we investigate *Fine computable functions* and the *effective Fine convergence* in this article. For this extension, we first need an approximation theorem which asserts that, for a Fine computable function f, we can obtain an approximating computable sequence of dyadic step functions which Fine converges effectively to f. To complete the extension theory, we need furthermore effectivization of integration theory. We will treat it in a sequel.

In Section 2, we first review briefly the Fine metric and Fine convergence. We define weakly locally uniform Fine convergence (Definition 2.3) and the Fine convergence (Definition 2.4) and prove their equivalence (Proposition 2.2). Either is stronger than the pointwise convergence and weaker than the locally uniform Fine convergence. They preserve Fine continuity (Proposition 2.4). Their relation to the continuous convergence is also discussed.

Section 3 is devoted to preliminaries to computabilities on the Fine space.

In Section 4, we define Fine computable sequences of functions (Definition 4.1). In this definition, effective Fine continuity depends on the choice of an effective separating set. We prove that the notion of effective Fine continuity of functions does not depend on the choice of an effective separating set (Theorem 3).

In Section 5, we introduce the effective Fine convergence of functions (Definition 5.1). We prove that the limit of an effectively Fine continuous sequence of functions under this effective Fine convergence is also effectively Fine continuous (Theorem 7) and that the effective Fine limit of a Fine computable sequence of functions is Fine computable (Theorem 8). We also define the notion of a computable sequence of dyadic step functions and prove that a function f is Fine computable if and only if there exists a computable sequence of dyadic step functions which Fine converges effectively to f (Theorem 9).

In Section 6, we treat the example of Brattka, which is Fine computable but not locally uniformly Fine continuous. We prove that his example satisfies a recursive functional equation, which is related to self-similarity. We modify this equation and obtain other examples (Theorems 12 and 13).

2 Fine metric and Fine convergence

The *Fine metric* d_F on $[0,1)$ is defined as follows: Put $\Omega = \{0,1\}^{\mathbb{N}^+}$, where $\mathbb{N}^+ = \mathbb{N}\backslash\{0\} = \{1,2,\ldots\}$. Let Ω_0 be the set of all elements of Ω with infinitely many zeros. We first define $\mu(x)$ to be the binary representation of $x \in [0,1)$, under the restriction that it has infinitely many zeros. For $x, y \in [0,1)$, let $\mu(x) = \sigma_1\sigma_2\cdots$ and $\mu(y) = \tau_1\tau_2\cdots$. Then, the Fine metric $d_F(x,y)$ is defined by

$$d_F(x,y) = \sum_{k=1}^{\infty} |\sigma_k - \tau_k| 2^{-k}. \tag{1}$$

The following properties of the Fine metric is well known.

Lemma 2.1 (i) *$d_F(x,y) < 2^{-n}$ implies that the first n bits of $\mu(x)$ and $\mu(y)$ coincide.*

(ii) *If the first n bits of $\mu(x)$ and $\mu(y)$ coincide, then $d_F(x,y) < 2^{-(n-1)}$.*

(iii) *For a dyadic rational r, $d_F(r,y) < 2^{-n}$ is equivalent to the coincidence of the first n bits of $\mu(r)$ and $\mu(y)$.*

The Fine space is totally bounded. However, it is not complete, since, for any dyadic rational r, the sequence $\{r - 2^{-n}\}$ is a Fine Cauchy sequence but does not Fine converge.

A left-closed right-open interval with dyadic rational end points is called a *dyadic interval*. It is easy to see that a dyadic interval is open and closed with respect to the Fine metric. This property corresponds to prohibition of left convergence to dyadic rationals and makes some $\mathbb{E}$-discontinuous functions Fine continuous. It also induces the existence of a finite open disjoint covering of $[0,1)$, where the maximum of the diameters of the open sets in the covering is arbitrarily small.

We use the following notations for special dyadic intervals.

$I(n,k) = [k\,2^{-n}, (k+1)2^{-n}),\ 0 \leqslant k \leqslant 2^n - 1,$
$J(x,n) =$ such $I(n,k)$ that includes x.

Since the intervals $\{I(n,k)\}_k$ are disjoint and $\cup_{k=0}^{2^n-1} I(n,k) = [0,1)$, $J(x,n)$ is uniquely determined for each n and x.

We call $I(n,k)$ a *fundamental dyadic interval* (*of level* n) and $J(x,n)$ a *dyadic neighborhood of* x (*of level* n). It is obvious that $I(n,k) = \{x \mid d_F(x, k\,2^{-n}) < 2^{-n}\}$ holds. We state a simple property for later use.

Lemma 2.2 *The following three are equivalent for any $x, y \in [0,1)$ and any positive integer n.*
(i) $y \in J(x,n)$.
(ii) $x \in J(y,n)$.
(iii) $J(x,n) = J(y,n)$.

It is obvious that the sequence $\{J(x,n)\}_n$ forms a fundamental system of neighborhoods of x and the set of all fundamental dyadic intervals becomes an open base for the topology introduced by the Fine metric. If we define $V_n(x) = J(x,n)$, then it is easy to show that $\{V_n\}$ satisfies the axioms of an effective uniform topology ([18] Definition 3.1). It holds that $J(x, n+1) \subset \{y \mid d_F(x,y) < 2^{-n}\}$. So the topology induced by $\{V_n\}$ is equivalent to that induced by the Fine metric.

In the rest of this section, we discuss classical notions of continuity and convergence with respect to the Fine metric.

We remark that most of the arguments below can be carried over to more general topological spaces. Although, we need separability and a countable fundamental system of neighborhoods for Definition 2.2 below. We can define the notions of continuity, locally uniform continuity and uniform continuity for functions on topological spaces in general.

The continuity on the Fine space can be formulated as follows.

Definition 2.1 (t-Fine continuity) *A function f is said to be* t-Fine continuous *if for each k and x there exists an integer $N(k,x)$ such that*
$y \in J(x, N(k,x))$ *implies* $|f(y) - f(x)| < 2^{-k}$.

For a classical theory on the Fine space, the above definition is sufficient. For the sake of effectivization, we define the following Fine continuity using an enumeration of all dyadic rationals $\{e_i\}$. We remark that, we can select a sequence of dyadic rationals which Fine converges to x for each $x \in [0,1)$, and that we can select an e_i such that $x \in J(e_i, n)$ or $e_i \in J(x,n)$ for each x and n.

Definition 2.2 (Fine continuity) *A function f is said to be* Fine continuous *if for each k and i there exists an integer $N(k,i)$ such that*
(a) $x \in J(e_i, N(k,i))$ *implies* $|f(x) - f(e_i)| < 2^{-k}$.
(b) $\bigcup_i J(e_i, N(k,i)) = [0,1)$.

Proposition 2.1 *The t-Fine continuity and the Fine continuity are equivalent.*

Proof. Suppose first that f is Fine continuous with respect to $N_2(k,i)$. Then, by (b), for each k and x, there exists an i such that $x \in J(e_i, N_2(k+1,i))$. Define, for such an i, $N_1(k,x) = N_2(k+1,i)$. Recall that $J(e_i, N_2(k+1,i)) = J(x, N_2(k+1,i))$ holds. If

$$y \in J(x, N_1(k,x)) = J(x, N_2(k+1,i)),$$

then $y \in J(e_i, N_2(k+1,i))$. Since $x \in J(e_i, N_2(k+1,i))$, by (a) for x and y, it follows

$$|f(y) - f(x)| \leqslant |f(y) - f(e_i)| + |f(x) - f(e_i)| < 2^{-(k+1)} + 2^{-(k+1)} = 2^{-k}.$$

So, f is t-Fine continuous with respect to $N_1(k,x)$.

Conversely, assume that f is t-Fine continuous with respect to $N_1(k,x)$. Define

$$N_2(k,i) = \min\{N_1(k+1,x) \,|\, e_i \in J(x, N_1(k+1,x)), x \in [0,1)\}.$$

Notice that the minimum is attained by some z. For such a z, $N_2(k,i) = N_1(k+1,z)$. Now suppose $x \in J(e_i, N_2(k,i)) = J(e_i, N_1(k+1,z))$. Notice that $J(e_i, N_1(k+1,z)) = J(z, N_1(k+1,z))$. Then $x \in J(z, N_1(k+1,z))$ and $e_i \in J(z, N_1(k+1,z))$. So, using the Fine continuity twice, we have,

$$|f(x) - f(e_i)| \leqslant |f(x) - f(z)| + |f(z) - f(e_i)| < 2^{-(k+1)} + 2^{-(k+1)} = 2^{-k}.$$

This proves (a)

Notice next that, for each k and x, there is an e_i such that $e_i \in J_1(x, N_1(k+1,x))$. Take a z as above. Then, since $N_1(k+1,z) \leqslant N_1(k+1,x)$,

$$x \in J(e_i, N_1(k+1,x)) \subset J(e_i, N_1(k+1,z))$$

and

$$x \in J(x, N_1(k+1,x)) \subset J(z, N_1(k+1,z)) = J(e_i, N_2(k,i)).$$

Hence, we obtain $x \in J(e_i, N_2(k,i))$. This proves (b). ∎

Remark 2.1 We can also define the Fine continuity with respect to any separating set (a countable dense subset) $\{s_i\}$, by replacing $\{e_i\}$ by $\{s_i\}$. The above proof is valid also for this modification. From this fact, we can deduce that the Fine continuity does not depend on the choice of a separating set.

The uniform convergence and the locally uniform convergence are fundamental concepts of the calculus. It is well known that the limit of a sequence of uniformly continuous functions under the uniform convergence is uniformly continuous and that the limit of a sequence of locally uniformly continuous functions under the locally uniform convergence is locally uniformly continuous.

We give a simple example of the Fine continuous function. A *dyadic step function* is defined to be a finite linear combination of indicator functions of dyadic intervals.

By definition, a function f is a dyadic step function if and only if there exists a positive integer n such that f is constant on each fundamental dyadic interval $I(n, j)$, $0 \leqslant j < 2^n$. Therefore, a dyadic step function is uniformly Fine continuous.

It is easy to prove that a function f is uniformly Fine continuous if and only if there exists a sequence of dyadic step functions which converges uniformly to f and that f is locally uniformly Fine continuous if and only if there exists a sequence of dyadic step functions which Fine converges locally uniformly to f. An approximating sequence of f by dyadic step functions is given by

$$\varphi_n(x) = \sum_{j=0}^{2^n-1} f(j2^{-n})\chi_{I(n,j)}(x). \tag{2}$$

The example of Brattka, which we treat in Section 6, is Fine continuous but not locally uniformly Fine continuous. Therefore, we need a weaker notion of convergence of functions in order to approximate a Fine continuous function by a sequence of dyadic step functions. By weakening the locally uniform Fine convergence, we obtain the following two notions of convergence.

Definition 2.3 (Weakly locally uniform Fine convergence) *A sequence of functions* $\{f_n\}$ *is said to* Fine converge weakly locally uniformly *to* f *if, for each* k *and* x, *there exist integers* $N(k, x)$ *and* $M(k, x)$ *such that*

$y \in J(x, N(k, x))$ and $n \geqslant M(k, x)$ imply $|f_n(y) - f(y)| < 2^{-k}$.

Let us remark that we obtain the locally uniform Fine convergence from Definition 2.3 if $N(k, x)$ does not depend on k. We also define the following convergence, for the sake of effectivization, similarly to Definition 2.2.

Definition 2.4 (Fine convergence) *A sequence of functions* $\{f_n\}$ *is said to* Fine converge *to* f *if, for each* k *and* i, *there exist integers* $N(k, i)$ *and* $M(k, i)$ *such that*

(a) *$x \in J(e_i, N(k, i))$ and $n \geqslant M(k, i)$ imply $|f_n(x) - f(x)| < 2^{-k}$,*

(b) *$\bigcup_{i=1}^{\infty} J(e_i, N(k, i)) = [0, 1)$ for each k.*

Similarly to Proposition 2.1, we can prove the following proposition.

Proposition 2.2 *The weakly locally uniform Fine convergence and the Fine convergence are equivalent.*

For a sequence of Fine continuous functions, we obtain the following proposition.

Proposition 2.3 *If a sequence of t-Fine continuous functions Fine converges weakly locally uniformly to f, then f is also t-Fine continuous.*

Proof. Let $\{f_n\}$ be a sequence of Fine continuous functions with respect to $N_1(n,k,x)$ and suppose that it converges weakly locally uniformly to f with respect to $N_2(k,x)$ and $M(k,x)$. Define $N(k,x) = \max\{N_2(k+2,x), N_1(M(k+2,x), k+2, x))\}$. Then $y \in J(x, N(k,x))$ implies

$$
\begin{aligned}
& |f(y) - f(x)| \\
\leqslant\ & |f(y) - f_{M(k+2,x)}(y)| + |f_{M(k+2,x)}(y) - f_{M(k+2,x)}(x)| + |f_{M(k+2,x)}(x) - f(x)| \\
<\ & 32^{-(k+2)} < 2^{-k}.
\end{aligned}
$$

∎

From Proposition 2.1, Proposition 2.2 and Proposition 2.3, we obtain the following proposition.

Proposition 2.4 *If a sequence of Fine continuous functions Fine converges to f, then f is also Fine continuous.*

We also obtain the following proposition.

Proposition 2.5 *A function f is Fine continuous if and only if there exists a sequence of dyadic step functions which Fine converges to f or, equivalently, weakly locally uniformly Fine converges to f.*

The weakly locally uniform convergence reminds us of the continuous convergence. According to Binz ([2]), the continuous convergence is equivalent to the compact uniform convergence in the case of locally compact topological spaces. In [4], it is pointed out that metrizable $c{=}lu$ spaces are locally compact, where c means continuous convergence and lu means locally uniform convergence. Accordingly, the continuous convergence and the locally uniform convergence do not coincide on the Fine space.

Schröder ([16]) investigated the notion of continuous convergence of a function sequence in relation to the admissible representation of the space of all continuous functions and to the sequentialization of the compact uniform topology. In general, continuous convergence is defined to be the coarsest convergence structure on the space of continuous functions which makes the evaluation map $(f,x) \to f(x)$ continuous. We discuss briefly continuous convergence following the definition in [16].

Definition 2.5 (Continuous Fine Convergence) $\{f_n\}$ *is said to* Fine converge continuously *to f if $\{f_n(x_n)\}$ $\mathbb{E}$-converges to $f(x)$ for every sequence $\{x_n\}$ which Fine converges to x.*

As stated in Introduction, $\mathbb{E}$-convergence means the convergence with respect to the Euclidean topology.

Remark 2.2 The continuous Fine convergence is equivalent to the following.

For each k and x there exist integers $N(k,x)$ and $M(k,x)$ which satisfy that $y \in J(x, N(k,x))$ and $n \geqslant M(k,x)$ imply $|f_n(y) - f(x)| < 2^{-k}$.

The proposition below follows.

Proposition 2.6 *If a sequence of Fine continuous functions $\{f_n\}$ Fine converges continuously to f, then f is Fine continuous.*

Proof. Let us assume that a sequence $\{x_m\}$ Fine converges to x. For each m, $\{f_n(x_m)\}_n$ $\mathbb{E}$-converges to $f(x_m)$ by virtue of continuous convergence. So, we can choose a strictly increasing sequence of positive integers $\{n_m\}$ such that $|f_{n_m}(x_m) - f(x_m)| < 2^{-m}$. If we define $y_n = x_m$ if $n = n_m$ for some m and $= x$ otherwise, then $\{y_n\}$ Fine converges to x. From the continuous convergence, $\{f_n(y_n)\}$ $\mathbb{E}$-converges to $f(x)$. Hence, the subsequence $\{f_{n_m}(x_m)\}$ $\mathbb{E}$-converges to $f(x)$. From the above inequality, $\{f(x_m)\}$ also $\mathbb{E}$-converges to $f(x)$. $\square$

It seems to be difficult to discuss the relation between the continuous Fine convergence and the weakly locally uniform Fine convergence in a general setting. Let us assume that a sequence of Fine continuous functions converges under one of them. Then, from the last two propositions, the limit function is Fine continuous. So, we can obtain the other convergence by changing $N(k, x)$ in a suitable manner, using the Fine continuity of the limit function.

Proposition 2.7 *For a sequence of Fine continuous functions, the weakly locally uniform Fine convergence and the continuous Fine convergence are equivalent.*

3 Preliminaries on computability

A sequence of rationals $\{r_n\}$ is called *recursive* if there exist recursive functions $\alpha(n)$, $\beta(n)$ and $\gamma(n)$ which satisfy $r_n = (-1)^{\gamma(n)}\frac{\beta(n)}{\alpha(n)}$. We will subsequently treat the computability of sequences from the Fine space and the computability of functions on the Fine space. So, we assume that a number x or a sequence $\{x_n\}$ is in $[0, 1)$ unless otherwise stated.

A double sequence $\{x_{n,m}\}$ is said to *Fine converge effectively* to a sequence $\{x_n\}$ if there exists a recursive function $\alpha(n, k)$ such that $x_{n,m} \in J(x_n, k)$ for all n, k and all $m \geqslant \alpha(n, k)$.

A sequence $\{x_n\}$ is said to be *Fine computable* if there exists a recursive double sequence of rationals $\{r_{n,m}\}$, which Fine converges effectively to $\{x_n\}$. For this definition, it is sufficient to take a recursive sequence of dyadic rationals instead of a recursive sequence of rationals in general. A single element x is called Fine computable if the sequence $\{x, x, x, \ldots\}$ is Fine computable. The definition of Fine computability can be extended to multiple sequences in an obvious manner.

A Fine computable sequence which is dense in $[0, 1)$ is called an *effective separating set*.

If we use the Euclidean metric instead of the Fine metric in the above, then we obtain the usual notion of computability on the real line. We call this computability

$\mathbb{E}$-*computability.* Notice that a single real number is $\mathbb{E}$-computable if and only if it is Fine computable, and that a Fine computable sequence of real numbers is also an $\mathbb{E}$-computable sequence ([3, 11, 23]). But the converse of the latter does not hold. It also holds that a recursive sequence of rationals is Fine computable, while an $\mathbb{E}$-computable sequence of rationals is not necessarily Fine computable ([3, 11]).

In the subsequent sections, we treat computability and effective convergence of real valued functions on the Fine space. In the scheme of Pour-El and Richards ([15]) for computable analysis, computability of a real function is formulated in terms of two properties: (i) *Sequential computability* and (ii) *Effective continuity.* (i) claims that computable sequences are mapped to computable sequences by the function, and (ii) claims that the function has a recursive modulus of continuity. They used the effective uniform continuity for functions on bounded closed intervals and the effectively compact uniform continuity for functions on the real line. In the representation based approach developed by Weihrauch ([19]), continuity is also a necessary condition for the computability of functions.

There have been several approaches to weaker notions of computable functions in order to make some simple $\mathbb{E}$-discontinuous functions computable. We quote only some recent works, which are closely related to this article: [3, 11, 12, 18, 23, 20, 24]. In the last two, the computability on the range is weakened by replacing the recursive modulus of convergence with the limitting recursive one in the definition of computable sequences of reals. Another method is that the topology on the domain of definition is replaced by the Fine metric, which is stronger than the Euclidean metric: [3, 11, 12]. The latter approach is generalized to the computability with respect to an effective uniformity in [18, 23]. Various examples of effective uniformities, which make other types of discontinuous functions computable, are listed in [23].

The uniform Fine computability of a function is introduced in [11]. The locally uniform Fine computability is treated in [12] together with the effective locally uniform Fine convergence. A similar but slightly different notion of computability is also introduced in [18] for functions on a space with an effective uniform topology.

In this section, we review the above two definitions of computability for functions on the Fine space, together with the corresponding effective convergence. Another will be introduced in the next section. For this purpose, we take a recursive enumeration of all dyadic rationals in $[0, 1)$, denoted by $\{e_i\}$, as an effective separating set and use it through this article. An effective separating set is defined to be a computable sequence which is a dense subset.

Roughly speaking, we define the effective Fine continuity and the effective Fine convergence by requiring that $N(k, i)$ and $M(k, i)$ in Definition 2.2 and Definition 2.4 are recursive functions.

Definition 3.1 (Uniformly Fine computable sequence of functions [11]) *A sequence of functions* $\{f_n\}$ *is said to be* uniformly Fine computable *if* (i) *and* (ii) *below hold.*

(i) (Sequential Fine computability) *The double sequence* $\{f_n(x_m)\}$ *is* $\mathbb{E}$*-computable for any Fine computable sequence* $\{x_m\}$.

(ii) (Effectively uniform Fine continuity) *There exists a recursive function* $\alpha(n,k)$ *such that, for all* n,k *and all* $x,y \in [0,1)$, $y \in J(x,\alpha(n,k))$ *implies* $|f_n(x) - f_n(y)| < 2^{-k}$.

The uniform Fine computability of a single function f is defined by that of the sequence $\{f,f,\ldots\}$.

Notice that the computability of the sequence $\{f_n(x_m)\}$ in (i) is $\mathbb{E}$-computability.

Definition 3.2 (Effectively uniform convergence of functions [11]). *A sequence of functions* $\{f_n\}$ *is said to* converge effectively uniformly *to a function* f *if there exists a recursive function* $\alpha(k)$ *such that, for all* n *and* k,

$$n \geqslant \alpha(k) \quad \textit{implies} \quad |f_n(x) - f(x)| < 2^{-k} \quad \textit{for all} \quad x.$$

Closedness of the space of uniformly Fine computable functions under the effectively uniform convergence is claimed in ([12]). The proof is similar to that of the corresponding theorem in [15].

Theorem 1 *If a uniformly Fine computable sequence of functions* $\{f_n\}$ *converges effectively uniformly to a function* f, *then* f *is also uniformly Fine computable.*

Definition 3.3 (Locally uniformly Fine computable sequence of functions, [10]) *A sequence of functions* $\{f_n\}$ *is said to be* locally uniformly Fine computable *if the following* (i) *and* (ii) *hold.*

(i) $\{f_n\}$ *is sequentially Fine computable.*

(ii) (Effectively locally uniform Fine continuity) *There exist recursive functions* $\alpha(n,k,i)$ *and* $\beta(n,i)$ *which satisfy the following* (ii-a) *and* (ii-b).

(ii-a) *For all* i, n *and* k, $|f_n(x) - f_n(y)| < 2^{-k}$ *if* $x,y \in J(e_i,\beta(n,i))$ *and* $y \in J(x,\alpha(n,k,i))$.

(ii-b) $\bigcup_{i=1}^{\infty} J(e_i,\beta(n,i)) = [0,1)$ *for each* n.

It is proved in Example 4.1 of [10] that the function f defined by $f(x) = \frac{1}{1-2x}$ if $x < \frac{1}{2}$ and $= 0$ if $x \geq \frac{1}{2}$ is locally uniformly Fine computable but not uniformly Fine continuous, since it diverges at $\frac{1}{2}$.

A weakened notion of effective convergence is defined as follows.

Definition 3.4 (Effectively locally uniform Fine convergence, [10]). *A sequence of functions* $\{f_n\}$ *is said to* Fine converge effectively locally uniformly *to a function* f *if there exist a recursive function* $\gamma(i)$ *and a recursive function* $\delta(k,i)$ *such that*

(a) $|f_n(x) - f(x)| < 2^{-k}$ *for* $x \in J(e_i,\gamma(i))$ *and* $n \geq \delta(k,i)$,

(b) $\cup_{i=1}^{\infty} J(e_i,\gamma(i)) = [0,1)$.

Theorem 2 ([10]) *If a locally uniformly Fine computable sequence of functions* $\{f_n\}$ *Fine converges effectively locally uniformly to* f*, then* f *is locally uniformly Fine computable.*

Theorem 2 can be proved similarly to the proof of Theorem 8 in Section 4.

Remark 3.1 The above two definitions of computable functions can be carried over to an effectively separable metric space with a computability structure or to a space with effective uniformity. The latter case is treated in [18].

4 Fine computable functions

The notion of the Fine computable functions is introduced as that of (ρ_F, ρ_E)-computable functions by Brattka ([3]) for a single function. Here, ρ_E is an admissible standard representation of the real numbers with respect to the Euclidean metric and ρ_F is the Fine representation (cf. [3, 19]), that is, the inverse of μ. He proved that (ρ_F, ρ_E)-computability is equivalent to the following Fine computability. We extend Brattka's definition to that of a function sequence, and prove a theorem similar to Theorem 2.

Recall that $\{e_i\}$ is a recursive enumeration of all dyadic rationals in $[0, 1)$ and notice that it is an effective separating set.

Definition 4.1 (Fine computable sequence of functions) *A sequence of functions* $\{f_n\}$ *is said to be* Fine computable *if it satisfies the following.*

(i) $\{f_n\}$ *is sequentially Fine computable.*

(ii) (Effective Fine Continuity) *There exists a recursive function* $\alpha(n, k, i)$ *such that*

(ii-a) $x \in J(e_i, \alpha(n, k, i))$ *implies* $|f_n(x) - f_n(e_i)| < 2^{-k}$,

(ii-b) $\bigcup_{i=1}^{\infty} J(e_i, \alpha(n, k, i)) = [0, 1)$ *for each* n, k.

The Fine computability of a single function f is defined by replacing $\alpha(n, k, i)$ with $\alpha(k, i)$. It is equivalent to saying that the sequence $\{f, f, \ldots\}$ is computable. Effective Fine continuity is the "effectivization" of Fine continuity, that is, we require that $N(k, i)$ in Definition 2.2 is a recursive function,

Brattka ([3]) proved the existence of a Fine computable (hence Fine continuous) function which is not locally uniformly Fine continuous. We extend his result in Section 6. It appears to be easy to prove that effectively locally uniform Fine continuity implies effective Fine continuity, but Iiduka has pointed out that the proof is not trivial. He proved this property in a more general setting ([9]).

Definition 4.2 (Effective Fine continuity with respect to $\{r_i\}$) *If the requirement* (ii) *in Definition 4.1 holds for a Fine computable sequence* $\{r_i\}$ *instead of* $\{e_i\}$*, we say that* f *is* effectively Fine continuous with respect to $\{r_i\}$.

In [18], we proposed a slightly different notion of computability of functions on an effective uniform topological space, that is, we required the sequential computability, the effective continuity with respect to some effective separating set and the relative computability. On the Fine space, we can prove that the effective Fine continuity of a function sequence does not depend on the choice of an effective separating set.

Theorem 3 *If f is effectively Fine continuous with respect to an effective separating set $\{r_i\}$, then f is effectively Fine continuous with respect to any effective separating set $\{t_j\}$.*

For the proof of this theorem, we prepare some elementary properties concerning dyadic intervals. Classically, they are self-evident. We will explain how some of classical proofs can be effectivized. We say that a sequence of dyadic intervals $I_j = [a_j, b_j)$ $(a_j < b_j)$ is an *effective dyadic covering of a dyadic interval* I if $\{a_j\}$ and $\{b_j\}$ are recursive sequences of dyadic rationals, each I_j is a subinterval of I and $\cup_j I_j = I$.

Lemma 4.1 *The following hold.*

(i) *Let I be a dyadic interval, that is, $I = [i2^{-m}, j2^{-n})$ for some positive integers i, j, m and n, and let x be Fine computable. Then, we can decide effectively whether $x \in I$ or $x \notin I$.*

(ii) *Let I and J be dyadic intervals. Then, we can decide effectively whether $I \cap J = \phi$ or not, and whether $I \subseteq J$ or not.*

(iii) *Let $\{s_i\}$ be an effective separating set. Then for any n and k, we can find effectively an i such that $s_i \in I(n, k)$, that is , there is a recursive function of n and k which computes i. In this case, $I(n, k) = J(s_i, n)$.*

(iv) *Let $\{I_j\}$ be an effective dyadic covering of $[0, 1)$ and let $\{x_n\}$ be Fine computable. Then we can select effectively some $j = j(n)$ such that $x_n \in I_j$.*

(v) *If a dyadic interval $[a, b)$ is not a fundamental dyadic interval, then we can decompose it effectively into finitely many disjoint fundamental dyadic intervals.*

From the condition (ii-b) in Definition 4.1, it follows that the set of dyadic neighborhoods $\{J(e_i, \alpha(n, k, i))\}_i$ is an effective dyadic covering of $[0, 1)$ for each n, k.

For a covering consisting of dyadic intervals, the following lemma holds.

Lemma 4.2 *Let $\{J_p\}$ be an effective dyadic covering of a dyadic interval I. Then, we can construct an effective dyadic covering $\{I_q\}$ of I, which satisfies the following conditions.*

(i) *Each I_q is a fundamental dyadic interval.*

(ii) *I_q is a subinterval of J_p for some p.*

(iii) *I_q's are disjoint.*

Proof. Let us first note that we can perform the following (a) and (b) effectively by using Lemma 4.1:

(a) The complement of a dyadic interval, say $[a,b)^C$, is equal to $[0,a) \cup [b,1)$.

(b) The complement of a finite union of dyadic intervals $(\cup_{i=1}^{n}[a_i,b_i))^C = \cap_{i=1}^{n}[a_i,b_i)^C$ can be represented by a finite disjoint union of fundamental dyadic intervals.

We only explain the construction of $\{I_q\}$ according to a routine procedure in measure theory. The construction itself will explain that the whole procedure can be done effectively.

First, J_1 is a dyadic interval by definition. So, we can decompose it into finitely many disjoint fundamental dyadic intervals, say, $I_1, \ldots, I_{\tau_1}$ by Lemma 4.1 (v).

Second, consider $(J_2 \cap (J_1)^C) = (J_2 \cap (\cup_{q=1}^{\tau_1} I_q)^C)$. It is a finite disjoint union of dyadic intervals by (b) just above. So, we decompose them and obtain a finite sequence of fundamental dyadic intervals $I_{\tau_1+1}, \ldots, I_{\tau_1+\tau_2}$, the union of which is $(J_2 \cap (J_1)^C)$.

Next, try the same for $(J_3 \cap (J_1 \cup J_2)^C) = (J_3 \cap (\cup_{q=1}^{\tau_1+\tau_2} I_q)^C)$, and so on. If we continue the above process, we obtain $\{I_q\}$, which is the desired sequence.

The construction above suggests the following: if $J_p = J(r_p, \alpha(p))$ for some recursive function $\alpha(p)$ and a recursive sequence of dyadic rationals $\{r_p\}$, then we can obtain recursive functions $\beta(q)$ and $\gamma(q)$ $(0 \leqslant \gamma(q) \leqslant 2^{\beta(q)} - 1)$ so that $I_q = I(\beta(q), \gamma(q))$. $\square$

Proposition 4.1 *Let $\{r_i\}$ be an effective separating set and let f be a function on $[0,1)$. Then, f is effectively Fine continuous with respect to $\{r_i\}$ if and only if there exist a Fine computable double sequence $\{s_{k,q}\}$ and a recursive function $\delta(k,q)$ which satisfy the following.*

(a) *$\{s_{k,q}\}_q$ is a subset of $\{r_i\}$ for each k.*

(b) *$\{J(s_{k,q}, \delta(k,q))\}_q$ is disjoint for each k.*

(c) *$x \in J(s_{k,q}, \delta(k,q))$ implies $|f(x) - f(s_{k,q})| < 2^{-k}$.*

(d) *$\bigcup_{q=1}^{\infty} J(s_{k,q}, \delta(k,q)) = [0,1)$ for each k.*

Proof. First, we prove the "if" part. For each k and i, we can find effectively such q that $r_i \in J(s_{k+1,q}, \delta(k+1,q))$. It is sufficient to take $\alpha(k,i) = \delta(k+1,q)$, since

$$|f(x) - f(r_i)| \leqslant |f(x) - f(s_{k+1,q})| + |f(s_{k+1,q}) - f(r_i)| < 2^{-k}$$

for $x \in J(r_i, \alpha(k,i)) = J(s_{k+1,q}, \delta(k+1,q))$.

To prove the "only if" part, let $\alpha(k,i)$ be a recursive modulus of continuity of f and let us consider $\{J(r_p, \alpha(k+1,p))\}_p$ for each k. If we apply Lemma 4.2 to this sequence with $I = [0,1)$, then we obtain recursive functions $\beta(k,q)$ and $\gamma(k,q)$ so that the sequence $\{I_{k,q}\} = \{I(\beta(k,q), \gamma(k,q))\}$ is an effective dyadic covering of $[0,1)$ and satisfies (i) to (iii) of Lemma 4.2 for each k. We define $\delta(k,q) = \gamma(k,q)$. For each q, we can select p and r_i so that $r_i \in I_{k,q} \subseteq J(r_p, \alpha(k+1,p))$. If we put $s_{k,q} = r_i$, then it holds that

$$|f(x) - f(s_{k,q})| \leqslant |f(x) - f(r_p)| + |f(r_p) - f(r_i)| < 2^{-k},$$

for $x \in J(s_{k,q}, \delta(k,q)) = I_{k,q}$. $\square$

Proof of Theorem 3. Assume that f is effectively Fine continuous with respect to an effective separating set $\{r_i\}$ and that $\{t_j\}$ is an effective separating set. Let $\{s_{k,q}\}$ and $\delta(k,q)$ satisfy the requirements (a) to (d) in Proposition 4.1. For each k, q, choose some $t_j \in J(s_{k+1,q}, \delta(k+1,q))$ and denote it by $u_{k,q}$. (We can do this effectively, hence $\{u_{k,q}\}$ is computable). It holds that $J(s_{k+1,q}, \delta(k+1,q)) = J(u_{k,q}, \delta(k+1,q))$ and

$$|f(y) - f(u_{k,q})| \leqslant |f(y) - f(s_{k+1,q})| + |f(s_{k+1,q}) - f(u_{k,q})| < 2^{-k}$$

for $y \in J(u_{k,q}, \delta(k+1,q))$. If we define $\tilde{\delta}(k,q) = \delta(k+1,q)$, then $\tilde{\delta}(k,q)$ is recursive and the conditions (a) to (d) of Proposition 4.1 hold for $\{u_{k,q}\}$ and $\tilde{\delta}(k,q)$ with respect to $\{t_j\}$. If we apply Proposition 4.1 again, we obtain that f is effectively Fine continuous with respect to $\{t_j\}$. $\square$

As a corollary to Theorem 3, it follows that the computability of functions in [18] is equivalent to the Fine computability on the Fine space.

Let us consider the maximum of a Fine computable function. On the unit interval $[0,1]$, Pour-El and Richards proved that the maximum of a uniformly $\mathbb{E}$-computable function is an $\mathbb{E}$-computable real ([15]). This property also holds on an effectively compact metric space with a computability structure ([13]). Since the Fine space is not complete, a Fine continuous function does not necessarily attain its maximum. Nevertheless, we obtain the following weaker property ([12]).

Theorem 4 *A uniformly Fine computable function has the $\mathbb{E}$-computable supremum.*

This theorem is proved by using the following theorem and its proof in [11].

Theorem 5 *A function f on $[0,1)$ is uniformly Fine computable if and only if there exists a uniformly computable function g on (Ω, d_C) such that $f(x) = g(\mu(x))$ for all $x \in [0,1)$, where d_C is the Cantor metric on Ω, that is, $d_C(\sigma,\tau) = \sum_{k=1}^{\infty} 2^{-k}|\sigma_k - \tau_k|$ for $\sigma, \tau \in \Omega$.*

Notice that $\sup_{x\in[0,1)} f(x) = \max_{\sigma\in\Omega} g(\sigma)$ holds.

For locally uniformly Fine computable functions, the corresponding property does not hold.

In the rest of this section, we treat some examples of Fine computability of a function. For this purpose, we introduce special dyadic step functions:

$$\chi_c(x) = \chi_{[0,c)}(x),\ \tilde{\chi}_n(x) = \chi_{[1-2^{-(n-1)},1-2^{-n})}(x). \tag{3}$$

It is obvious that $\tilde{\chi}_n$ is uniformly Fine computable and the same holds for χ_c if c is a dyadic rational.

Proposition 4.2 *There exists a bounded locally uniformly Fine computable function, the supremum of which is not $\mathbb{E}$-computable.*

Proof. Let a be a one-to-one recursive function from $\mathbb{N}^+$ to $\mathbb{N}^+$, whose range $a(\mathbb{N}^+)$ is not recursive. Define $c_n = \sum_{k=1}^{n} 2^{-a(k)}$. Then $\{c_n\}$ is an $\mathbb{E}$-computable sequence of real numbers, which is monotonically increasing and converges to a non-$\mathbb{E}$-computable limit c ([15]). Define also $f(x) = \sum_{n=1}^{\infty} c_n \tilde{\chi}_n(x)$. $I_n = [1 - 2^{-(n-1)}, 1 - 2^{-n}) = [\frac{2^n-2}{2^n}, \frac{2^n-1}{2^n})$ is a fundamental dyadic interval and $\{I_n\}$ is a partition of $[0, 1)$. Let us define $\beta(i) = \alpha(k, i) = n$ if $e_i \in I_n$. Then f is locally uniformly Fine computable with respect to β and α. But $\sup_{0 \leqslant x < 1} f(x) = c$ is not $\mathbb{E}$-computable. $\square$

We give a simple example of a function which is not Fine computable. In the following proposition, $\frac{1}{3}$ is not essential, and the proposition remains valid if we replace $\frac{1}{3}$ with any dyadic irrational.

Proposition 4.3 *$\chi_{\frac{1}{3}}$ satisfies the following:*

(i) *It is not Fine continuous.*

(ii) *It is not sequentially Fine computable.*

(iii) *For any $\alpha(k, i)$ which satisfies Definition 4.1* (ii-a)*,* (ii-b) *does not hold.*

Proof. (i) is obvious, since $\frac{1}{3}$ is not a dyadic rational and a Fine continuous function must be continuous with respect to the Euclidean metric at every dyadic irrational.

(ii) Let a be a one-to-one recursive function, whose range $A = a(\mathbb{N})$ is not recursive and each value of which is greater than 1. Define $\{x_n\}$ and $\{x_{n,k}\}$ respectively by

$$x_n = \begin{cases} \frac{1}{3} - \frac{1}{2^{m+1}} & \text{if there exists an } m \text{ such that } a(m) = n \\ \frac{1}{3} & \text{otherwise} \end{cases},$$

and

$$x_{n,k} = \begin{cases} \frac{1}{3} - \frac{1}{2^{m+1}} & \text{if there exists } m \text{ s.t. } \; m \leqslant k \; \text{ and } \; n = a(m) \\ \frac{1}{3} & \text{otherwise} \end{cases}.$$

Then, $x_{n,k} \in J(x_n, p)$ if $k \geqslant p + 2$ for all n, and $\{x_{n,k}\}$ Fine converges effectively to $\{x_n\}$. Therefore, $\{x_n\}$ is Fine computable.

To show that $\{\chi_{\frac{1}{3}}(x_n)\}$ is not $\mathbb{E}$-computable, recall that $\chi_{\frac{1}{3}}(x) = 1$ if $0 \leqslant x < \frac{1}{3}$ and $= 0$ if $\frac{1}{3} \leqslant x < 1$. By the definition, $x_n < \frac{1}{3}$ if $n \in A$ and $x_n = \frac{1}{3}$ if $n \notin A$. So, $\chi_{\frac{1}{3}}(x_n) = 1$ if and only if $n \in A$. If $\{\chi_{\frac{1}{3}}(x_n)\}$ were $\mathbb{E}$-computable, then it would be a recursive sequence of natural numbers. So A would be recursive, yielding a contradiction.

(iii) Assume that $\frac{1}{3} \in J(e_i, \alpha(k, i))$ for some i. Since $J(e_i, \alpha(k, i))$ is a dyadic interval and $\frac{1}{3}$ is not a dyadic rational, $\frac{1}{3}$ is not an end point. So, there exists x in $J(e_i, \alpha(k, i))$, which satisfies $x < \frac{1}{3}$. By the definition of χ, $\chi_{\frac{1}{3}}(x) - \chi_{\frac{1}{3}}(\frac{1}{3}) = 1$. $\square$

5 Effective Fine convergence

In this section, we define the effective Fine convergence of a sequence of functions, and prove that the space of effectively Fine continuous functions is closed with respect to this convergence.

Definition 5.1 (Effective Fine convergence of functions) *We say that a sequence of functions* $\{f_n\}$ Fine converges effectively *to a function* f *if there exist recursive functions* $\beta(k,i)$ *and* $\gamma(k,i)$ *which satisfy*

(a) $x \in J(e_i, \beta(k,i))$ *and* $n \geqslant \gamma(k,i)$ *imply* $|f_n(x) - f(x)| < 2^{-k}$,

(b) $\bigcup_{i=1}^{\infty} J(e_i, \beta(k,i)) = [0,1)$ *for each* k.

The above definition is the effectivization of Definition 2.4.

Notice that a dyadic step function which takes only $\mathbb{E}$-computable values is uniformly Fine computable.

Definition 5.2 (Computable sequence of dyadic step functions, [11]) *A sequence of functions* $\{\varphi_n\}$ *is called a* computable sequence of dyadic step functions *if there exist a monotonically increasing recursive function* $\delta(n)$ *and an* $\mathbb{E}$*-computable sequence of reals* $\{c_{n,j}\}$ $(0 \leqslant j < 2^{\delta(n)},\ n = 1, 2, \ldots)$ *such that*

$$\varphi_n(x) = \sum_{j=0}^{2^{\delta(n)}-1} c_{n,j} \chi_{I(\delta(n),j)}(x). \tag{4}$$

A computable sequence of dyadic step functions is a uniformly Fine computable sequence of functions since $\varphi_n(x) = \varphi_n(y)$ if $x, y \in I(\delta(n), j)$ for some j. Typical examples of computable sequences of dyadic step functions are the system of Walsh functions, that of Haar functions and that of Rademacher functions.

Theorem 6 *Let* f *be a Fine computable function. Define a computable sequence of dyadic step functions* $\{\varphi_n\}$ *from* f *as (2) in Section 2, that is,*

$$\varphi_n(x) = \sum_{j=0}^{2^n-1} f(j2^{-n}) \chi_{I(n,j)}(x).$$

Then $\{\varphi_n\}$ *Fine converges effectively to* f.

Proof. Let f be a Fine computable function with respect to $\alpha(k,i)$.

If $n \geqslant \alpha(k+1,i)$, then $J(e_i, \alpha(k+1,i)) = \bigcup_{j2^{-n} \in J(e_i, \alpha(k+1,i))} I(n,j)$. Assume further that $x \in J(e_i, \alpha(k+1,i))$. Then, $x \in I(n,j)$ for some j which satisfies $j2^{-n} \in J(e_i, \alpha(k+1,i))$ and $\varphi_n(x) = f(j2^{-n})$. So we obtain

$$\begin{aligned} |\varphi_n(x) - f(x)| &= |f(j2^{-n}) - f(x)| \leqslant |f(j2^{-n}) - f(e_i)| + |f(e_i) - f(x)| \\ &< 2^{-(k+1)} + 2^{-(k+1)} = 2^{-k}. \end{aligned}$$

Therefore, $\{\varphi_n\}$ Fine converges effectively to f with respect to $\gamma(k,i) = \beta(k,i) = \alpha(k+1,i)$. $\square$

Remark 5.1 If f is uniformly Fine computable or locally uniformly Fine computable, then the convergence can be replaced by the effectively uniform convergence or the effectively locally uniform Fine convergence respectively ([10, 11]).

Similarly to the proof of Proposition 4.1, we can prove the following proposition.

Proposition 5.1 *A sequence of functions $\{f_n\}$ Fine converges effectively to f if and only if there exist a recursive sequence of dyadic rationals $\{s_{k,i}\}$ and recursive functions $\beta(k,i)$ and $\gamma(k,i)$ which satisfy the following:*

(a) *$x \in J(s_{k,i}, \beta(k,i))$ and $n \geqslant \gamma(k,i)$ imply $|f_n(x) - f(x)| < 2^{-k}$.*

(b) *$\bigcup_{i=1}^{\infty} J(s_{k,i}, \beta(k,i)) = [0,1)$ for each k.*

(c) *$\{J(s_{k,i}, \beta(k,i))\}_i$ is a set of pairwise disjoint dyadic neighborhoods for each k.*

We can also define the notion of effective Fine convergence with respect to any effective separating set $\{r_i\}$, and prove that the notion of effective Fine convergence does not depend on the choice of an effective separating set.

Now, we prove the closedness of the space of Fine computable functions under effective Fine convergence.

Theorem 7 *If an effectively Fine continuous sequence of functions $\{f_n\}$ Fine converges effectively to f, then f is effectively Fine continuous.*

Proof. Let $\{f_n\}$ be effectively Fine continuous with respect to $\alpha(n,k,p)$, that is, $x \in J(e_p, \alpha(n,k,p))$ implies $|f_n(x) - f_n(e_p)| < 2^{-k}$ and $\bigcup_{p=1}^{\infty} J(e_p, \alpha(n,k,p)) = [0,1)$ for each n, k. From the effective Fine convergence, we obtain $\{s_{k,i}\}$, $\beta(k,i)$ and $\gamma(k,i)$ satisfying the conditions (a), (b) and (c) in Proposition 5.1. In particular, the dyadic neighborhoods $\{J(s_{k,i}, \beta(k,i))\}$ are mutually disjoint with respect to i.

From the requirement (ii-b) of Definition 4.1 for $\alpha(n,k,p)$, we have

$$J(s_{k+2,i}, \beta(k+2,i)) \subseteq [0,1) = \bigcup_{p=1}^{\infty} J(e_p, \alpha(\gamma(k+2,i), k+2, p)).$$

If we set $I = J(s_{k+2,i}, \beta(k+2,i))$ and $\{J_{k,i,p}\}_p = \{J(e_p, \alpha(\gamma(k+2,i), k+2, p)) \cap I\}_p$, and apply Lemma 4.2, we obtain an effective dyadic covering of $J(s_{k+2,i}, \beta(k+2,i))$, say $\{I_{k,i,q}\} = \{I(\xi(k,i,q), \eta(k,i,q))\}$, which satisfies (i)$\sim$(iii) of Lemma 4.2 for each pair k, i . Let us remark that $I_{k,i,q}$ is a subinterval of $J_{k,i,p}$ for some p, and that $\xi(k,i,q)$ and $\eta(k,i,q)$ are recursive functions.

For each k, i and q, we can find effectively some $p = p(k,i,q)$ such that $e_p \in I_{k,i,q}$.

Define $r_{k,i,q} = e_p$ and $\delta(k,i,q) = \xi(k,i,q)$, and assume $x \in J(r_{k,i,q}, \delta(k,i,q)) = I_{k,i,q}$. Since $J(r_{k,i,q}, \delta(k,i,q)) \subseteq J(s_{k+2,i}, \beta(k+2,i))$, $|f(x) - f_{\gamma(k+2,i)}(x)| < 2^{-(k+2)}$ and $|f(r_{k,i,q}) - f_{\gamma(k+2,i)}(r_{k,i,q})| < 2^{-(k+2)}$ hold. So

$$\begin{aligned}
& |f(x) - f(r_{k,i,q})| \\
\leqslant\ & |f(x) - f_{\gamma(k+2,i)}(x)| + |f_{\gamma(k+2,i)}(x) - f_{\gamma(k+2,i)}(r_{k,i,q})| + |f_{\gamma(k+2,i)}(r_{k,i,q}) - f(r_{k,i,q})| \\
<\ & |f_{\gamma(k+2,i)}(x) - f_{\gamma(k+2,i)}(r_{k,i,q})| + 2^{-(k+1)}.
\end{aligned}$$

On the other hand, $I_{k,i,q} = J(r_{k,i,q}, \delta(k,i,q)) \subseteq J(e_p, \alpha(\gamma(k+2,i), k+2, p)) = J_{k,i,p}$ and $r_{k,i,q} = e_p$ imply that

$$|f_{\gamma(k+2,i)}(x) - f_{\gamma(k+2,i)}(r_{k,i,q})| = |f_{\gamma(k+2,i)}(x) - f_{\gamma(k+2,i)}(e_p)| < 2^{-(k+2)}.$$

Therefore, $x \in J(r_{k,i,q}, \delta(k,i,q))$ implies $|f(x) - f(r_{k,i,q})| < 2^{-k}$.

Furthermore, $\cup_i \cup_q J(r_{k,i,q}, \delta(k,i,q)) = \cup_i J(s_{k+2,i}, \beta(k+2,i)) = [0,1)$ due to the assumption for $\{s_{k,i}\}$, β and γ.

We can perform the above procedure effectively in i. So, taking some recursive pairing function, $\langle i,q \rangle = i + \frac{1}{2}(i+q)(i+q+1)$ for example, define $r_{k,\ell} = r_{k,i,q}$ and $\delta(k,\ell) = \delta(k,i,q)$, for $\ell = \langle i,q \rangle$. Then, the necessary condition of Proposition 4.1 (with respect to k and ℓ) holds for this $r_{k,\ell}$ and $\delta(k,\ell)$ for each i. $\square$

Theorem 8 *If a Fine computable sequence of functions $\{f_n\}$ Fine converges effectively to f, then f is Fine computable.*

Proof. Effective Fine continuity is guaranteed by Theorem 7.

Let us assume that $\{f_n\}$ Fine converges effectively to f with respect to $\beta(k,i)$ and $\gamma(k,i)$. To prove the sequential computability, let $\{x_m\}$ be Fine computable. For each k, m, we can find effectively an $i = i(k,m)$ so that $x_m \in J(e_i, \beta(k,i))$. If $n \geqslant \gamma(k,i)$, then $|f_n(x_m) - f(x_m)| < 2^{-k}$. So the $\mathbb{E}$-computable sequence $\{f_n(x_m)\}_n$ converges effectively to $\{f(x_m)\}$, and hence $\{f(x_m)\}_m$ is an $\mathbb{E}$-computable sequence. $\square$

Combining Theorem 8 with Theorem 6, we obtain the following theorem.

Theorem 9 (Necessary and sufficient condition for Fine computable function) *A function f is Fine computable if and only if there exists a computable sequence of dyadic step functions, which Fine converges effectively to f.*

We can extend Theorem 8 to the case where a computable double sequence $\{f_{m,n}\}$ Fine converges effectively to a sequence $\{f_m\}$, by suitably extending the notions of the Fine computable sequence, the effective Fine convergence and the computable sequence of dyadic step functions. This can be done by adding a new argument m to the relevant recursive functions.

Theorem 10 *If a Fine computable double sequence of functions $\{f_{m,n}\}$ Fine converges effectively to a sequence $\{f_m\}$, then $\{f_m\}$ is Fine computable.*

Theorem 11 *A sequence of functions $\{f_m\}$ is Fine computable if and only if there exists a computable double sequence of dyadic step functions $\{\varphi_{m,n}\}$, which Fine converges effectively to $\{f_m\}$.*

Example 5.1 Let us consider $\chi_{\frac{1}{3}}$ in Proposition 4.3. Define x_n to be $\frac{1}{3}(1-4^{-n})$. Then $\{x_n\}$ is a Fine computable sequence of reals and Fine converges to $\frac{1}{3}$. Hence, χ_{x_n} converges pointwise to $\chi_{\frac{1}{3}}$. Moreover, $\{\chi_{x_n}\}$ is a computable sequence of dyadic step functions (Definition 5.2). However, the convergence is neither Fine nor continuous due to Propositions 2.4 and 2.6.

6 Recursive functional equations and Fine computable functions

In this section, we provide several examples concerning Fine computability of functions. Some of them are represented as linear combinations of $\chi_c(x)$'s and $\tilde{\chi}_n(x)$'s, which have been introduced in Section 4 (Equation (3)).

Example 6.1 Let us define $f_n = \displaystyle\sum_{i=1}^{n} 2^{-i}\chi_{e_i}$ and $f = \displaystyle\sum_{i=1}^{\infty} 2^{-i}\chi_{e_i}$.
Then, for $n < m$, $|f_n(x) - f_m(x)| \leqslant \sum_{i=n+1}^{m} 2^{-i} < 2^{-n}$ holds and $\{f_n\}$ converges effectively uniformly to f. So, f is uniformly Fine computable by Theorem 1. On the other hand, f is $\mathbb{E}$-discontinuous at every dyadic rational, since $f(x) - f(e_i) \geqslant 2^{-i}$ for any $x < e_i$.

There is an example of a computable sequence of dyadic step functions, which converges classically but the convergence is not effectively Fine.

Example 6.2 Let a be a one-to-one recursive function from $\mathbb{N}^+$ to $\mathbb{N}^+$, whose range $a(\mathbb{N}^+)$ is not recursive and let us define $f_n(x) = \displaystyle\sum_{k=1}^{n} \tilde{\chi}_{a(k)}(x)$, $f(x) = \displaystyle\sum_{k=1}^{\infty} \tilde{\chi}_{a(k)}(x)$. Then, $\{f_n\}$ is a computable sequence of dyadic step functions. Classically, $\{f_n\}$ converges to f and f is Fine continuous. However, f does not satisfy the sequential computability, since $f(1 - 2^{-m}) = 1$ if $m = a(k)$ for some $k \in \mathbb{N}$ and $= 0$ otherwise. So, the convergence is not effectively Fine. On the other hand, the limit function f is effectively locally uniformly Fine continuous.

The existence of an example which is Fine computable but not locally uniformly Fine computable has been proved by Brattka.

Example 6.3 (Brattka [3]) The example of Brattka is the following:

$$v(x) = \begin{cases} \displaystyle\sum_{i=0}^{\infty} (\ell_i \bmod 2) 2^{-n_i - \sum_{j=0}^{i-1}(n_j + \ell_j)} \\ \qquad\qquad \text{if } \mu(x) = 0^{n_0} 1^{\ell_0} 0^{n_1} 1^{\ell_1} 0^{n_2} \cdots \\ \displaystyle\sum_{i=0}^{m} (\ell_i \bmod 2) 2^{-n_i - \sum_{j=0}^{i-1}(n_j + \ell_j)} \\ \qquad\qquad \text{if } \mu(x) = 0^{n_0} 1^{\ell_0} 0^{n_1} 1^{\ell_1} 0^{n_2} \cdots 1^{\ell_m} 0^{\omega} \end{cases}, \tag{5}$$

where, $n_0 \geqslant 0$, $n_i > 0$ for $i > 0$ and $\ell_i > 0$ for all $i \geqslant 0$.

For investigation of this example and its generalizations, we introduce the following fundamental dyadic intervals and mappings.

$$
\begin{aligned}
A_\ell &= [1-2^{-(\ell-1)}, 1-2^{-\ell}) \\
B_\ell &= [1-2^{-\ell}, 1) = \textstyle\bigcup_{j=\ell+1}^{\infty} A_j \\
S_\ell(t) &= 1-2^{-(\ell-1)}+2^{-\ell}t \,:\, [0,1) \to A_\ell \\
R_\ell(t) &= 1-2^{-\ell}+2^{-\ell}t \,:\, [0,1) \to B_\ell.
\end{aligned}
$$

Obviously, $\{A_\ell\}_{\ell=1}^{\infty}$ is an infinite partition of $[0,1)$ and $\{A_1,\ldots,A_j,B_j\}$ is a finite partition of $[0,1)$ for each j. Furthermore, S_ℓ is a bijection from $[0,1)$ to A_ℓ and $S_\ell^{-1}x = 2^\ell(x-(1-2^{-(\ell-1)}))$. R_ℓ is a bijection from $[0,1)$ to B_ℓ and $R_\ell^{-1}x = 2^\ell(x-(1-2^{-\ell}))$.

We note that $x \in A_\ell$ is equivalent to that $\mu(x)$ is represented as $1^{\ell-1}0**\cdots$.

First, we treat the approximating sequence of dyadic step functions $\{v_n\}$, which is obtained from v by Equation (2) in Theorem 6. Since v is known to be Fine computable, $\{v_n\}$ Fine converges effectively to v by virtue of Theorem 6.

The proof by Brattka [3] that the v defined by Equation (5) is not locally uniformly Fine continuous assures that it is not locally uniformly Fine continuous. It is easy to prove that the limit of a sequence of locally uniform Fine continuous functions under locally uniformly Fine convergence is also locally uniformly Fine continuous. If the convergence of $\{v_n\}$ to v were effectively locally uniformly Fine, then v would be locally uniformly Fine continuous by virtue of Theorem 2. So the convergence is not locally uniformly Fine,

It is easy to see that the sequence $\{v_n\}$ satisfies the following recurrence equation.

$$
\begin{aligned}
v_1(x) &= \begin{cases} 0 & \text{if } x \in A_1 = [0, \frac{1}{2}) \\ 1 & \text{if } x \in B_1 = [\frac{1}{2}, 1) \end{cases}, \\
v_n(x) &= \begin{cases} \frac{1+(-1)^i}{2} + 2^{-i} v_{n-i}(S_i^{-1}x) & \text{if } x \in A_i \, (1 \leqslant i \leqslant n-1) \\ \frac{1+(-1)^n}{2} & \text{if } x \in A_n \\ \frac{1+(-1)^{n+1}}{2} & \text{if } x \in B_n \end{cases}.
\end{aligned}
\tag{6}
$$

It also holds that $v_n(k2^{-n}) = v(k2^{-n})$ for each n and k.

We illustrate the first four of $\{v_n\}$ in Figure 1. Let us examine the graph of v_4. The restriction of v_4 to $A_1 = [0, \frac{1}{2})$ is the contraction of the graph of v_3 with scale $\frac{1}{2}$. The restriction of v_4 to $A_2 = [\frac{1}{2}, \frac{3}{4})$ is the vertical translation of the contraction of the graph of v_2 with scale $\frac{1}{4} = 2^{-2}$. The restriction of v_4 to $A_3 = [\frac{3}{4}, \frac{7}{8})$ is the contraction of the graph of v_1 with scale 2^{-3}. $v_4(x) = 1$ if $x \in A_4 = [\frac{7}{8}, \frac{15}{16})$ and $v_4(x) = 0$ if $x \in B_4 = [\frac{15}{16}, 1)$.

By definition, it holds that $v_\ell(k2^{-n}) = v_n(k2^{-n})$ for $\ell \geqslant n$, and hence they are equal to $v(k2^{-n})$ for any natural number k which is less than 2^n. This shows that the value $v(x)$ is determined by $v_n(x)$ if x is a dyadic rational of level n.

In Figure 2, we draw line from $(k2^{-6}, v(k2^{-6}))$ to $((k+1)2^{-6}, v(k2^{-6}))$ for $0 \leqslant k \leqslant 2^6 - 1$.

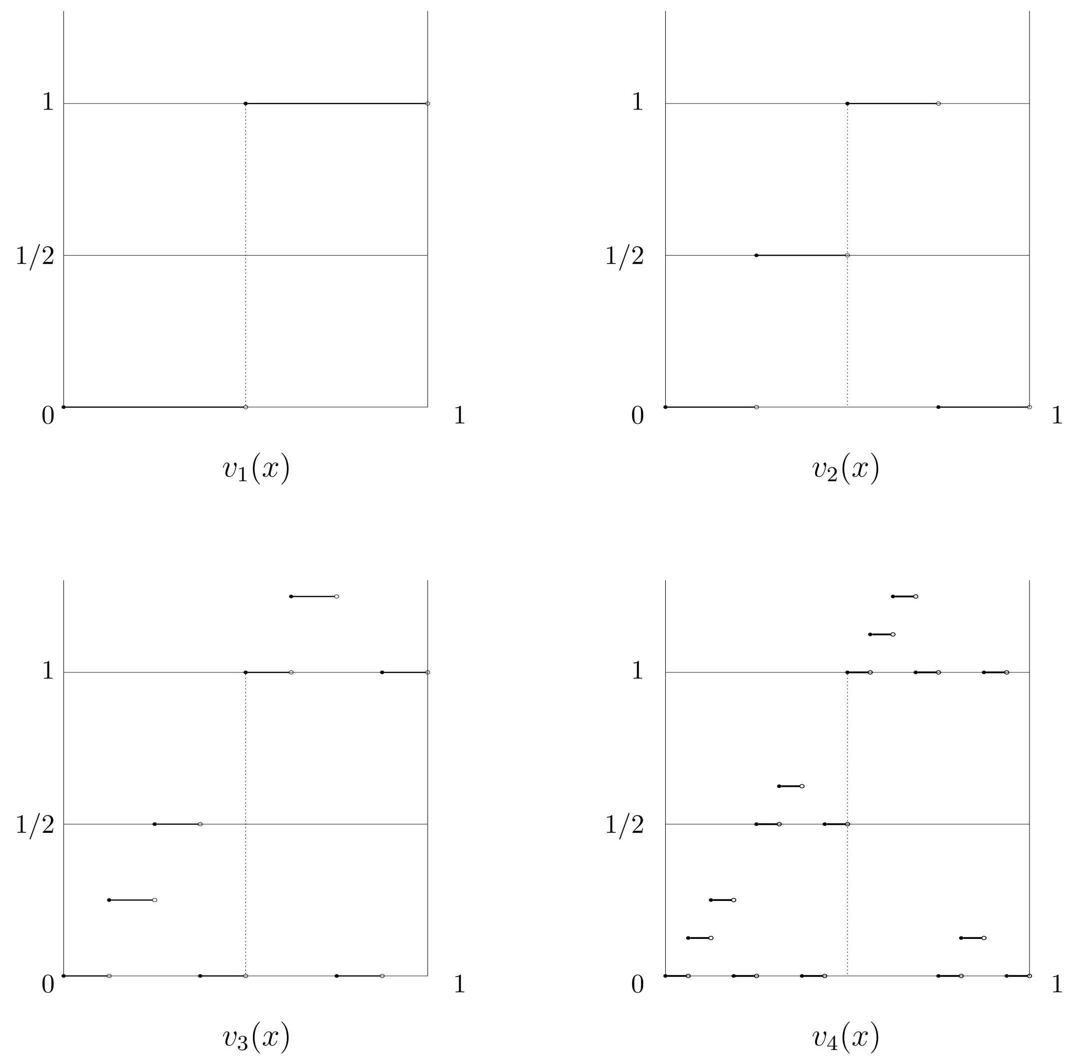

Figure 1: $v_n(x)$ for $n = 1, 2, 3, 4$

To prove some properties of the function v, we derive a simple recurrence equation.

It is easily proved that $v(x)$ defined by Equation (5) satisfies $v(0) = 0$ and the following functional equation

$$v(x) = \tfrac{1+(-1)^\ell}{2} + 2^{-\ell} v(S_\ell^{-1} x) \quad \text{if} \quad x \in A_\ell \quad (\ell = 1, 2, \ldots). \tag{7}$$

Properties of fractals deduced from Equation (7) will be discussed in a forthcoming paper.

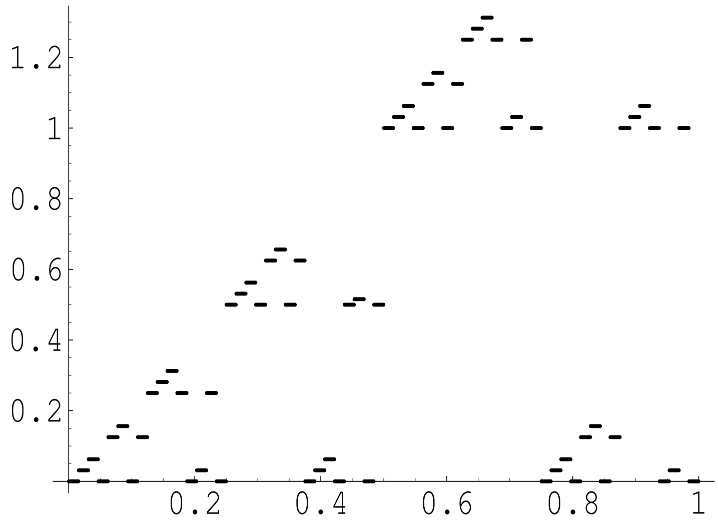

Figure 2: $v(x)$ for $x = k2^{-6}, 0 \leqslant k \leqslant 2^6 - 1$

If we replace the first term in the right hand side of Equation (7) with a recursive function, we can obtain other examples of Fine computable functions.

Theorem 12 *Assume that* $\{h(\ell)\}_\ell$ *is an* $\mathbb{E}$*-computable sequence from* $[0,1]$ *and that* $h(1) = 0$.

(i) *The equation*

$$f(x) = h(\ell) + 2^{-\ell} f(S_\ell^{-1} x) \quad \textit{if} \quad x \in A_\ell \ (\ell = 1, 2, \ldots) \tag{8}$$

has a unique Fine computable solution in the space of bounded functions on $[0,1)$.

(ii) *If* $\liminf_{\ell\to\infty} h(\ell) \neq \limsup_{\ell\to\infty} h(\ell)$*, then the bounded solution of Equation* (8) *is not locally uniformly Fine continuous.*

(iii) *If* $\liminf_{\ell\to\infty} h(\ell) = \limsup_{\ell\to\infty} h(\ell) = a$ *and the convergence is effective, then the bounded solution of Equation* (8) *is uniformly Fine computable.*

If h is given by $h(\ell) = 0$ for an odd ℓ and $= 1$ for an even ℓ, then we obtain the example of Brattka.

We can also get Fine computable functions by the following equation, which is similar to Equation (8) but slightly different.

Theorem 13 *Let h satisfy the assumption of Theorem 12.*

(i) *The equation*

$$f(x) = h(\ell) + \frac{1}{2} f(S_\ell^{-1} x) \quad \textit{if} \quad x \in A_\ell \ (\ell = 1, 2, \ldots) \tag{9}$$

has a unique Fine computable solution in the space of bounded functions on $[0,1)$.

(ii) *If h is not constant, then the bounded solution of Equation* (9) *is not locally uniformly Fine continuous.*

For the proof of Theorems 12 and 13, we introduce the following notations.

For each $x \in [0,1)$, we can define an infinite sequence $\{\ell_i(x)\}_{i=1}^{\infty}$ in $\mathbb{N}^+$ by $x \in A_{\ell_1(x)}$ and $S^{-1}_{\ell_i(x)} \cdots S^{-1}_{\ell_1(x)}(x) \in A_{\ell_{i+1}(x)}$.

We also define $L_0(x) = 0$ and $L_j(x) = \ell_1(x) + \ell_2(x) + \cdots + \ell_j(x)$ for $j > 0$.

Finally, for a dyadic rational r, we define its level by

$$lev(r) = \min\{n \in \mathbb{N} \mid \exists j. r = j2^{-n}\}. \tag{10}$$

We have defined the level of a fundamental dyadic interval I in Section 2. We also denote this as $lev(I)$.

If $\{r_n\}$ is a recursive sequence of dyadic rationals, then $\{lev(r_n)\}_n$ is recursive.

We list up some properties concerning $\{S_\ell\}$ and $\{\ell_i(x)\}$.

Fact 1: *$\ell_j(x) \geqslant 1$ and $L_j(x) \geqslant j$.*

Fact 2: *For any positive integers $\ell_1, \ell_2, \ldots, \ell_k$, we define $L_k = \ell_1 + \ldots + \ell_k$. Then $S_{\ell_1} S_{\ell_2} \cdots S_{\ell_k}[0,1) = [1 - 2^{-L_1} - 2^{-L_2} - \cdots - 2^{-(L_k - 1)}, 1 - 2^{-L_1} - 2^{-L_2} - \cdots - 2^{-L_k})$ is a fundamental dyadic interval of level L_k.*

Fact 3: *For any positive integers $\ell_1, \ell_2, \ldots, \ell_k$, if $x \in S_{\ell_1} S_{\ell_2} \cdots S_{\ell_k}[0,1)$, then $\ell_i(x) = \ell_i$, $i = 1, 2, \ldots, k$.*

Fact 4: *If a dyadic rational r is of level n and lies in A_ℓ, then the level of $S_\ell^{-1} r$ is equal to or less than $n - \ell$. Hence, if $L_j(r) > lev(r)$ then $S^{-1}_{\ell_j(r)} \cdots S^{-1}_{\ell_2(r)} S^{-1}_{\ell_1(r)} r = 0$.*

Fact 5: *If $\{x_n\}$ is a Fine computable sequence of reals, then the double sequence of integers $\{\ell_i(x_n)\}$ is computable. (In fact, it is recursive for any fixed $\{x_n\}$.)*

Fact 6: *Let f be a solution of Equation (8). Put $t = S^{-1}_{\ell_j(x)} \cdots S^{-1}_{\ell_2(x)} S^{-1}_{\ell_1(x)} x$ for $x \in [0,1)$. Then we obtain*

$$f(x) = h(\ell_1(x)) + 2^{-L_1(x)} h(\ell_2(x)) + \cdots + 2^{-L_{j-1}(x)} h(\ell_j(x)) + 2^{-L_j(x)} f(t). \tag{11}$$

Moreover, if r is dyadic rational and $L_j(r) > lev(r)$, then it holds by Fact 4 that

$$f(r) = h(\ell_1(r)) + 2^{-L_1(r)} h(\ell_2(r)) + \cdots + 2^{-L_{j-1}(r)} h(\ell_j(r)). \tag{12}$$

Fact 7: *Let f satisfy Equation (9). Put $t = S^{-1}_{\ell_j(x)} \cdots S^{-1}_{\ell_2(x)} S^{-1}_{\ell_1(x)} x$ for $x \in [0,1)$. Then, we obtain*

$$f(x) = h(\ell_1(x)) + 2^{-1} h(\ell_2(x)) + \cdots + 2^{-(j-1)} h(\ell_j(x)) + 2^{-j} f(t). \tag{13}$$

and

$$f(r) = h(\ell_1(r)) + 2^{-1} h(\ell_2(r)) + \cdots + 2^{-(j-1)} h(\ell_j(r)). \tag{14}$$

for dyadic rational r with $L_j(r) > lev(r)$.

Subsequently, $||f||$ will denote the supremum of a function f (when it exists).

Proof of Theorem 13 (i). Let f be a bounded solution of Equation (9) (or Equation (8)). Since, $0 \in A_1$ and $S_1^{-1}(0) = 0$, we obtain $f(0) = \frac{1}{2}f(0)$ and hence $f(0) = 0$.

From Equation (13) (or Equation (11)) and the assumption that $h(\ell) \in [0, 1]$, we obtain

$$|f(x)| \leqslant 1 + 2^{-1} + \cdots + 2^{-(j-1)} + 2^{-j}||f||.$$

Letting j to infinity, we obtain

$$|f(x)| \leqslant \sum_{j=0}^{\infty} 2^{-j} = 2. \tag{15}$$

Existence: Since $||h|| \leqslant 1$, $\sum_{j=1}^{\infty} 2^{-(j-1)} h(\ell_j(x))$ converges absolutely and uniformly in x. If we denote this limit function by f, then it is easy to prove that f satisfies Equation (9).

Uniqueness: Suppose that f and g are bounded solutions of Equation (9) or of Equation (8). Then, from Equation (13) (or from Equation (11)),

$$|f(x) - g(x)| \leqslant 2^{-j}(||f|| + ||g||)$$

holds for all j. Since the right-hand side tends to zero as k tends to infinity, we obtain $f = g$.

Remark 6.1 From *Existence* and *Uniqueness*, the unique bounded solution of Equation (9) is given by

$$f(x) = \sum_{j=1}^{\infty} 2^{-(j-1)} h(\ell_j(x)). \tag{16}$$

The convergence in the right-hand side is effectively uniform.

Effective Fine Continuity: We fix an arbitrary k. From the definition of $\{S_\ell\}$ and Fact 2, the set of intervals $\{S_{\ell_1} S_{\ell_2} \cdots S_{\ell_{k+3}}[0, 1)\}_{\ell_1, \ell_2, \ldots, \ell_{k+3}}$ is a partition of $[0, 1)$ consisting of fundamental dyadic intervals. Therefore, each e_i is contained in some $I = S_{\ell_1} S_{\ell_2} \cdots S_{\ell_{k+3}}[0, 1)$. Note that we can find such I effectively in k and i. If we define $\gamma(k, i)$ to be the level of I, then $J(e_i, \gamma(k, i)) = I$ and γ is recursive.

Assume that $x \in J(e_i, \gamma(k, i))$. Then, $\ell_j(x) = \ell_j(e_i) = \ell_j$ for $1 \leqslant j \leqslant k + 3$ by Fact 3, and we obtain by Equation (13)

$$\begin{aligned} f(x) &= h(\ell_1) + 2^{-1}h(\ell_2) + \cdots + 2^{-(k+2)}h(\ell_{k+3}) + 2^{-(k+3)}f(t), \\ f(e_i) &= h(\ell_1) + 2^{-1}h(\ell_2) + \cdots + 2^{-(k+2)}h(\ell_{k+3}) + 2^{-(k+3)}f(s), \end{aligned}$$

where, $t = S_{\ell_{k+3}}^{-1} \cdots S_{\ell_2}^{-1} S_{\ell_1}^{-1}(x)$ and $s = S_{\ell_{k+3}}^{-1} \cdots S_{\ell_2}^{-1} S_{\ell_1}^{-1}(e_i)$. Therefore,

$$|f(x) - f(e_i)| \leqslant 2\, 2^{-(k+3)}||f|| \leqslant 4\, 2^{-(k+3)} < 2^{-k}.$$

This proves the effective Fine continuity of f.

Sequential Computability: Let $\{x_n\}$ be a Fine computable sequence in $[0,1)$. Define

$$y_{n,m} = h(\ell_1(x_n)) + 2^{-1}h(\ell_2(x_n)) + \cdots + 2^{-(m-1)}h(\ell_m(x_n)).$$

Then, the double sequence $\{y_{n,m}\}$ is $\mathbb{E}$-computable by Fact 5 and $\mathbb{E}$-converges effectively to $\{f(x_n)\}$ by Remark 6.1. Therefore, $\{f(x_n)\}$ is an $\mathbb{E}$-computable sequence of reals.

Theorem 12 (i) can be proved similarly.

Proof of Theorem 12 (ii). Let us assume that $\liminf_{m\to\infty} h(m) \neq \limsup_{m\to\infty} h(m)$ and that f is locally uniformly Fine continuous with respect to functions $\alpha(k,i)$ and $\beta(i)$, that is, for all k, $|f(x)-f(y)| < 2^{-k}$ if $x,y \in J(e_i,\beta(i))$ and $y \in J(x,\alpha(k,i))$, and $\bigcup_{i=1}^{\infty} J(e_i,\beta(i)) = [0,1)$.

Put $\delta = \limsup_{\ell\to\infty} h(\ell) - \liminf_{\ell\to\infty} h(\ell)$ and consider any fixed i and the corresponding $J(e_i,\beta(i))$.

Now, take k so large that $2^{-k} < \delta\, 2^{-(\beta(i)+1)}$. From the definition of δ, there exist $m_1 > \alpha(k,i)$ and $m_2 > \alpha(k,i)$ such that $h(m_2) - h(m_1) > \frac{\delta}{2}$.

Let z be the left end point of $J(e_i,\beta(i))$. Then it holds that $lev(z) \leqslant \beta(i)$. Define further $x = z + 2^{-(\beta(i)+1)}(1-2^{-(m_1-1)})$ and $y = z + 2^{-(\beta(i)+1)}(1-2^{-(m_2-1)})$. Then z, x and y are dyadic rationals and z can be written as $j2^{-\beta(i)}$ for some integer j. From the last property above, there exists an integer n such that $L_n(z) = \beta(i)+1$. In this case, $\ell_j(z) = \ell_j(x) = \ell_j(y)$ if $j \leqslant n$, $\ell_j(z) = 1$ if $j > n$, $\ell_{n+1}(x) = m_1$, $\ell_{n+1}(y) = m_2$ and $\ell_j(x) = \ell_j(y) = 1$ if $j > n+1$.

By Equation (12) and Fact 1, we obtain

$$f(y) - f(x) = 2^{-L_n(z)}(h(m_2) - h(m_1)) > 2^{-(\beta(i)+1)}\delta. \tag{17}$$

From Equation (17) and the choice of k, $f(y) - f(x) > 2^{-k}$ holds.

On the other hand, $x,y \in J(e_i,\beta(i))$ and $y \in J(x,\alpha(k,i))$ hold. This implies, from the assumption, $|f(x)-f(y)| < 2^{-k}$, contradicting Equation (17). f is thus not locally uniformly Fine continuous.

Proof of Theorem 13 (ii). Assume that $h(m_1) < h(m_2)$. For any i, there exists an integer n such that $L_n(e_i) = lev(e_i)+1$. Put, for any m,

$$x = e_i + 2^{-(lev(e_i)+1)}(1-2^{-(m-1)}) + 2^{-(m+lev(e_i)+1)}(1-2^{-(m_1-1)})$$

and

$$y = e_i + 2^{-(lev(e_i)+1)}(1-2^{-(m-1)}) + 2^{-(m+lev(e_i)+1)}(1-2^{-(m_2-1)}).$$

Then, x and y are dyadic rationals and satisfy

$$\ell_{n+1}(x) = \ell_{n+1}(y) = m,\ \ell_{n+2}(x) = m_1,\ \ell_{n+2}(y) = m_2,$$
$$\ell_{n+3}(x) = \ell_{n+3}(y) = 1,\ \ell_{n+4}(x) = \ell_{n+4}(y) = 1,\ \cdots.$$

So we obtain

$$f(x) = f(e_i) + 2^{-(lev(e_i)+1)}h(m) + 2^{-(lev(e_i)+2)}h(m_1)$$

and

$f(y) = f(e_i) + 2^{-(lev(e_i)+1)}h(m) + 2^{-(lev(e_i)+2)}h(m_2)$
by Equation (14). Hence,

$$f(y) - f(x) = 2^{-(lev(e_i)+2)}(h(m_2) - h(m_1)) > 0.$$

On the other hand, it holds that $x, y \in J(z, lev(e_i) + m)$ and $y \in J(x, lev(e_i) + m)$. If f were locally uniformly Fine continuous, then $f(y) - f(x)$ would be arbitrarily small for sufficiently large m, contradicting the last inequality. ∎

Proof of Theorem 12 (iii). For any $\ell_1, \ell_2, \ldots, \ell_j$ and $x \in [0, 1)$, define $t = S_{\ell_j}^{-1} \cdots S_{\ell_1}^{-1} x$. Then it holds that $\ell_i(x) = \ell_i$ for $1 \leqslant i \leqslant j$ and we obtain

$$f(x) = h(\ell_1) + 2^{-L_1}h(\ell_2) + \cdots + 2^{-L_{j-1}}h(\ell_j) + 2^{-L_j}f(t). \tag{18}$$

Let $\alpha(k)$ be a modulus of convergence of h, that is, α is a recursive function which satisfies that $\ell \geqslant \alpha(k)$ implies $|h(\ell) - a| < 2^{-k}$. We can assume that $\alpha(k) \geqslant k$.

Let us consider the finite partition of $[0, 1)$ consisting of all sets of the form $U_1U_2 \cdots U_{k+3}[0, 1)$, where U_i is chosen from $\{S_1, S_2, \ldots, S_{\alpha(k+3)}, R_{\alpha(k+3)}\}$. By Fact 2, each $U_1U_2 \cdots U_{k+3}[0, 1)$ is a fundamental dyadic interval. So we can define $\beta(k)$ to be the maximum of their levels.

Suppose $y \in J(x, \beta(k))$. Then x and y are contained in some $U_1U_2 \cdots U_{k+3}[0, 1)$.

If $R_{\alpha(k+3)}$ does not appear in $U_1, U_2, \ldots, U_{k+3}$, then it holds that $\ell_i(x) = \ell_i(y)$ for $1 \leqslant i \leqslant k + 3$ from Fact 3. So we obtain by Equation (18)

$$|f(x) - f(y)| \leqslant 2\,2^{-L_{k+3}}||f|| \leqslant 4\,2^{-(k+3)} < 2^{-k}.$$

Otherwise, there exists at least one $R_{\alpha(k+3)}$ in $U_1, U_2, \ldots, U_{k+3}$. Let U_j be the first appearance of $R_{\alpha(k+3)}$. (U_1 may be $R_{\alpha(k+3)}$.) If $j \geqslant 2$, then $\ell_i(x) = \ell_i(y)$ for $1 \leqslant i \leqslant j - 1$. Since $R_{\alpha(k+3)}[0, 1) = B_{\alpha(k+3)} = \bigcup_{i=\alpha(k+3)+1}^{\infty} A_i$, we obtain for some $t, s \in [0, 1)$

$$\begin{aligned} f(x) &= h(\ell_1(x)) + 2^{-L_1(x)}h(\ell_2(x)) + \cdots + 2^{-L_{j-2}(x)}h(\ell_{j-1}(x)) \\ &\qquad + 2^{-L_{j-1}(x)}h(\ell_j(x)) + 2^{-L_j(x)}f(t), \\ f(y) &= h(\ell_1(y)) + 2^{-L_1(y)}h(\ell_2(y)) + \cdots + 2^{-L_{j-2}(y)}h(\ell_{j-1}(y)) \\ &\qquad + 2^{-L_{j-1}(y)}h(\ell_j(y)) + 2^{-L_j(y)}f(s). \end{aligned}$$

It holds that $\ell_j(x), \ell_j(y) \geqslant \alpha(k + 3) \geqslant k + 3$. So we obtain

$$\begin{aligned} |f(x) - f(y)| &\leqslant 2^{-L_{j-1}(x)}|h(\ell_j(x)) - h(\ell_j(y))| + 2^{-L_j(x)}|f(t)| + 2^{-L_j(y)}|f(s)| \\ &\leqslant 2^{-(k+3)} + 4\,2^{-\alpha(k+3)} \leqslant 5\,2^{-(k+3)} < 2^{-k}. \end{aligned}$$

Therefore, $y \in J(x, \beta(k))$ implies $|f(x) - f(y)| < 2^{-k}$, and the effectively uniform Fine continuity holds. ∎

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