Integral of Fine Computable functions and Walsh Fourier series

Takakazu Mori [*] Mariko Yasugi [†] Yoshiki Tsujii [‡]

April 6, 2007

Abstract

We define the effective integrability of Fine-computable functions and effectivize some fundamental limit theorems in the theory of Lebesgue integral such as Bounded Convergence Theorem and Dominated Convergence Theorem. It is also proved that the Walsh-Fourier coefficients of an effectively integrable Fine-computable function form an $\mathbb{E}$-computable sequence of reals and converge effectively to zero. The latter fact is the effectivization of Walsh-Riemann-Lebesgue Theorem. The article is closed with the effective version of Dirichlet's test.

keyword: Fine-computable function, Fine convergence, Walsh Fourier series, effective integrability, Dirichlet's test

1 Introduction

In this article, we make an introductory step to reconstruct effectively the theory of Walsh-Fourier series ([3], [12]). Although Walsh-Fourier series and Haar wavelets have become important tools in digital processing nowadays, it seems that Haar, Rademacher, Walsh, Fine and others had already investigated these subjects in the middle of the twentieth century from mathematical interest. The theory of Walsh-Fourier series is treated similarly to that of Fourier series by replacing trigonometric functions with Walsh functions.

Let $S_n(f)$ be the nth partial sum of the Walsh-Fourier series of a function f. A major problem concerning $S_n(f)$ is to find a sufficient condition for the convergence of $\{S_n(f)\}$ to f. Many types of convergence, such as pointwise convergence, uniform convergence, almost everywhere convergence and L^p-convergence, are treated in [3] and [12].

From the standpoint of computable analysis, it is more appropriate if the pointwise convergence of $\{S_n(f)\}$ to f be replaced by some kind of effective convergence, which is stronger than pointwise convergence. We have adopted "effective Fine convergence" in [10] for Fine-computable functions.

Our present objective is to effectivize Dirichlet's test for Fine-computable functions with respect to effective Fine convergence. For this purpose, we need to reformulate integration

[*]Faculty of Science, Kyoto Sangyo University: morita@cc.kyoto-su.ac.jp

[†]Faculty of Science, Kyoto Sangyo University: yasugi@cc.kyoto-su.ac.jp; This work has been supported in part by Scientific Foundations of JSPS No. 6340028.

[‡]Faculty of Science, Kyoto Sangyo University: tsujiiy@cc.kyoto-su.ac.jp

theory in an effective way and prove effective versions of some fundamental theorems such as Bounded Convergence Theorem and Dominated Convergence Theorem of Lebesgue (Theorems 8, 10). These are treated in Section 3. A Fine-computable function is Fine continuous, and hence is Euclidean continuous at dyadic irrationals. This means that such a function is measurable, and so the classical theorems hold for it. Therefore, the effectivization of integration theory is reduced essentially to the replacement of "convergence" by "effective convergence".

In Section 4, we prove $\mathbb{E}$-computability of the indefinite integral and the second mean value theorem (Theorem 16) for an effectively integrable Fine-computable function (Theorem 20).

In Section 5, we prove the effectivizations of Walsh Riemann Lebesgue Theorem (Theorem 19) and Dirichlet's Test.

For the reader's convenience, we review some basics of Fine metric, Fine-computable functions and Fine convergence, and some fundamental theorems of integration.

We assume the knowledge of computability of the real number sequences and the real function sequences with respect to the Euclidean topology. See [11] for details.

2 Preliminaries

The *Fine-metric* on $[0,1)$ was introduced in [2]. It is defined by

$$d_F(x,y) = \sum_{k=1}^{\infty} |\sigma_k - \tau_k| 2^{-k}, \tag{1}$$

where, $\sigma_1\sigma_2\cdots$ and $\tau_1\tau_1\cdots$ are dyadic expansions of x and y respectively with infinitely many $0'$s.

A left-closed right-open interval with dyadic rational end points is called a *dyadic interval*. It is easy to see that a dyadic interval is open with respect to the Fine metric.

We use the following notations for special dyadic intervals.

$$I(n,k) = [k\,2^{-n}, (k+1)2^{-n}),\ 0 \leqslant k \leqslant 2^n - 1,$$
$$J(x,n) = \text{such } I(n,k) \text{ that includes } x.$$

We call $I(n,k)$ a *fundamental dyadic interval* (*of order* n) and $J(x,n)$ a *dyadic neighborhood of* x (*of order* n).

The topology generated by $\{J(x,n)\,|\,x \in [0,1), n = 1,2,3,\cdots\}$ is equivalent to that induced by the Fine metric. We call this topological space the *Fine space*. We put "Fine-" to the topological notions with respect to this topology. For topological notions with respect to the usual Euclidean metric, we put prefix "$\mathbb{E}$-".

We cite the following lemma from [10] concerning $I(n,k)$ and $J(x,n)$.

Lemma 2.1 ([10]) *The following three conditions are equivalent for any $x,y \in [0,1)$ and any positive integer n.*

(i) $y \in J(x,n)$.

(ii) $x \in J(y,n)$.

(iii) $J(x,n) = J(y,n)$.

A sequence of dyadic rationals $\{r_n\}$ in $[0,1)$ is called *recursive* if there exist recursive functions $\alpha(n)$ and $\beta(n)$ which satisfy $r_n = \alpha(n)2^{-\beta(n)}$. A double sequence $\{x_{m,n}\}$ in $[0,1)$ is said to *Fine-converge effectively* to a sequence $\{x_m\}$ from $[0,1)$ if there exists a recursive function $\alpha(m,k)$ such that, for all m,k, $x_{m,n} \in J(x_m,k)$ for all $n \geqslant \alpha(m,k)$.

A sequence $\{x_m\}$ in $[0,1)$ is said to be *Fine-computable* if there exists a recursive sequence of dyadic rationals $\{r_{m,n}\}$ which Fine-converges effectively to $\{x_m\}$.

Definition 2.1 (Uniformly Fine-computable sequence of functions, [6]) *A sequence of functions $\{f_n\}$ is said to be uniformly Fine-computable if* (i) *and* (ii) *below hold.*

(i) (Sequential Fine-computability) *The double sequence $\{f_n(x_m)\}$ is $\mathbb{E}$-computable for any Fine-computable sequence $\{x_m\}$.*

(ii) (Effectively uniform Fine-continuity) *There exists a recursive function $\alpha(n,k)$ such that, for all n,k and all $x,y \in [0,1)$, $y \in J(x,\alpha(n,k))$ implies $|f_n(x) - f_n(y)| < 2^{-k}$.*

The Fine-computability of a single function f is defined by that of the sequence $\{f,f,\ldots\}$.

Notice that the computability of the sequence $\{f_n(x_m)\}$ in (i) is $\mathbb{E}$-computability.

Throughout this article, we fix an effective enumeration of all dyadic rationals in $[0,1)$ and denote it with $\{e_i\}$.

Definition 2.2 (Effectively uniform convergence of functions, [6]). *A double sequence of functions $\{g_{m,n}\}$ is said to converge effectively uniformly to a sequence of functions $\{f_m\}$ if there exists a recursive function $\alpha(m,k)$ such that, for all m,n and k,*

$$n \geqslant \alpha(m,k) \quad \textit{implies} \quad |g_{m,n}(x) - f_m(x)| < 2^{-k} \quad \textit{for all} \quad x.$$

Theorem 1 ([6]) *If a uniformly Fine-computable sequence of functions $\{f_n\}$ Fine-converges effectively uniformly to a function f, then f is also uniformly Fine-computable.*

We can treat weakened notions of computability and convergence as follows.

Definition 2.3 (Locally uniformly Fine-computable sequence of functions, [7]) *A sequence of functions $\{f_n\}$ is said to be locally uniformly Fine-computable if the following* (i) *and* (ii) *hold.*

(i) *$\{f_n\}$ is sequentially Fine-computable.*

(ii) (Effectively locally uniform Fine-continuity) *There exist recursive functions $\alpha(n,i,k)$ and $\beta(n,i)$ which satisfy the following* (ii-a) *and* (ii-b).

(ii-a) *For all i, n and k, $|f_n(x)-f_n(y)| < 2^{-k}$ if $x,y \in J(e_i,\beta(n,i))$ and $y \in J(x,\alpha(n,i,k))$.*

(ii-b) *$\bigcup_{i=1}^{\infty} J(e_i,\beta(n,i)) = [0,1)$ for each n.*

Definition 2.4 (Effectively locally uniform Fine-convergence, [7]). *A double sequence of functions $\{g_{m,n}\}$ is said to Fine-converge effectively locally uniformly to a sequence of functions $\{f_m\}$ if there exist recursive functions $\alpha(m,i)$ and $\beta(m,i,k)$ such that*

(a) *$|g_{m,n}(x) - f_m(x)| < 2^{-k}$ for $x \in J(e_i,\alpha(m,i))$ and $n \geq \beta(m,i,k)$,*

(b) *$\cup_{i=1}^{\infty} J(e_i,\alpha(m,i)) = [0,1)$.*

Theorem 2 ([7]) *If a locally uniformly Fine-computable sequence of functions $\{f_n\}$ Fine-converges effectively locally uniformly to f, then f is locally uniformly Fine-computable.*

Definition 2.5 (Fine-computable sequence of functions) *A sequence of functions* $\{f_n\}$ *is said to be Fine-computable if it satisfies the following.*

(i) $\{f_n\}$ *is sequentially Fine-computable.*

(ii) (Effective Fine-Continuity) *There exists a recursive function* $\alpha(n,k,i)$ *such that*

(ii-a) $x \in J(e_i, \alpha(n,k,i))$ *implies* $|f_n(x) - f_n(e_i)| < 2^{-k}$,

(ii-b) $\bigcup_{i=1}^{\infty} J(e_i, \alpha(n,k,i)) = [0,1)$ *for each* n, k.

Definition 2.6 (Effective Fine-convergence of functions) *We say that a double sequence of functions* $\{g_{m,n}\}$ *Fine-converges effectively to a sequence of functions* $\{f_m\}$ *if there exist recursive functions* $\alpha(m,k,i)$ *and* $\beta(m,k,i)$, *which satisfy*

(a) $x \in J(e_i, \alpha(m,k,i))$ *and* $n \geqslant \beta(m,k,i)$ *imply* $|g_{m,n}(x) - f_m(x)| < 2^{-k}$,

(b) $\bigcup_{i=1}^{\infty} J(e_i, \alpha(m,k,i)) = [0,1)$ *for each* m *and* k.

Definition 2.7 (Computable sequence of dyadic step functions, [6]) *A sequence of functions* $\{\varphi_n\}$ *is called a computable sequence of dyadic step functions if there exist a recursive function* $\alpha(n)$ *and a* $\mathbb{E}$*-computable sequence of reals* $\{c_{n,j}\}$ $(0 \leqslant j < 2^{\alpha(n)}$, $n = 1, 2, \ldots)$ *such that*

$$\varphi_n(x) = \sum_{j=0}^{2^{\alpha(n)}-1} c_{n,j} \chi_{I(\alpha(n),j)}(x),$$

where χ_A *denotes the indicator (characteristic) function of* A.

Proposition 2.1 *Let* f *be a Fine-computable function. Define a computable sequence of dyadic step functions* $\{\varphi_n\}$ *by*

$$\varphi_n(x) = \sum_{j=0}^{2^n-1} f(j2^{-n}) \chi_{I(n,j)}(x). \tag{2}$$

Then $\{\varphi_n\}$ *Fine-converges effectively to* f.

Remark 2.1 If f is uniformly Fine-computable or locally uniformly Fine-computable, then the convergence can be replaced by the effectively uniform Fine-convergence or the effectively locally uniform Fine-convergence respectively([7, 6]).

Theorem 3 ([10]) *If a Fine-computable sequence of functions* $\{f_n\}$ *Fine-converges effectively to* f, *then* f *is Fine-computable.*

Now, we review the theory of Lebesgue integral for functions on $[0,1)$. In the following, we will say simply "measurable" or "integrable" instead of "Lebesgue measurable" or "Lebesgue integrable" respectively.

A function $\varphi(x)$ is called a simple function if it is represented as a finite linear combination of indicator functions of some measurable sets, that is, if $\varphi(x) = \sum_{i=0}^{n-1} a_i \chi_{E_i}(x)$, where a_i's are real numbers and E_i's are mutually disjoint measurable sets satisfying $\bigcup_{i=0}^{n-1} E_i = [0,1)$. The integral $\int_0^1 \varphi dx$ is defined by $\sum_{i=0}^{n-1} a_i |E_i|$, where $|E_i|$ is the Lebesgue measure of the set E_i.

For a bounded measurable function f, there exists a sequence of simple functions $\{\varphi_n\}$ which converges pointwise to f. In this case, $\int_0^1 \varphi_n dx$ converges and we denote this limit as $\int_0^1 f dx$. It holds that, if $\{\psi_n\}$ is another approximating sequence of simple functions of f, then $\lim_{n\to\infty} \int_0^1 \varphi_n dx = \lim_{n\to\infty} \int_0^1 \psi_n dx$ and hence the above definition is sound.

For a positive function f, we say that f is integrable if the limit $\lim_{n\to\infty} \int_0^1 f \wedge 2^n dx$ exists, and we denote this limit as $\int_0^1 f dx$, where $(f \wedge 2^n)(x) = \min\{f(x), 2^n\}$. A general measurable function f is called integrable if $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$ are both integrable. We define $\int_0^1 f dx = \int_0^1 f^+ dx - \int_0^1 f^- dx$.

For the reader's convenience, we cite two fundamental theorems from [4] and [5].

Theorem 4 *A bounded function f is Riemann integrable if and only if the Lebesgue measure of the set of all discontinuous points is zero. In this case, f is also Lebesgue integrable and the both integrals have the same value.*

From Theorem 4, a bounded Fine-computable function is Riemann integrable and also Lebesgue integrable.

Theorem 5 (Bounded convergence theorem) *Let $\{f_n\}$ be a uniformly bounded sequence of measurable functions which converges pointwise to a function f. Then $\lim_{n\to\infty} \int_0^1 f_n dx = \int_0^1 f dx$. (Uniformly boundedness means that there exists a constant M such that $|f_n(x)| \leqslant M$ for all n and x.)*

Theorem 6 (Dominated convergence theorem) *Let $\{f_n\}$ be a sequence of integrable functions which converges pointwise to a function f. Suppose further that there exists an integrable function g such that $|f_n(x)| \leqslant g(x)$ for all n and x. Then, f is integrable and $\lim_{n\to\infty} \int_0^1 f_n dx = \int_0^1 f dx$.*

3 Effective integrability of Fine-computable functions

In this section, we discuss the effective computability of integrals for Fine-computable functions on the Fine space. The main objective is the effectivization of Theorems 5 and 6. A Fine-continuous function is $\mathbb{E}$-continuous at every dyadic irrational, and so the Lebesgue measure of the set of all discontinuous points is zero, and hence Theorems 5 and 6 are valid for Fine-computable functions. Therefore the proofs of effectivizations of these theorems are reduced to effective convergence. Since the Fine space does not include the point 1, we write the integral of f on $[0,1)$ as $\int_{[0,1)} f(x)dx$ rather than $\int_0^1 f(x)dx$ or $\int_0^{1-0} f(x)dx$.

Effective integrability of a Fine-computable function is defined as follows. We note that $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$ are Fine-computable if f is Fine-computable.

Definition 3.1 (Effective integrability of a function)

(i) *A bounded Fine-computable function f is called effectively integrable if $\int_{[0,1)} f(x)dx$ is an $\mathbb{E}$-computable number.*

(ii) *A nonnegative Fine-computable function f is said to be effectively integrable if it is integrable and $\int_{[0,1)} f(x)dx$ is an $\mathbb{E}$-computable number.*

(iii) *A Fine-computable function f is called effectively integrable if it is integrable and $\int_{[0,1)} f^+(x)dx$ and $\int_{[0,1)} f^-(x)dx$ are $\mathbb{E}$-computable numbers.*

We also need these definitions for a sequence of functions.

Definition 3.2 (Effective integrability of a sequence of functions)

(i) *A sequence of bounded Fine-computable functions* $\{f_n\}$ *is said to be effectively integrable if each* f_n *is integrable and the sequence* $\{\int_{[0,1)} f_n(x)dx\}$ *forms an* $\mathbb{E}$*-computable sequence of reals.*

(ii) *A sequence of nonnegative Fine-computable functions is said to be effectively integrable if each* f_n *is integrable and* $\{\int_{[0,1)} f_n(x)dx\}$ *is an* $\mathbb{E}$*-computable sequence of reals.*

(iii) *A sequence of Fine-computable functions* $\{f_n\}$ *is called effectively integrable if each* f_n *is integrable and* $\{\int_{[0,1)} f_n^+(x)dx\}$ *and* $\{\int_{[0,1)} f_n^-(x)dx\}$ *are* $\mathbb{E}$*-computable sequences of real numbers.*

Definition 3.3 (i) *Let* E *be a finite union of of dyadic intervals. Then, a Fine-computable function* f *is said to be effectively integrable on* E *if* f *is integrable on* E *and* $\int_E f(x)dx = \int_{[0,1)} \chi_E(x)f(x)dx$ *is an* $\mathbb{E}$*-computable number.*

(ii) *Suppose that* $\{E_m\}$ *is a computable sequence of finite unions of dyadic intervals, that is, there exists a recursive function* $\alpha(m)$ *and recursive sequences of dyadic rationals* $\{a(m,i)\}$ *and* $\{b(m,i)\}$ *such that* $E_m = \bigcup_{i=1}^{\alpha(m)}[a(m,i), b(m,i))$. *Then, a Fine-computable function* f *is said to be effectively integrable on* $\{E_m\}$ *if* f *is integrable on each* E_m *and* $\{\int_{E_m} f(x)dx\}$ *is an* $\mathbb{E}$*-computable sequence of reals.*

(iii) *Effective integrability of a sequence of Fine-computable functions on* E *and* $\{E_n\}$ *are defined similarly.*

For a computable dyadic step function φ of the form $\varphi(x) = \sum_{i=0}^{2^k-1} c_i\chi_{[i2^{-k},(i+1)2^{-k})}(x)$, its integral $\int_{[0,1)} \varphi(x)dx$ is equal to $2^{-k}\sum_{i=0}^{2^k-1} c_i$. $\{\int_{[0,1)} \varphi_n(x)dx\}$ is hence an $\mathbb{E}$-computable sequence of reals if $\{\varphi_n\}$ is a computable sequence of dyadic step functions due to Definition 2.7.

For a uniformly Fine-computable function, the following theorem is essentially proved in the proof of Proposition 4.5 in [8].

Theorem 7 *A uniformly Fine-computable function is effectively integrable.*

The proof goes as follows. Let f be uniformly Fine-computable and let $\{\varphi_n\}$ be an approximating computable sequence of dyadic step functions defined by Equation (2). Then f is bounded and integrable. In addition, $\int_{[0,1)} \varphi_n(x)dx$ converges effectively uniformly to $\int_{[0,1)} f(x)dx$, since

$$|\int_{[0,1)} \varphi_n(x)dx - \int_{[0,1)} f(x)dx| \leqslant \int_{[0,1)} |\varphi_n(x) - f(x)|dx \leqslant \sup_{x\in[0,1)} |\varphi_n(x) - f(x)|.$$

Therefore $\int_{[0,1)} f(x)dx$ is computable.

For a locally uniformly Fine-computable function, we have the following counter-example.

Example 3.1 (Brattka [1]) Let α be an injective recursive function whose range is not recursive. Define

$$\varphi(x) = 2^k 2^{-\alpha(k)} \text{ if } 1 - 2^{-(k-1)} \leqslant x < 1 - 2^{-k}, k = 1, 2, \ldots.$$

Then φ is locally uniformly Fine-computable but $\int_{[0,1)} \varphi(x)dx = \sum_{k=1}^{\infty} 2^{-\alpha(k)}$ is not $\mathbb{E}$-computable.

Let us further note the following. Define

$$\varphi_n(x) = \begin{cases} 2^k 2^{-\alpha(k)} & \text{if } 1-2^{-(k-1)} \leqslant x < 1-2^{-k}, k=1,2,\ldots,n \\ 0 & \text{if } x \geqslant 1-2^{-n} \end{cases}$$

Then $\{\varphi_n\}$ is effectively integrable.

Classically, $\{\int_{[0,1)} \varphi_n\}$ converges to $\int_{[0,1)} \varphi(x)dx$, but the convergence is not effective.

This counter-example shows that the requirement on the computability of the integral is not redundant in the definition of effective integrability of a Fine-computable function.

The next example shows that the computability of $\int_{[0,1)} f(x)dx$ is generally not sufficient for effective integrability.

Example 3.2 Let α be an injective recursive function whose range is not recursive. Put

$$\varphi(x) = \begin{cases} 2^k 2^{-\alpha(k)} & \text{if } 1-2^{-(2k-2)} \leqslant x < 1-2^{-(2k-1)} \\ -2^k 2^{-\alpha(k)} & \text{if } 1-2^{-(2k-1)} \leqslant x < 1-2^{-2k} \end{cases} \qquad (k=1,2,\ldots).$$

Then φ^+ and φ^- are not effectively integrable but $\int_{[0,1)} \varphi(x)dx = 0$.

We need the following lemmas.

Lemma 3.1 (Monotone convergence [11]) *Let $\{x_{n,k}\}$ be an $\mathbb{E}$-computable sequence of reals which $\mathbb{E}$-converges monotonically to $\{x_n\}$ as k tends to infinity for each n. Then $\{x_n\}$ is $\mathbb{E}$-computable if and only if the $\mathbb{E}$-convergence is effective.*

Lemma 3.2 *$|A|$ will denote the Lebesgue measure of a set A.*

Let $\{[a_k, b_k)\}$ be a recursive sequence of dyadic intervals, that is, $\{a_k\}$ and $\{b_k\}$ are recursive sequence of dyadic rationals. If we define $E_n = \bigcup_{k=1}^{n}[a_k, b_k)$, then $\{|E_n|\}$ is an $\mathbb{E}$-computable sequence of reals. Assume that $\{E_n\}$ converges to $[0,1)$, i.e. $\bigcup_{k=1}^{\infty}[a_k, b_k) = [0,1)$. Then $\{|E_n|\}$ $\mathbb{E}$-converges effectively, i.e., there exists a recursive function $\alpha(p)$ such that $|E_n| > 1-2^{-p}$ (or $|E_n{}^C| < 2^{-p}$) for $n \geqslant \alpha(p)$, where A^C denotes the complement of a set A.

Proof For a dyadic interval $[a,b)$, $|[a,b)| = b-a$. E_n can be represented as the union of finite mutually disjoint dyadic intervals whose ends-points are determined effectively from a_k's and b_k's. Therefore, $\{|E_n|\}$ is an $\mathbb{E}$-computable sequence of reals and converges monotonically to 1. $\square$

Theorem 8 *(Effective bounded convergence theorem) Let $\{g_n\}$ be a bounded Fine-computable sequence of functions which is effectively integrable and Fine-converges effectively to f. Then, f is Fine-computable and $\{\int_{[0,1)} g_n(x)dx\}$ $\mathbb{E}$-converges effectively to $\int_{[0,1)} f(x)dx$. As a consequence, f is integrable.*

Proof Suppose that $\{g_n\}$ Fine-converges effectively to f with respect to $\alpha(i,k)$ and $\beta(i,k)$ and, for some integer M, $|g_n(x)| \leqslant M$. Then Theorem 3 yields that f is Fine-computable. Since $\{g_n(x)\}$ converges to $f(x)$, $|f(x)| \leqslant M$ holds, and $\{\int_{[0,1)} g_n(x)dx\}$ converges to $\int_{[0,1)} f(x)dx$ by virtue of Theorem 5.

We denote $\bigcup_{i=1}^{m} J(e_i, \alpha(i,k))$ with $E_{k,m}$. By Definition 2.6, $\bigcup_{m=1}^{\infty} E_{k,m} = [0,1)$. So, for each k, we can find effectively an $m = m(k)$ such that $|E_{k,m}| > 1 - 1/(2^{(k+2)}M)$ from Lemma 3.2. If we take $n \geq \delta(k) = \max\{\alpha(1,k+1), \alpha(2,k+1), \ldots, \alpha(m(k),k+1)\}$, then

$$\begin{aligned} \int_{[0,1)} |g_n(x) - f(x)|dx &\leqslant \int_{E_{k,m}} |g_n(x) - f(x)|dx + \int_{E_{k,m}^C} |g_n(x)| + \int_{E_{k,m}^C} |f(x)|dx \\ &< 2^{-(k+1)} + 2^{-(k+2)} + 2^{-(k+2)} = 2^{-k}. \end{aligned} \qquad \square$$

If f is a bounded Fine-computable function, then the sequence of dyadic step functions $\{\varphi_n\}$ defined by Equation 2 is also bounded. So, the assumption of Theorem 8 holds for f and $\{\varphi_n\}$, and we obtain the following theorem.

Theorem 9 *A bounded Fine-computable function is effectively integrable.*

From the definition of Lebesgue integral and Lemma 3.1, we obtain the following proposition.

Let us here note that, if f is Fine-computable, then $\{f \wedge 2^n\}$ is a Fine-computable sequence of functions.

Proposition 3.1 *Let f be a nonnegative integrable Fine-computable function. Then f is effectively integrable if and only $\{\int_{[0,1)} f \wedge 2^n\}$ $\mathbb{E}$-converges effectively to $\{\int_{[0,1)} f(x)dx\}$.*

Proposition 3.2 *Let f be an effectively integrable Fine-computable function and let I_n be a sequence of dyadic intervals such that $\bigcup_{n=1}^{\infty} I_n = [0,1)$. Put $E_n = \bigcup_{i=0}^{n} I_i$. Then, $\int_{E_n} f(x)dx$ converges effectively to $\int_{[0,1)} f(x)dx$, or equivalently, $\int_{E_n^C} f(x)dx$ converges effective to zero.*

Proof. Since $|\int_E f(x)dx| \leqslant \int_E f^+(x)dx + \int_E f^-(x)dx$, it is sufficient to prove the case where f is nonnegative. Put $f_n = f \wedge 2^n$. Then $\int_{[0,1)} f_n(x)dx$ converges effectively to $\int_{[0,1)} f(x)dx$ due to Proposition 3.1. Hence, there exists a recursive function $\beta(k)$ such that $n \geq \beta(k)$ implies $0 \leqslant \int_{[0,1)} f(x)dx - \int_{[0,1)} f_n(x)dx < 2^{-k}$. In particular, we get

$$0 \leqslant \int_{[0,1)} f(x)dx - \int_{[0,1)} f_{\beta(k+1)}(x)dx < 2^{-(k+1)}.$$

By virtue of Lemma 3.2, there exists a recursive function $\delta(k)$ such that $m \geq \delta(k)$ implies $|E_m^C| < 2^{-k}$. If we take $m \geq \delta(\beta(k+1)+k+1)$, then

$$\begin{aligned} & \int_{[0,1)} f(x)dx - \int_{E_m} f(x)dx = \int_{E_m^C} f(x)dx \\ \leqslant\ & \int_{E_m^C} (f(x) - f_{\beta(k+1)}(x))dx + \int_{E_m^C} f_{\beta(k+1)}(x)dx \\ \leqslant\ & \int_{[0,1)} (f(x) - f_{\beta(k+1)}(x))dx + \int_{E_m^C} f_{\beta(k+1)}(x)dx \\ <\ & 2^{-(k+1)} + 2^{\beta(k+1)} 2^{-(\beta(k+1)+k+1)} = 2^{-k}. \end{aligned} \qquad \square$$

Theorem 10 (Effective dominated convergence theorem) *Let $\{g_n\}$ be an effectively integrable Fine-computable sequence, which Fine converges effectively to f. Suppose that there exists an effectively integrable Fine-computable function h such that $|g_n(x)| \leqslant h(x)$. Then, $\{\int_{[0,1)} g_n(x)dx\}$ converges effectively to $\int_{[0,1)} f(x)dx$.*

Proof. From Theorem 6, f is integrable and $\{\int_{[0,1)} g_n(x)dx\}$ converges to $\int_{[0,1)} f(x)dx$. It also holds that $|f(x)| \leqslant h(x)$.

Suppose that $\{g_n\}$ Fine converges effectively to f with respect to $\alpha(k,i)$ and $\beta(k,i)$. Then,

$$x \in J(e_i, \alpha(k,i)) \text{ and } n \geqslant \beta(k,i) \text{ imply } |g_n(x) - f(x)| < 2^{-k},$$
$$\bigcup_{i=1}^{\infty} J(e_i, \alpha(k,i)) = [0,1) \text{ for each } k.$$

Put $I_i = J(e_i, \alpha(k+1,i))$ and $E_m = \bigcup_{i=1}^{m} I_i$. From Proposition 3.2, we can obtain a recursive function $\delta(k)$ which satisfies that $\int_{E_m^C} h(x)dx < 2^{-k}$ for $m \geqslant \delta(k)$.

Suppose that $n \geqslant \max\{\beta(k+1,1), \dots, \beta(k+1,\delta(k+2))\}$. Then

$$\begin{aligned} & |\textstyle\int_{[0,1)} g_n(x)dx - \int_{[0,1)} f(x)dx| \\ \leqslant\ & \textstyle\int_{E_{\delta(k+2)}} |g_n(x) - f(x)|dx + \int_{(E_{\delta(k+2)})^C} |g_n(x)|dx + \int_{(E_{\delta(k+2)})^C} |f(x)|dx \\ \leqslant\ & 2^{-(k+1)} + 2\,2^{-(k+2)} = 2^{-k} \end{aligned}$$

□

We have stated and proved the theorems and the propositions for a single Fine-computable function up to now. We can easily extend them to the case of a Fine-computable sequence of functions.

The sequentializations of Theorems 8, 9 and 10 can be stated as follows.

Theorem 11 (Sequential effective bounded convergence theorem) *Let $\{f_n\}$ be a Fine-computable sequence and let $\{g_{n,m}\}$ be a bounded Fine-computable sequence of functions which is effectively integrable and Fine-converges effectively to $\{f_n\}$. Assume also that there exists an $\mathbb{E}$-computable sequence of reals $\{M_n\}$ such that $|g_{n,m}(x)| \leqslant M_n$. Then, $\{\int_{[0,1)} g_{n,m}(x)dx\}$ $\mathbb{E}$-converges effectively to $\{\int_{[0,1)} f_n(x)dx\}$.*

Theorem 12 *Let $\{f_n\}$ be Fine-computable and effectively bounded, that is, there exists an $\mathbb{E}$-computable sequence of reals $\{M_n\}$ such that $|f_n(x)| \leqslant M_n$. Then $\{f_n\}$ is effectively integrable.*

Theorem 13 (Sequential effective dominated convergence theorem) *Let $\{g_{m,n}\}$ be an effectively integrable Fine-computable sequence which Fine converges effectively to $\{f_m\}$. Suppose that there exists an effectively integrable Fine-computable sequence $\{h_m\}$ such that $|g_{m,n}(x)| \leqslant h_m(x)$. Then, $\{\int_{[0,1)} g_{m,n}(x)dx\}$ converges effectively to $\int_{[0,1)} f_m(x)dx$.*

4 Indefinite integral and mean value theorem

In this section, we consider $\mathbb{E}$-computability of the indefinite integral $\int_0^x f(x)dx$ $(x \in [0,1])$ and the second mean value theorem for a Fine-computable and effectively integrable function f. We will first establish a fundamental fact.

Theorem 14 *Suppose $\{f_n\}$ is a Fine-computable and effectively integrable sequence of functions. Define $F_n(x) = \int_0^x f_n(x)dx$. Then $\{F_n\}$ is a uniformly $\mathbb{E}$-computable sequence of functions on $[0,1]$.*

Proof. We prove the theorem for the case of a single function f. It is only a mater of routine to modify the proof to the case of a function sequence. We note that $F(1) = \int_{[0,1)} f(x)dx$.

(i) (*Effective uniform $\mathbb{E}$-continuity*): If f is bounded, then there is an integer N such that $|f(x)| \leqslant 2^N$. So, if $|x-y| < 2^{-(k+N)}$, then

$$|F(x) - F(y)| \leqslant \int_x^y 2^N dx = 2^N|x-y| < 2^{-k}.$$

Suppose that f is nonnegative. Then, $\{\int_{[0,1)} g_n(x)dx\}$ $\mathbb{E}$-converges effectively to $\int_{[0,1)} f(x)dx$ by Proposition 3.1, where $g_n = f \wedge 2^n$. So, there exists a recursive function α such that $n \geqslant \alpha(k)$ implies $|\int_{[0,1)} f(x)dx - \int_{[0,1)} g_n(x)dx| < 2^{-k}$. If $|x-y| < 2^{-(k+\alpha(k+1)+1)}$ then

$$\begin{aligned}|F(x) - F(y)| &\leqslant |\int_{[0,1)} f(x)dx - \int_{[0,1)} g_{\alpha(k+1)}(x)dx| + |\int_x^y g_{\alpha(k+1)}(x)dx| \\ &\leqslant 2^{-(k+1)} + |x-y|2^{\alpha(k+1)} < 2^{-k}.\end{aligned}$$

The general case follows from the inequality $|F(x) - F(y)| \leqslant \int_x^y |f(x)|dx$ and the nonnegative case.

(ii) (*Sequential computability*): Let $\{x_n\}$ be any $\mathbb{E}$-computable sequence of reals in $[0,1)$. Then, there exists a recursive sequence of dyadic rationals $\{r_{n,m}\}$ which $\mathbb{E}$-converges effectively to $\{x_n\}$.

By definition, $F(r_{n,m}) = \int_{[0,1)} \chi_{[0,r_{n,m})}(x)f(x)dx$, $\{\chi_{[0,r_{n,m})}(x)f(x)\}$ is a Fine-computable sequence of functions and $|\chi_{[0,r_{n,m})}(x)f(x)| \leqslant |f(x)|$. $\{F(r_{n,m})\}$ is hence an $\mathbb{E}$-computable sequence of reals by Theorem 13. From the effective uniform $\mathbb{E}$-continuity proved in (i), $\{F(r_{n,m})\}$ $\mathbb{E}$-converges effectively to $\{F(x_n)\}$. Therefore $\{F(x_n)\}$ is an $\mathbb{E}$-computable sequence of reals. □

Remark 4.1 From (i) of the proof above, the following effective absolute continuity holds: if f is Fine-computable and effectively integrable, then there exists a recursive function such that $|\int_E f(x)dx| < 2^{-k}$ for a measurable set E with $|E| < 2^{-\alpha(k)}$.

Theorem 15 (Effective intermediate value theorem, Theorem 8 in Section 0.6 of [11]) *Let $[a,b]$ be an interval with $\mathbb{E}$-computable endpoints, and let f be an $\mathbb{E}$-computable function on$[a,b]$ such that $f(a) < f(b)$. Let s be an $\mathbb{E}$-computable real with $f(a) < s < f(b)$. Then there exists an $\mathbb{E}$-computable point c in (a,b) such that $f(c) = s$.*

It is pointed out in [11] that the sequential version of Theorem 15 does not hold.

We can prove the following variation of Theorem 15.

Theorem 15′ *Let $[a,b]$ be an interval with rational endpoints, and let f be $\mathbb{E}$-computable and non-constant on $[a,b]$. Put $m = \min_{x\in[a,b]} f(x)$ and $M = \max_{x\in[a,b]} f(x)$. For an $\mathbb{E}$-computable real number s with $m < s < M$, there exists a $\mathbb{E}$-computable point c in $[a,b]$ such that $f(c) = s$.*

Proof. Define

$$m_n = \min_{0 \leqslant i \leqslant n-1}\{f(a + i(b-a)/n\} \text{ and } M_n = \max_{0 \leqslant i \leqslant n-1}\{f(a + i(b-a)/n\}.$$

Then $\{m_n\}$ $\mathbb{E}$-converges effectively to m and $\{M_n\}$ $\mathbb{E}$-converges effectively to M ([11]).

Suppose s is a computable real number such that $m < s < M$. Then one can find effectively n_1 and n_2 such that $m_{n_1} < s < M_{n_2}$. For such n_1 and n_2, there exist $i_1 < n_1$ and $i_2 < n_2$ satisfying the following conditions. If we put $x_{n_1} = a + i_1(b-a)/n_1$ and $y_{n_2} = a + i_2(b-a)/n_2$, then $f(x_{n_1}) = m_{n_1}$ and $f(y_{n_2}) = M_{n_2}$ hold. If we apply Theorem 15 to the interval $[x_{n_1}, y_{n_2}]$ (or $[y_{n_2}, x_{n_1}]$), we obtain the desired c. (Notice that x_{n_1} and y_{n_2} are computable, although i_1 and i_2 may not be effectively found.) □

Since a Fine-computable function may be $\mathbb{E}$-discontinuous, the (first) mean value theorem does not hold. On the other hand, the second mean value theorem applies to $\mathbb{E}$-discontinuous functions. To effectivize this theorem, we need the following proposition, which can be proved easily following the classical proof.

Proposition 4.1 *Let f be Fine-computable and effectively integrable, and let g be bounded and Fine-computable. Then fg is also effectively integrable.*

Theorem 16 (Effective second mean value theorem) *Let f be Fine-computable and effectively integrable. Suppose that a and b are dyadic rationals satisfying $0 \leqslant a < b < 1$.*

(i) *Let g be Fine-computable, nonnegative and strictly decreasing. Then, there exists an $\mathbb{E}$-computable point $c \in [a, b]$ which satisfies*

$$\int_a^b g(t)f(t)dt = g(a)\int_a^c f(t)dt. \tag{3}$$

(ii) *If g is Fine-computable and strictly monotone, then there exists an $\mathbb{E}$-computable point $c \in [a, b]$ which satisfies*

$$\int_a^b g(t)f(t)dt = g(a)\int_a^c f(t)dx + g(b)\int_c^b f(t)dt. \tag{4}$$

Proof. Let us note that g is bounded on $[a, b]$.

(i) Define $F(x) = \int_a^x f(t)dt$. Then F is $\mathbb{E}$-computable by Theorem 14. Put $M = \max_{x\in[a,b]} F(x)$ and $m = \min_{x\in[a,b]} F(x)$.

The following holds by integration by parts and the absolute continuity of F ([13]).

$$\int_a^b f(t)g(t)dt = F(b)g(b) - \int_a^b F(t)dg(t), \tag{5}$$

where $\int F(t)dg(t)$ denotes Lebesgue-Stieltjes integral.

If $m = M$, then F is constant, and hence $f(x) = 0$ for all $x \in [a, b]$. So, Equation (3) holds for all $c \in [a, b]$. Now, we assume that $m < M$.

From the assumption on g and the definitions of m and M, we obtain

$$m\int_a^b dg(t) > \int_a^b F(t)dg(t) > M\int_a^b dg(t). \tag{6}$$

From Equation (5) and Inequality (6),

$$\begin{aligned} \int_a^b f(t)g(t)dt &> F(b)g(b) - m\int_a^b dg(t) = F(b)g(b) - m(g(b) - g(a)) \\ &= (F(b) - m)g(b) + mg(a) \geqslant mg(a). \end{aligned}$$

In the same way, we can prove $\int_a^b g(t)f(t)dt < Mg(a)$. Since $g(a) > 0$, we obtain

$$m < \frac{1}{g(a)} \int_a^b f(t)g(t)dt < M.$$

If we apply Theorem 15′ to F, there exists an $\mathbb{E}$-computable point c in $[a, b]$ such that

$$F(c) = \frac{1}{g(a)} \int_a^b g(t)f(t)dt.$$

This implies Equation (3).

(ii) If g is strictly decreasing, then $g - g(b)$ is nonnegative and strictly decreasing. So, we obtain from (i) an $\mathbb{E}$-computable point c in (a, b) such that

$$\int_a^b (g(x) - g(b))f(x)dx = (g(a) - g(b)) \int_a^c f(x)dx.$$

From this, we obtain

$$\int_a^b g(x)f(x)dx = g(a) \int_a^c f(x)dx + g(b) \int_c^b f(x)dx.$$

If g is strictly increasing, we can obtain the same result by applying the above result to $-g$. □

5 Effective Fine convergence of Walsh Fourier series

The system of Walsh functions $\{w_n\}$ is defined on $[0, 1)$ by

$$w_n(x) = (-1)^{\sum_{i=0}^{k} \sigma_{i+1} n_{-i}}, \tag{7}$$

where, $\sigma_1\sigma_2\cdots$ is the dyadic expansion of x with infinitely many $0'$s and $n = n_0 + n_{-1}2 + \cdots + n_{-k}2^k$ is the dyadic expansion of a positive integer n.

It can be easily shown that $\{w_n\}$ is a Fine-computable sequence of functions, and that, if f is Fine-computable and effectively integrable, then so is the sequence $\{fw_n\}$.

Theorem 17 *(Computability of Walsh Fourier coefficients) If f is Fine-computable and effectively integrable, then the sequence of Walsh Fourier coefficients $\{\int_{[0,1)} f(x)w_n(x)dx\}_{n=0}^{\infty}$ is an $\mathbb{E}$-computable sequence of reals.*

Proof. Put $f_n(x) = (f(x) \wedge 2^n) \vee (-2^n)$.

The sequence of Fine-computable functions $\{f_n w_m\}$ satisfies the assumption of Theorem 12. So $\{\int_{[0,1)} f_n(x)w_m(x)dx\}$ is an $\mathbb{E}$-computable (double) sequence of reals.

Then, $\{g_{m,n}\} = \{f_n w_m\}$ satisfies the assumption of Theorem 13 with $h_m(x) = f(x)$ and Fine-coverges effectively to fw_m. So, $\{\int_{[0,1)} f_n(x)w_m(x)dx\}$ $\mathbb{E}$-converges effectively to $\{\int_{[0,1)} f(x)w_m(x)dx\}$ and hence the latter is an $\mathbb{E}$-computable sequence of reals. □

Definition 5.1 *The partial sum $S_n(f)$ and modified Dirichlet kernel $D_n(x, t)$ are defined by*

$$S_n(f)(x) = \sum_{i=0}^{n-1} c_i w_i(x),\ D_n(x, t) = \sum_{i=0}^{n-1} w_i(x)w_i(t),$$

where $\{c_i\}$ is the Walsh Fourier coefficients of f, i.e. $c_i = \int_{[0,1)} f(t)w_i(t)dt$.

It is well known that

$$S_n(f)(x) = \int_{[0,1)} f(t) D_n(x,t) dt. \tag{8}$$

Remark 5.1 In the theory of classical Walsh Fourier series, $D_n(x \oplus t)$ is usually used instead of $D_n(x,t)$, where $D_n(x) = D_n(x,0) = D_n(0,x)$ ([3], [12]). Since the *dyadic sum* $x \oplus t$ is not defined for all x and t in $[0,1)$, we do not use the dyadic sum $x \oplus t$.

Lemma 5.1 (Paley) ([3])

$$D_{2^n}(x,t) = \begin{cases} 2^n & \textit{if} \;\; t \in J(x,n) \\ 0 & \text{otherwise} \end{cases}.$$

We can prove the following Theorems in a manner similar to the proof of Proposition 4.5 in [6]. The Fine-convergence of $\{S_{2^n} f\}$ can be proved similarly to the proof of Proposition 4.5 in [6] using the Paley's lemma.

Theorem 18 *If f is Fine-computable and effectively integrable, then $S_{2^n} f$ Fine-converges effectively to f.*

Proof. Recall that

$$S_{2^n} f(x) = \int_{[0,1)} f(t) D_{2^n}(x,t) dt = \int_{J(x,n)} f(t) D_{2^n}(x,t) dt.$$

Now, from Paley's Lemma,

$$S_{2^n} f(x) - f(x) = \int_{J(x,n)} (f(t) D_{2^n}(x,t) - 2^n f(x)) dt = 2^n \int_{J(x,n)} (f(t) - f(x)) dt.$$

Suppose that f is Fine continuous with respect to $\gamma(k,i)$. If $x \in J(e_i, \gamma(k+1,i))$ and $n \geqslant \gamma(k+1,i)$, then $t \in J(e_i, \gamma(k+1,i))$ for $t \in J(x,n)$. In this case, we obtain

$$|f(t) - f(x)| \leqslant |f(t) - f(e_i)| + |f(e_i) - f(x)| < 2^{-k}.$$

Hence, we obtain $|S_{2^n} f(x) - f(x)| < 2^{-k}$. If we define $\alpha(k,i) = \gamma(k+1,i)$ and $\beta(k,i) = \gamma(k+1,i)$, then $S_{2^n}(f)$ Fine converges effectively to f with respect to α and β. $\square$

The effective version of the Walsh Riemann Lebesgue theorem ([12]) can be stated and proved as follows.

Theorem 19 (Effective Walsh Riemann Lebesgue theorem) *If f is Fine-computable and effectively integrable, then its Walsh-Fourier coefficients $\{c_n\}$ converges effectively to zero.*

Proof. (i) First, we assume that f is bounded. Let $\{\varphi_m\}$ be the approximating sequence of dyadic step functions defined by Equation (2) and put $d_{m,n} = \int_{[0,1)} \varphi_m(x) w_n(x) dx$. Then $\{d_{m,n}\}$ is $\mathbb{E}$-computable by Theorem 12 and

$$|d_{m,n} - c_n| = |\int_{[0,1)} (\varphi_m(x) - f(x)) w_n(x) dx| \leqslant \int_{[0,1)} |\varphi_m(x) - f(x)| dx.$$

The right-hand side $\mathbb{E}$-converges effectively to zero by Theorem 8. So, $\{d_{m,n}\}$ $\mathbb{E}$-converges effectively to $\{c_n\}$ as m tends to infinity uniformly in n. This means that there exists a recursive function γ such $m \geqslant \gamma(k)$ implies $|d_{m,n} - c_n| < 2^{-k}$ for all n.

The sign of $w_n(x)$ on $[2j2^{-(n+1)}, (2j+1)2^{-(n+1)})$ and that on $[(2j+1)2^{-(n+1)}, (2j+2)2^{-(n+1)})$ are opposite to each other with absolute value 1 for $j = 0, 1, \ldots, 2^n - 1$. On the other hand, $\varphi_m(x)$ is constant on each $[i2^{-m}, (i+1)2^{-m})$. So, $d_{m,n} = 0$ if $n \geqslant m$.

From the discussion above, if $n \geqslant \gamma(k)$, then we obtain that $d_{\gamma(k),n} = 0$ and hence $|c_n| < 2^{-k}$. This proves that $\{c_n\}$ $\mathbb{E}$-converges effectively to zero.

(ii) For a general f, $f = f^+ - f^-$ and $c_n = \int_0^1 f^+(x)w_n(x)dx - \int_0^1 f^-(x)w_n(x)dx$. Therefore, it is sufficient to prove the case where f is nonnegative.

Put $f_\ell = f \wedge 2^\ell$ and $c_{\ell,n} = \int_{[0,1)} f_\ell(x)w_n(x)dx$. $\{f_\ell w_n\}$ is Fine-computable and effectively integrable as a double sequence of functions and $|f_\ell w_n| \leqslant 1$. $\{c_{\ell,n}\}$ is $\mathbb{E}$-computable by an extended version of Theorem 12.

Notice that the proof of (i) can be modified for effectively bounded sequence of functions $\{f_\ell\}$. This means that there exists a recursive function $\delta(\ell, k)$ such that $n \geqslant \delta(\ell, k)$ implies $|c_{\ell,n}| < 2^{-k}$. Similarly to (i), we obtain

$$|c_{\ell,n} - c_n| = |\int_{[0,1)}(f_\ell(x) - f(x))w_n(x)dx| \leqslant \int_{[0,1)}(f(x) - f_\ell(x))dx.$$

The right-hand side $\mathbb{E}$-converges effectively to zero by Proposition 3.1. So, $\{c_{\ell,n}\}$ $\mathbb{E}$-converges effectively to $\{c_n\}$ as ℓ tends to infinity uniformly in n. Let β be the modulus of this convergence. Then it holds that $\ell \geqslant \beta(k)$ implies $|c_{\ell,n} - c_n| < 2^{-k}$ for all n.

If $n \geqslant \delta(\beta(k+1), k+1)$, then

$$|c_n| \leqslant |c_{\beta(k+1),n} - c_n| + |c_{\beta(k+1),n}| < 2^{-(k+1)} + 2^{-(k+1)} = 2^{-k}.$$

This proves that $\{c_n\}$ $\mathbb{E}$-converges effectively to zero with respect to $\alpha(k) = \delta(\beta(k+1), k+1)$. ∎

To prove an effective version of Dirichlet's test, we need the following two lemmas. The second one is the effectivization of the fundamental lemma which is used in proving pointwise convergence of the partial sums $S_n(f)$ to f (cf. [12]).

Lemma 5.2 ([12]) $D_n(x,t) = w_n(x)w_n(t)\sum_{j=0}^N n_{-j}\phi_j(x)\phi_j(t)D_{2^j}(x,t)$, *where $\phi_j(x)$ is the j-th Radmacher function and $n = n_0 + n_1 2 + \cdots + n_N 2^N$ is the dyadic expansion of n.*

Prior to the next lemma, let us make the following remark. In the classical case, one can use $w_j(x \oplus t)$ instead of $w_j(x)w_j(t)$, and this fact leads us to the desired conclusion quickly. Here, however, we cannot use $w_j(x \oplus t)$, and hence we need some elaborate work.

Lemma 5.3 (Key lemma) *If f is Fine-computable and effectively integrable, then $F_{M,n}(x) = \int_{[0,1)\setminus J(x,M)} f(t)D_n(x,t)dt$ Fine converges effectively to zero effectively in M uniformly in x. This means that there exist a recursive function $\alpha(M,k)$ which satisfies that $n \geqslant \alpha(M,k)$ implies $|F_{M,n}(x)| < 2^{-k}$.*

Proof. (i) Let f be bounded Fine-computable, $\{\varphi_m\}$ be the approximating sequence of dyadic step functions defined by Equation (2) and $\Phi_{M,m,n}(x)$ be $\int_{[0,1)\setminus J(x,M)} \varphi_m(t)D_n(x,t)dt$.

(i-a) First, we show that $\{\Phi_{M,m,n}\}$ is Fine-computable: Let $\{x_\ell\}$ be a Fine-computable sequence in $[0,1)$. Then, $\chi_{[0,1)\setminus J(x_\ell,M)}(t)\varphi_m(t)D_n(x_\ell,t)$ is a bounded Fine-computable (quadruple) sequence of dyadic step functions of t. So, $\{\Phi_{M,m,n}(x_\ell)\}$ is $\mathbb{E}$-computable by Theorem 12. This proves the sequential computability of $\{\Phi_{M,m,n}\}$.

By the definition of $D_n(x,t)$, it is constant on a rectangle $I(\ell,j)\times I(\ell',j')$ if $n\geqslant \ell$ and $n\geqslant \ell'$ as a function of x and t, where $I(\ell,j)=[j2^{-\ell},(j+1)2^{-\ell})$. For such an n, $D_n(x,t)=D_n(y,t)$ if $y\in J(x,\ell)$.

If $\ell\geqslant M$ and $y\in J(x,\ell)$, then $J(x,M)=J(y,M)$. It also holds that $I(\ell,j)\subset J(x,M)$ or $I(\ell,j)\subset[0,1)\backslash J(x,M)$. Therefore, if $\ell\geqslant\max\{[\log_2 n],M\}$ and $y\in J(x,\ell)$, then $\Phi_{M,m,n}(x)=\Phi_{M,m,n}(y)$.

(i-b) $\{\Phi_{M,m,n}\}$ converges effectively to zero as n tends to infinity uniformly in x effectively in $m\geqslant M$: First, we have

$$D_n(x,t)=D_{2^M}(x,t)+\textstyle\sum_{j=2^M}^{n-1}w_j(x)w_j(t)$$

if $n\geqslant 2^M$, and $D_{2^M}(x,t)=0$ on $[0,1)\backslash J(x,M)$ by Paley's Lemma.

On the other hand, φ_m is constant on $[i2^{-m},(i+1)2^{-m})$ for each i. If $n\geqslant 2^m(\geqslant 2^M)$, the sign of $w_n(t)$ on $[2j2^{-([\log_2 n]+1)},(2j+1)2^{-([\log_2 n]+1)})$ and that on $[(2j+1)2^{-([\log_2 n]+1)},(2j+2)2^{-([\log_2 n]+1)})$ are opposite with absolute value 1. ($[x]$ denotes the integer part of x.) Therefore, $\Phi_{M,m,n}(x)=0$ if $n\geqslant 2^m\geqslant 2^M$. In other words, $\Phi_{M,m,n}(x)=0$ if $n\geqslant 2^m$ and $m\geqslant M$.

(i-c) $\{\Phi_{M,m,n}\}$ converges to $F_{M,n}$ as m tends to infinity uniformly in n effectively in M: By Lemma 5.2,

$$\begin{aligned}&\Phi_{M,m,n}(x)-F_{M,n}(x)=\textstyle\int_{[0,1)\backslash J(x,M)}(\varphi_m(t)-f(t))D_n(x,t)dt\\ =&\textstyle\sum_{j=0}^{M-1}n_{-j}w_n(x)\phi_j(x)\int_{[0,1)\backslash J(x,M)}(\varphi_m(t)-f(t))w_n(t)\phi_j(t)D_{2^j}(x,t)dt,\end{aligned}$$

and so

$$\begin{aligned}|\Phi_{M,m,n}(x)-F_{M,n}(x)| &\leqslant \textstyle\sum_{j=0}^{M-1}2^j\int_{[0,1)\backslash J(x,M)}|\varphi_m(t)-f(t)|dt\\ &\leqslant 2^M\textstyle\int_{[0,1)}|\varphi_m(t)-f(t)|dt.\end{aligned}$$

By Theorem 8, $\int_{[0,1)}|\varphi_m(t)-f(t)|dt$ $\mathbb{E}$-converges effectively to zero. Let δ be the modulus of this convergence. Then $|\Phi_{M,m,n}(x)-F_{M,n}(x)|<2^{-k}$ if $m\geqslant\delta(k+M)$.

(i-d) $F_{M,n}$ Fine converges effectively to zero as n tends to infinity effectively in M uniformly in x: Define $\alpha(M,k)=2^{\delta(k+M)\vee M}$ and put $m_0=\delta(k+M)\vee M$. Then, by virtue of (i-c), $|\Phi_{M,m_0,n}(x)-F_{M,n}(x)|<2^{-k}$ for all n and x. On the other hand, $\Phi_{M,m_0,n}(x)=0$ if $n\geq\alpha(M,k)$. So, for $n\geq\alpha(M,k)$,

$$|F_{M,n}(x)|\leqslant|\Phi_{M,m_0,n}(x)-F_{M,n}(x)|+|\Phi_{M,m_0,n}(x)|<2^{-k}.$$

(ii) The case where f is Fine-computable, nonnegative and effectively integrable: Put $f_\ell=f\wedge 2^\ell$ and $F_{M,\ell,n}(x)=\int_{[0,1)\backslash J(x,M)}f_\ell(t)D_n(x,t)dt$. Then, $\{f_\ell\}$ is Fine-computable and effectively bounded. As in the proof of (i-d), there exists a recursive function $\alpha(M,\ell,k)$ such that $n\geqslant\alpha(M,\ell,k)$ implies $|F_{M,\ell,n}(x)|<2^{-k}$. We have also

$$\begin{aligned}&|F_{M,\ell,n}(x)-F_{M,n}(x)|=|\textstyle\int_{[0,1)\backslash J(x,M)}(f(t)-f_\ell(t))D_n(x,t)dt|\\ \leqslant\ &2^M\textstyle\int_{[0,1)}(f(t)-f_\ell(t))dt.\end{aligned}$$

$\int_{[0,1)}(f(t)-f_\ell(t))dt$ $\mathbb{E}$-converges effectively to zero by Proposition 3.1. Let $\gamma(k)$ be the modulus of this convergence. If $n\geqslant\beta(M,k)=\alpha(M,\gamma(k+M+1),k+1)$, then

$$|F_{M,n}(x)| \leqslant |F_{M,\gamma(k+M+1),n}(x) - F_{M,n}(x)| + |F_{M,\gamma(k+M+1),n}(x)| < 2^{-(k+1)} + 2^{-(k+1)} = 2^{-k}.$$

This proves that $\{F_{M,n}\}$ converges to zero effectively as n tends to infinity effectively in M uniformly in x.

(iii) For a general f, $f = f^+ - f^-$ holds, and hence the lemma follows from (ii). □

Before we treat the final objective, the effectivization of the Dirichlet's test, we study the computability of the variation of a Fine-computable function.

Zheng, Rettinger and Braunmühl investigated functions of bounded variation and Jordan decomposability ([15]). They showed an example that is effectively absolutely continuous but not effectively Jordan decomposable.

Subsequently $V_0^x(f)$ denotes the variation of f in $[0,x]$ $(0 \leqslant x < 1)$. $V_0^1(x)$ is defined to be $\sup_{0\leqslant x<1} V_0^x(f)$.

The following example is a modification of Proposition 4.2 in [10].

Example 5.1 Let α be an injective recursive function whose range is not recursive. Define

$$f(x) = e^{-\alpha(n)} \text{ if } \tfrac{1}{2} - 2^{-n} \leqslant x < \tfrac{1}{2} - 2^{-(n+1)} \ (n = 1, 2, \ldots).$$

Then $V_0^x(f) = \sum_{n=1}^{\infty} e^{-\alpha(n)}$ for $x \geqslant \frac{1}{2}$, and $\sum_{n=1}^{\infty} e^{-\alpha(n)}$ is not $\mathbb{E}$-computable.

According to Example 5.1, sequential computability of the variation fails. However, we can prove easily effective Fine continuity of $V_0^x(f)$ if it is finite.

Definition 5.2 *A Fine-computable function is said to be effectively Jordan decomposable if there exist monotone increasing Fine-computable functions ψ_1 and ψ_2 such that $f = \psi_1 - \psi_2$.*

Theorem 20 (Effective Dirichlet's test) *Let f be Fine-computable, effectively integrable and effectively Jordan decomposable. Then $\{S_n(f)\}$ Fine converges effectively to f.*

Proof. It is sufficient to prove the case where f is monotone increasing and Fine-computable.

According to [12], the following holds classically: Put $\Theta(x,M) = \sup_{t\in J(x,M)} |f(t) - f(x)|$, $\Delta = \sup_{x,y\in[0,1),n} |\int_{[0,y)} D_n(x,t)dt|$ and $U_{n,M}(x,f) = \int_{J(x,M)}(f(t) - f(x))D_n(x,t)dt$. Then

$$|U_{n,M}(x,f)| \leqslant 4\Delta\Theta(x,M) \text{ and } |\Delta| \leqslant 2$$

holds. We remark that the right-hand side of the above inequality does not depend on n.

If $n \geqslant 2^M$, then

$$\begin{aligned} & S_n(f)(x) - f(x) = \textstyle\int_{[0,1)}(f(t) - f(x))D_n(x,t)dt \\ \leqslant \ & U_{n,M}(x,f) + \textstyle\int_{[0,1)\setminus J(x,M)} f(t)D_n(x,t)dt - f(x)\int_{[0,1)\setminus J(x,M)} D_n(x,t)dt. \end{aligned}$$

If f is Fine-computable with respect to $\alpha(k,i)$, then $x \in J(e_i, \alpha(k,i))$ implies $|f(x) - f(e_i)| < 2^{-k}$. If $x \in J(e_i, M)$, then $J(x,M) = J(e_i,M)$, and $t \in J(x,M)$ is equivalent to $t \in J(e_i,M)$. So, $\Theta(x,\alpha(k)) \leqslant 2 \cdot 2^{-k}$.

By Lemma 5.3, we obtain recursive functions $\beta(M,k)$ and $\gamma(M,k)$ which satisfy

$n \geqslant \alpha(M,k)$ implies $\int_{[0,1)\setminus J(x,M)} f(t)D_n(x,t)dt < 2^{-k}$

and

$n \geqslant \beta(M,k)$ implies $\int_{[0,1)\setminus J(x,M)} D_n(x,t)dt < 2^{-k}$.

If $x \in J(e_i, \alpha(k+6))$ and $n \geqslant \max\{\beta(\alpha(k+6), k+2), \gamma(k+6), k+2)\}$, then from the equations and inequalities above, we obtain

$$|S_n(f)(x) - f(x)| \leqslant 8 \cdot 2\Theta(x, \alpha(k+6)) + 2 \cdot 2^{-(k+2)} < 2^{-k}.$$

So the effective Fine convergence is proved. □

In Theorems 18 and 20, we can replace "Fine convergence" to "uniform Fine convergence" if f is uniformly Fine-computable, and to "locally uniform Fine convergence" if f is locally uniformly Fine-computable.

Theorems 18, 19 and Lemmas 5.1, 5.3 are effectivizations of corresponding classical Theorems and Lemmas. So the following classical version of Theorem 20 holds. (See [10] for terminologies.)

Theorem 21 *If f is Fine continuous, integrable and of bounded variation, then $S_n(f)$ Fine converges to f.*

It is pointed out in [10] that Fine convergence is weaker than locally uniform Fine-convergence and stronger than point wise convergence. For a sequence of Fine continuous functions, Fine convergence is equivalent to continuous convergence.

References

[1] V. Brattka. Some Notes on Fine Computability. *Journal of Universal Computer Science*, 8:382-395, 2002.

[2] N. J. Fine. On the Walsh Functions. *Trans. Amen. Math. Soc.*, 65:373-414, 1949.

[3] B. Golubov, A. Efimov and V. Skvortsov. *Walsh Series and Transforms*, Kluwer Academic, 1991.

[4] R. A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron, and Henstock.* 1994, Am. Math. Soc.

[5] K. Ito (ed.). *Encyclopedic Dictionary of Mathematics.* Second Edition. MIT Press, 1993.

[6] T. Mori. On the computability of Walsh functions. *Th. Comp. Sci.*, 284:419-436, 2002.

[7] T. Mori. Computabilities of Fine continuous functions. *Computability and Complexity in Analysis, (4th International Workshop, CCA2000. Swansea)*, ed. by Blanck, J. *et al.,* 200-221. Springer, 2001.

[8] T. Mori. Computabilities of Fine continuous functions. *Acta Humanistica et Scientifica Universitatis Sangio Kyotiensis, Natural Science Series I,* 31:163-220, 2002. (in Japanese)

[9] T. Mori, Y. Tsujii and M. Yasugi. Computability Structures on Metric Spaces. *Combinatorics, Complexity and Logic* (*Proceedings of DMTCS'96*), ed. by Bridges *et al.,* 351-362. Springer, 1996.

[10] T. Mori, Y. Tsujii and M. Yasugi. Fine-Computable Functions and Effective Fine-Convergence. CCA2005, Informatik Berichte 326-7/2005, FernUniversität in Hagen, 177-197; revized version available at

http://www.cc.kyoto-su.ac.jp/∼yasugi/page/recent.html

[11] M.B. Pour-El and J. I. Richards. *Computability in Analysis and Physics.* Springer, 1988.

[12] F. Schipp, W.R. Wade and P. Simon. *Walsh Series.* Adam Hilger, 1990.

[13] M. Tsuji. *Theory of real Functions.* Maki Shotenn, 1962. (in Japanese)

[14] M. Yasugi, Y. Tsujii and T.Mori. Metrization of the Uniform Space and Effective Convergence. Math. Log. Quart. 48 Suppl. 1:123-130, 2002.

[15] X. Zheng, R. Rettinger and B. v. Braunmühl. Effecttively Absolute Continuity and Effective Jordan Decomposability. *Proceedings of the 5th Workshop on Computability and Complexity in Analysis, CCA 2002, Malaga.* ed. by Brattka *et al.,* 223-234.