Kinetic Theory of Granular Gases

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Contents

Chapter 1

Introduction

1.1 Granular gases and kinetic theory

Granular gases and flows of granular particles have been studied well due to the importance in technology, geophysics, astrophysics, and science. The peculiar properties of the granular gases are mainly caused by the energy dissipation during the collisions of particles. Due to the dissipative collisions, the equilibrium state does not exist and the steady state or steady flow of granular particles can be retained only if the external force fields exist. In this sense, the granular gases are intrinsically *non-equilibrium* and quite different from molecular gases and usual fluids. To describe the macroscopic properties of the granular gases, *kinetic theory* of granular gases has been developed by many authors in the last decades [1, 2]. We know the kinetic theory of molecular gases predicts the transport coefficients starting from the *Boltzmann equation*. *Chapman-Enskog method*, *Sonine polynomials expansion* and *Grad expansion* are the successful tools to evaluate the distribution functions and the transport coefficients, and we find the kinetic theory bridges the gap between the microscopic princiles and the macroscopic properties of molecular gases [3, 4]. Although the application of kinetic theory to granular gases involves several problems, for example, the lack of scale separation, the long range correlations, etc [5], the *granular hydrodynamic equations* derived by the kinetic theory of granular gases well describe the dynamics of granular flows.

1.2 Basic assumption

To apply kinetic theory to granular gases, we need basic assumption as well as molecular gases. That is *binary collision* of granular particles, where we assume the statistical properties of granular gases are mainly determined by the binary collisions and collisions accompanied by more than three particles are too rare to take into account. This assumption is related to the density of granular gases, the duration time of particle collisions and rigidity of granular particles. To satisfy the basic assumption, the density ϕ has to be low enough. It is known that, above the *jamming density* ϕ_J, the mean numbebr of contacts per particle, i.e., the coordination number z, jumps to the isostatic value $z_c = 2d$ from zero in d-dimension. Clearly, ϕ should be much lower than ϕ_J and there may be a limit of the basic assumption at ϕ_m, thus, our density should be $\phi < \phi_m < \phi_J$. The duration time of each collision is also important, because the long duration time of collision causes to invite another particles to the multibody collision. The duration time is related to the rigidity of granular particles. The harder the granular particles are, the shorter the duration time is. Thus, our granular particles have to be *slightly inelastic* or *inelastic hardsphere* to avoid the multibody collisions.

1.3 Restitution coefficient

Inelastic collision of granular particles are characterized by the *restitution coefficient* e. In general, the restitution coefficient depends on the incident velocities of granular particles and many effects influence the value of e, for example, the surface tension, the asperities, the fracture, the shape of granular particles, etc. Because of these effects, the restitution coefficient can be changed in each collision, however, we restrict our study to the simplest case that $e = \text{const}$ in $0 < e < 1$. We also assume the granular particles are spherical and identical with the same mass and diameter. The kinetic theory with the velocity dependent restitution coefficient has been found in Ref. [1]. We also notice that in some cases, the restitution coefficient can be negative $e < 0$ [18, 21] and there is a critical point $e_c = 0$ [18]. This critical point could violate the Boltzmann equation and the kinetic theory of granular gases, because we will see the prefactor $1/e^2$ in the first term of the collision integral (see Eq. (2.30) in Chap. 2) and $1/e^2$ diverges at the critical point $e_c = 0$.

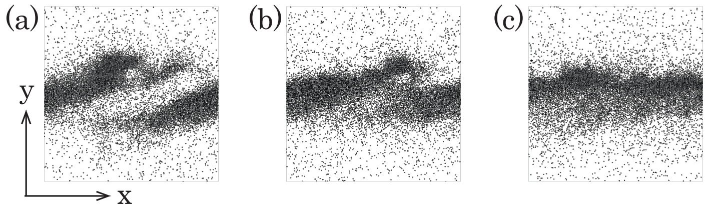

Figure 1.1: Development of spatial inhomogeneity in two-dimensional granular shear flow.

1.4 Instabilities of granular flows

Because of the inelastic collisions and the energy dissipation, homogeneous solutions of the granular hydrodynamic equations are often unstable. For example, *clustering* in the *homogeneous cooling state* [1], shear band in the granular shear flow [19, 22] and convections in the vibrated granular thin layer [23]. Figure 1.1 displays the time development of granular particles under the Lees-Edwards boundary condition, where the homogeneous flow becomes unstable by shear and the shear band is generated in the steady state. It is notable the time development of the hydrodynamic fields of granular shear flow can be well described by the numerical solutions of the granular hydrodynamic equations [19]. Such instabilities in granular flows can be expected by the *linear stability analysis* [24, 25]. More developed analyses, i.e., *weakly nonlinear analyses* are also paid much attention in these days [20, 26–28].

1.5 Organization of this lecture note

In this lecture, we explain the kinetic theory of granular gases. In Chap. 2, we derive the Boltzmann equation of pairs of granular particles. In Chap. 3, we explain the homogeneous cooling state of granular gases. In Chap. 4, we derive the hydrodynamic equations of granular gases by the Chapman-Enskog method. In Chap. 5, we explain the collisional transfer of granular particles which is important in the dense flows of granular particles. This lecture note is based on the books

[1] ”*Kinetic Theory of Granular Gases*”, N. V. Brilliantov and T. Pöschel
[2] ”*Granular Gases*”, T. Pöschel and S. Luding

and we also refer to the text books of kinetic theory of molecular gases

[3] ”*The Mathematical Theory of Non-uniform Gases*”, S. Chapman and T. G. Cowling,
[4] ”*Classical Kinetic Theory of Fluids*”, P. Résibois and M. de Leener.

For collisional transfer, we refer to some papers [6–8]. There are several applications of the kinetic theory of granular gases, for instance, the kinetic theory of frictional granular particles [9–11], binary mixtures [12, 13], polydisperse granular particles [14, 15], and the granular particles with the long range interaction [16]. We do not explain the important properties of the *linearized Boltzmann equation* in this lecture, however, this topic can be seen in the references [4, 17].

Chapter 2

Boltzmann equation

2.1 Distribution functions

If the number of granular particles are large enough, we may adopt the approach of the statistical mechanics for the description of the macroscopic properties of granular gases. Macroscopic properties of molecular gases are governed by the velocity distribution function and this function is determined by a certain integral equation so-called the *Boltzmann equation* first derived by Boltzmann in 1872. Because the collisions between granular particles are inelastic, we have to modify the velocity distribution function to take into account the inelasticity and the Boltzmann equation is also derived in a little different way from the case of molecular gases. At first, we introduce the *single distribution function* such that

$$f(\mathbf{r},\mathbf{v},t)d\mathbf{r}d\mathbf{v} \tag{2.1}$$

is the number of particles placed in a small volume $d\mathbf{r}$ around the position $\mathbf{r}$ and have velocities in a small element $d\mathbf{v}$ around the velocity $\mathbf{v}$ at time t. The number of particles in the system is given by

$$N = \int d\mathbf{r} \int d\mathbf{v} f(\mathbf{r},\mathbf{v},t) , \tag{2.2}$$

where the integrals are defined in $-\infty \le \{\mathbf{r},\mathbf{v}\} \le \infty$. If any particles do not collide during the time interval dt, the position and the velocity are respectively changed to

$$\mathbf{r} \;\to\; \mathbf{r} + \mathbf{v}dt , \tag{2.3}$$

$$\mathbf{v} \;\to\; \mathbf{v} + (\mathbf{F}/m)dt , \tag{2.4}$$

where m is the particle mass and

$$\mathbf{F} = m\frac{\partial \mathbf{v}}{\partial t} \tag{2.5}$$

is the external body force. Because there is no collision, all particles develop as Eqs. (2.3) and (2.4), thus the number of particles Eq. (2.1) does not change, i.e.,

$$\{f(\mathbf{r}+\mathbf{v}dt,\mathbf{v}+\mathbf{F}dt/m,t+dt) - f(\mathbf{r},\mathbf{v},t)\}\, d\mathbf{r}d\mathbf{v} = 0 . \tag{2.6}$$

In acutual case, some particles collide with each other during dt and the left-hand-side of Eq. (2.6) is not equal to zero. The change of the number of particles may be proportional to $d\mathbf{r}d\mathbf{v}dt$ and the left-hand-side of Eq. (2.6) can be written as

$$\{f(\mathbf{r}+\mathbf{v}dt,\mathbf{v}+\mathbf{F}dt/m,t+dt) - f(\mathbf{r},\mathbf{v},t)\}\, d\mathbf{r}d\mathbf{v} = \left(\frac{\partial f}{\partial t}\right)_{\mathrm{col}} d\mathbf{r}d\mathbf{v}dt , \tag{2.7}$$

where $(\partial f/\partial t)_{\mathrm{col}}$ represents the rate of change due to the collisions. Then, we divide Eq. (2.7) by $d\mathbf{r}d\mathbf{v}dt$ and find

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + \frac{\mathbf{F}}{m}\cdot\frac{\partial f}{\partial \mathbf{v}} = \left(\frac{\partial f}{\partial t}\right)_{\mathrm{col}} , \tag{2.8}$$

where $\nabla = \partial/\partial\mathbf{r}$ is the gradient. For simplicity, we do not take into account the external body force $\mathbf{F} = \mathbf{0}$, thus we find

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f = \left(\frac{\partial f}{\partial t}\right)_{\mathrm{col}} . \tag{2.9}$$

In the following, we assume the particles are spherical and identical with the same mass m and the same diameter σ.

For later use, we also introduce the *pair distribution function* such that

$$f(\mathbf{r}_1,\mathbf{v}_1;\mathbf{r}_2,\mathbf{v}_2;t)d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{v}_1 d\mathbf{v}_2 \tag{2.10}$$

is the number of pairs of particles placed in small volumes $d\mathbf{r}_1$ and $d\mathbf{r}_2$ around $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively, and have velocities in small elements $d\mathbf{v}_1$ and $d\mathbf{v}_2$ around $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively, at time t.

2.2 Inelastic collision

The collision between two granular particles is inelastic and the part of the kinetic energy is lost by each collision. Such an inelastic collision is characterized by the normal restitution coefficient e. In general, the normal restitution coefficient depends on the relative velocity and the duration time of colliding two granular particles, however, we consider the simplest case that e is a constant and satisfies $0 < e < 1$. If the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ of two granular particles are respectively changed to $\mathbf{v}'_1$ and $\mathbf{v}'_2$ by the collision, the normal restitution coefficient is defined as

$$\mathbf{g}' \cdot \mathbf{e} = -e(\mathbf{g} \cdot \mathbf{e}) \,, \tag{2.11}$$

where we introduced the relative velocities $\mathbf{g} \equiv \mathbf{v}_1 - \mathbf{v}_2$ and $\mathbf{g}' \equiv \mathbf{v}'_1 - \mathbf{v}'_2$, and $\mathbf{e}$ is the normal unit vector connecting the centers of the colliding two granular particles. In the same way, if the velocities $\mathbf{v}''_1$ and $\mathbf{v}''_2$ are respectively changed to $\mathbf{v}_1$ and $\mathbf{v}_2$ by the collision, we find

$$\mathbf{g}'' \cdot \mathbf{e} = -\frac{1}{e}(\mathbf{g} \cdot \mathbf{e}) \,, \tag{2.12}$$

where $\mathbf{g}'' \equiv \mathbf{v}''_1 - \mathbf{v}''_2$. Then, the tangential components of $\mathbf{g}$ and $\mathbf{g}''$ are given by

$$\begin{aligned} \mathbf{t} &= \mathbf{g} - (\mathbf{g} \cdot \mathbf{e})\mathbf{e} \,, && (2.13) \\ \mathbf{t}'' &= \mathbf{g}'' - (\mathbf{g}'' \cdot \mathbf{e})\mathbf{e} \,, && (2.14) \end{aligned}$$

respectively. If the granular particles are frictionless, the tangential velocity does not change by the collision $\mathbf{t} = \mathbf{t}''$, i.e.,

$$\mathbf{g} - (\mathbf{g} \cdot \mathbf{e})\mathbf{e} = \mathbf{g}'' - (\mathbf{g}'' \cdot \mathbf{e})\mathbf{e} \,. \tag{2.15}$$

Therefore, from Eq. (2.12), we find

$$\mathbf{g}'' = \mathbf{g} - \left(1 + \frac{1}{e}\right)(\mathbf{g} \cdot \mathbf{e})\mathbf{e} \,. \tag{2.16}$$

The velocity of the center-of-mass of the two granular particles $\mathbf{G} \equiv (\mathbf{v}_1 + \mathbf{v}_2)/2$ and $\mathbf{G}'' = (\mathbf{v}''_1 + \mathbf{v}''_2)/2$ also do not change by the collision $\mathbf{G} = \mathbf{G}''$, becuase of the conservation law of the momentum $m\mathbf{v}''_1 + m\mathbf{v}''_2 = m\mathbf{v}_1 + m\mathbf{v}_2$. From the definition, the relation between $\mathbf{G}$ and $\mathbf{g}$ is given by

$$\mathbf{G} = \mathbf{v}_1 - \frac{1}{2}\mathbf{g} \,. \tag{2.17}$$

Then, we convert $d\mathbf{v}_1 d\mathbf{v}_2$ to $d\mathbf{G}d\mathbf{g}$ as

$$d\mathbf{G}d\mathbf{g} = \left|\frac{\partial(\mathbf{G}, \mathbf{g})}{\partial(\mathbf{v}_1, \mathbf{v}_2)}\right| d\mathbf{v}_1 d\mathbf{v}_2 \,, \tag{2.18}$$

where the Jacobian is given by

$$\begin{aligned} \frac{\partial(\mathbf{G}, \mathbf{g})}{\partial(\mathbf{v}_1, \mathbf{v}_2)} &= \frac{\partial(\mathbf{v}_1, \mathbf{g})}{\partial(\mathbf{v}_1, \mathbf{v}_2)} - \frac{1}{2}\frac{\partial(\mathbf{g}, \mathbf{g})}{\partial(\mathbf{v}_1, \mathbf{v}_2)} \\ &= \frac{\partial(\mathbf{v}_1, \mathbf{v}_1)}{\partial(\mathbf{v}_1, \mathbf{v}_2)} - \frac{\partial(\mathbf{v}_1, \mathbf{v}_2)}{\partial(\mathbf{v}_1, \mathbf{v}_2)} \\ &= -\frac{\partial(\mathbf{v}_1, \mathbf{v}_2)}{\partial(\mathbf{v}_1, \mathbf{v}_2)} \\ &= -1 \,. \end{aligned} \tag{2.19}$$

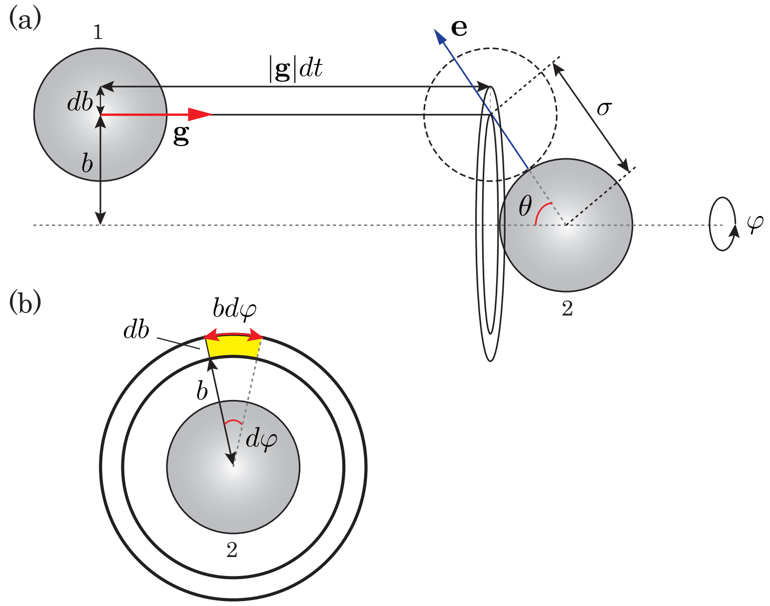

Figure 2.1: Collision cylinder.

Therefore, $d\mathbf{v}_1 d\mathbf{v}_2 = d\mathbf{G}d\mathbf{g}$. In the same way, we also find $d\mathbf{v}_1'' d\mathbf{v}_2'' = d\mathbf{G}'' d\mathbf{g}''$. If we choose the z-axis in the same direction of $\mathbf{e}$, i.e., $\mathbf{e} = (0, 0, 1)$, the components of $\mathbf{g}''$ given by Eq. (2.16) are written as

$$g_x'' = g_x\,, \qquad g_y'' = g_y\,, \qquad g_z'' = g_z - \left(1 + \frac{1}{e}\right) g_z\,, \tag{2.20}$$

respectively, and we find

$$d\mathbf{g}'' = \left|\frac{\partial(\mathbf{g}'')}{\partial(\mathbf{g})}\right| d\mathbf{g} = \mathrm{abs}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/e \end{vmatrix} d\mathbf{g} = \frac{1}{e} d\mathbf{g}\,. \tag{2.21}$$

From Eq. (2.21) and $d\mathbf{G}'' = d\mathbf{G}$, we find $d\mathbf{v}_1'' d\mathbf{v}_2'' = (1/e) d\mathbf{G} d\mathbf{g}$ and the conversion of $d\mathbf{v}_1'' d\mathbf{v}_2''$ into $d\mathbf{v}_1 d\mathbf{v}_2$ is given by

$$d\mathbf{v}_1'' d\mathbf{v}_2'' = \frac{1}{e} d\mathbf{v}_1 d\mathbf{v}_2\,. \tag{2.22}$$

2.3 Boltzmann equation

In this section, we show the microscopic expression of the change rate, $(\partial f/\partial t)_{\mathrm{col}}$, in Eq. (2.9) and derive the Boltzmann equation. Let us consider that the particle 1 with velocity $\mathbf{v}_1$ will collide with the particle 2 with velocity $\mathbf{v}_2$ during the time interval dt, where both of them are placed in a small volume $d\mathbf{r}$ around $\mathbf{r}$. Figure 2.1 shows the configuration of these two particles, where θ is the angle between $\mathbf{g}$ and $\mathbf{e}$, and is defined between $0 \leq \theta \leq \pi/2$ so that the two particles can collide with each other. In this figure, the volume of the *collision cylinder* is given by the area, $bd\varphi \times db$ (the yellow shaded area in Fig. 2.1(b)), multiplied by the height, gdt, i.e.

$$dV = bgdbd\varphi dt = \sigma^2 g \cos\theta \sin\theta d\theta d\varphi dt \equiv \sigma^2 |\mathbf{g}\cdot\mathbf{e}| d\Omega dt\,, \tag{2.23}$$

where we used $b = \sigma \sin\theta$, $db = \sigma\cos\theta d\theta$, and $|\mathbf{g}\cdot\mathbf{e}| = |g\cos(\pi-\theta)| = g\cos\theta$, and introduce a solid angle as $d\Omega \equiv \sin\theta d\theta d\varphi$. We call the particles 1 and 2 *collider* and *scatter*, respectively. The scatters are placed in a volume $d\mathbf{r}$ around $\mathbf{r}$ and have the velocities in an element $d\mathbf{v}_2$ around $\mathbf{v}_2$. On the other hand, the colliders are placed in the collision cylinder dV to collide with the scatters and have the velocities in an element $d\mathbf{v}_1$ around $\mathbf{v}_1$. Then, the number of pairs of such particles are given by using the pair distribution function as

$$f(\mathbf{r},\mathbf{v}_1;\mathbf{r},\mathbf{v}_2;t)dVd\mathbf{r}d\mathbf{v}_1 d\mathbf{v}_2 \ . \tag{2.24}$$

Strictly speaking, the positions of the colliders should be $\mathbf{r}+\sigma\mathbf{e}$ or somewhere in the collision cylinder, however, if the system is dilute, we can assume the velocity distribution function does not change with slightly changing the position $f(\mathbf{r}+\sigma\mathbf{e},\mathbf{v}_1;\mathbf{r},\mathbf{v}_2;t) \simeq f(\mathbf{r},\mathbf{v}_1;\mathbf{r},\mathbf{v}_2;t)$. Moreover, if the system is dilute, we can also assume the correlation between the colliders and the scatters can be negligible. Therefore, the pair distribution function can be decomposed into the single distribution functions as

$$f(\mathbf{r},\mathbf{v}_1;\mathbf{r},\mathbf{v}_2;t) \simeq f(\mathbf{r},\mathbf{v}_1,t)f(\mathbf{r},\mathbf{v}_2,t) \ . \tag{2.25}$$

Above assumptions are not correct if the system is dense and we will discuss the case of

$$f(\mathbf{r}+\sigma\mathbf{e},\mathbf{v}_1;\mathbf{r},\mathbf{v}_2;t) \neq f(\mathbf{r},\mathbf{v}_1,t)f(\mathbf{r},\mathbf{v}_2,t) \ . \tag{2.26}$$

in the final chapter ”Collisional transfer”. From Eqs. (2.23) and (2.25), the number of *direct collisions* Eq. (2.24) is given by

$$f(\mathbf{r},\mathbf{v}_1,t)f(\mathbf{r},\mathbf{v}_2,t)\sigma^2|\mathbf{g}\cdot\mathbf{e}|d\Omega d\mathbf{v}_1 d\mathbf{v}_2 d\mathbf{r}dt \ . \tag{2.27}$$

In the same way, the number of *inverse collisions* is given by

$$f(\mathbf{r},\mathbf{v}_1'',t)f(\mathbf{r},\mathbf{v}_2'',t)\sigma^2|\mathbf{g}''\cdot\mathbf{e}|d\Omega d\mathbf{v}_1'' d\mathbf{v}_2'' d\mathbf{r}dt \ . \tag{2.28}$$

From Eqs. (2.12) and (2.22), Eq. (2.28) is rewritten as

$$\frac{1}{e^2}f(\mathbf{r},\mathbf{v}_1'',t)f(\mathbf{r},\mathbf{v}_2'',t)\sigma^2|\mathbf{g}\cdot\mathbf{e}|d\Omega d\mathbf{v}_1 d\mathbf{v}_2 d\mathbf{r}dt \ . \tag{2.29}$$

The direct collisions contribute to decrease the number of particles $f(\mathbf{r},\mathbf{v}_1,t)d\mathbf{r}d\mathbf{v}_1$ and the inverse collisions contribute to increase $f(\mathbf{r},\mathbf{v}_1,t)d\mathbf{r}d\mathbf{v}_1$. If we integrate $\mathbf{v}_2$ and $\mathbf{e}$ in Eqs. (2.27) and (2.29), the right-hand-side of Eq. (2.7) is given by the difference of them

$$\begin{aligned}\left(\frac{\partial f}{\partial t}\right)_{\mathrm{col}} &= \sigma^2\int d\mathbf{v}_2\int_{\mathbf{g}\cdot\mathbf{e}<0} d\Omega|\mathbf{g}\cdot\mathbf{e}|\left\{\frac{1}{e^2}f(\mathbf{r},\mathbf{v}_1'',t)f(\mathbf{r},\mathbf{v}_2'',t) - f(\mathbf{r},\mathbf{v}_1,t)f(\mathbf{r},\mathbf{v}_2,t)\right\} \\ &\equiv I(f,f) \ ,\end{aligned} \tag{2.30}$$

where we divided the factor $d\mathbf{v}_1 d\mathbf{r}dt$ and the second integral is restricted to the case of $\mathbf{g}\cdot\mathbf{e} < 0$, otherwise the two particles do not collide with each other. Eq. (2.30) is the so-called *collision integrals*. From Eq. (2.9), the Boltzmann equation is given by

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f = I(f,f) \ . \tag{2.31}$$

Clealy, the collision integral $I(f,f)$ satisfies the following algebra

$$I(f+p,f+q) = I(f,f)+I(f,q)+I(p,f)+I(p,q) \ , \tag{2.32}$$

$$I(af,bf) = abI(f,f) \ , \tag{2.33}$$

where p and q are functions of the velocity, and a and b are constants. From Eqs. (2.32) and (2.33), we readily find

$$I(af+bp,cf+dq) = acI(f,f)+adI(f,q)+bcI(p,f)+bdI(p,q) \ , \tag{2.34}$$

where c and d are constants.

2.4 Collision invariant

If we multiply the function of the velocity $\psi(\mathbf{v}_1)$ to $I(f,f)$ and integrate over $\mathbf{v}_1$, we find

$$\begin{aligned}\int d\mathbf{v}_1\psi(\mathbf{v}_1)I(f,f) &= \sigma^2\int d\mathbf{v}_1\int d\mathbf{v}_2\int_{\mathbf{g}\cdot\mathbf{e}<0} d\Omega|\mathbf{g}\cdot\mathbf{e}|\frac{1}{e^2}f(\mathbf{r},\mathbf{v}_1'',t)f(\mathbf{r},\mathbf{v}_2'',t)\psi(\mathbf{v}_1)\\ &\quad -\sigma^2\int d\mathbf{v}_1\int d\mathbf{v}_2\int_{\mathbf{g}\cdot\mathbf{e}<0} d\Omega|\mathbf{g}\cdot\mathbf{e}|f(\mathbf{r},\mathbf{v}_1,t)f(\mathbf{r},\mathbf{v}_2,t)\psi(\mathbf{v}_1)\,. \end{aligned}\tag{2.35}$$

If we notice the relation

$$\frac{1}{e^2}|\mathbf{g}\cdot\mathbf{e}|d\mathbf{v}_1 d\mathbf{v}_2 = |\mathbf{g}''\cdot\mathbf{e}|d\mathbf{v}_1''d\mathbf{v}_2''\,,\tag{2.36}$$

the first term in the reft-hand-side of Eq. (2.35) becomes

$$\sigma^2\int d\mathbf{v}_1''\int d\mathbf{v}_2''\int_{\mathbf{g}''\cdot\mathbf{e}<0} d\Omega|\mathbf{g}''\cdot\mathbf{e}|f(\mathbf{r},\mathbf{v}_1'',t)f(\mathbf{r},\mathbf{v}_2'',t)\psi(\mathbf{v}_1)\,.\tag{2.37}$$

Since the relation $(\mathbf{v}_1'',\mathbf{v}_2'')\to(\mathbf{v}_1,\mathbf{v}_2)$ (the relation between the velocities before the collision and the velocities after the collision) is equivalent to $(\mathbf{v}_1,\mathbf{v}_2)\to(\mathbf{v}_1',\mathbf{v}_2')$, we can rewrite Eq. (2.37) as

$$\sigma^2\int d\mathbf{v}_1\int d\mathbf{v}_2\int_{\mathbf{g}\cdot\mathbf{e}<0} d\Omega|\mathbf{g}\cdot\mathbf{e}|f(\mathbf{r},\mathbf{v}_1,t)f(\mathbf{r},\mathbf{v}_2,t)\psi(\mathbf{v}_1')\,.\tag{2.38}$$

Eq. (2.38) is symmetric about $\mathbf{v}_1$ and $\mathbf{v}_2$ and does not change by exchanging the indices 1 and 2. Thus, Eq. (2.38) is equivalent to

$$\sigma^2\int d\mathbf{v}_1\int d\mathbf{v}_2\int_{\mathbf{g}\cdot\mathbf{e}<0} d\Omega|\mathbf{g}\cdot\mathbf{e}|f(\mathbf{r},\mathbf{v}_1,t)f(\mathbf{r},\mathbf{v}_2,t)\psi(\mathbf{v}_2')\,.\tag{2.39}$$

From Eqs. (2.38) and (2.39), the first term in the reft-hand-side of Eq. (2.35) is rewritten as

$$\frac{\sigma^2}{2}\int d\mathbf{v}_1\int d\mathbf{v}_2\int_{\mathbf{g}\cdot\mathbf{e}<0} d\Omega|\mathbf{g}\cdot\mathbf{e}|f(\mathbf{r},\mathbf{v}_1,t)f(\mathbf{r},\mathbf{v}_2,t)[\psi(\mathbf{v}_1')+\psi(\mathbf{v}_2')]\,,\tag{2.40}$$

which is completely symmetric about the indices 1 and 2. The second term in the reft-hand-side of Eq. (2.35) is also symmetric about $\mathbf{v}_1$ and $\mathbf{v}_2$, and in the same way with Eq. (2.40), this term is also rewritten as

$$\frac{\sigma^2}{2}\int d\mathbf{v}_1\int d\mathbf{v}_2\int_{\mathbf{g}\cdot\mathbf{e}<0} d\Omega|\mathbf{g}\cdot\mathbf{e}|f(\mathbf{r},\mathbf{v}_1,t)f(\mathbf{r},\mathbf{v}_2,t)[\psi(\mathbf{v}_1)+\psi(\mathbf{v}_2)]\,,\tag{2.41}$$

which is also symmetric about the indices 1 and 2. Then, Eq. (2.35) is given by

$$\int d\mathbf{v}_1\psi(\mathbf{v}_1)I(f,f) = \frac{\sigma^2}{2}\int d\mathbf{v}_1\int d\mathbf{v}_2\int_{\mathbf{g}\cdot\mathbf{e}<0} d\Omega|\mathbf{g}\cdot\mathbf{e}|\Delta[\psi(\mathbf{v}_1)+\psi(\mathbf{v}_2)]f_1 f_2\,,\tag{2.42}$$

where we denoted $f_a\equiv f(\mathbf{r},\mathbf{v}_a,t)$ with $a=1,2$ and we introduced $\Delta\psi(\mathbf{v})\equiv\psi(\mathbf{v}')-\psi(\mathbf{v})$. The function $\psi(\mathbf{v})$ defines the set of *collision invariant*

$$\psi(\mathbf{v}) = \left\{m, m\mathbf{v}, \frac{m}{2}\mathbf{v}^2\right\}\,,\tag{2.43}$$

where each component satisfies

$$\Delta\left[m+m\right] = 0\,,\tag{2.44}$$

$$\Delta\left[m\mathbf{v}_1+m\mathbf{v}_2\right] = 0\,,\tag{2.45}$$

$$\Delta\left[\frac{m}{2}\mathbf{v}_1^2+\frac{m}{2}\mathbf{v}_2^2\right] = -\frac{m}{4}(1-e^2)(\mathbf{g}\cdot\mathbf{e})^2\,,\tag{2.46}$$

corresponding to the *conservation of mass*, *the conservation of momentum*, and the *energy balance*, respectively.

Chapter 3

Homogeneous cooling state

3.1 Granular temperature

Even though the granular gases include a huge amount of granular particles, each granular particle is still macroscopic material and is not affected by the *thermal fluctuation*. Therefore, the temperature may not have so much importance in granular gases. However, we can introduce an analogical expression of ”temperature” in granular gases and keep consistency with the molecular gases. If the system is in a homogeneous state, where the density is uniform $n = \text{const}$ and the velocity fields are zero $\mathbf{u} = \mathbf{0}$, the distribution functions are independent on the space and *granular temperature* is defined as the average of the kinetic energy of granular particles

$$\int d\mathbf{v}\frac{m}{2}v^2 f(\mathbf{v},t) = \frac{3}{2}nT(t) \ . \tag{3.1}$$

Due to the inelastic collisions between granular particles, the kinetic energy decreases as time goes on. Thus, the granular temperature continuously decays if any external fields do not exist. *Homogeneous cooling state* is defined with $n = \text{const}$ and $\mathbf{u} = \mathbf{0}$, thus the system behaves like an equilibrium state at each time instant, except for the decay of the granular temperature. Then, the Boltzmann equation is written as

$$\frac{\partial}{\partial t} f(\mathbf{v},t) = I(f,f) \ . \tag{3.2}$$

Because the homogeneous cooling state is governed by the time dependent granular temperature $T(t)$, we scale the velocity of each granular particles by the *thermal velocity* $v_T(t) \equiv \sqrt{2T(t)/m}$ and the distribution function is scaled as

$$f(\mathbf{v},t) \equiv \frac{n}{v_T(t)^3}\tilde{f}(c) \ , \tag{3.3}$$

where we defined the scaled velocity $\mathbf{c} \equiv \mathbf{v}/v_T(t)$ and the absolute value $c \equiv |\mathbf{c}|$.

3.2 Sonine polynomials expansion

If the granular particles are slightly inelastic $e \lesssim 1$, $\tilde{f}(c)$ can be expanded around the Maxwell distribution function. Therefore, we assume $\tilde{f}(c)$ can be expanded into the series of the orthogonal function as

$$\tilde{f}(c) = \phi(c)\left\{1 + \sum_{p=1}^{\infty} a_p S_p(c^2)\right\} \ , \tag{3.4}$$

where the leading term is the Maxwell distribution function

$$\phi(c) \equiv \frac{1}{\pi^{3/2}} \exp(-c^2) \ , \tag{3.5}$$

and the series of the orthogonal function $S_p(x)$ represent the deviation from the elastic gases. In the following, we adopt the Sonine polynomials for the orthogonal function $S_p(x)$. The Sonine polynomials are defined as the associated Laguerre polynomials

$$S_p^{(m)}(x) = \sum_{n=0}^{p} \frac{(-1)^n (m+p)!}{(m+n)!(p-n)!n!} x^n \ , \tag{3.6}$$

where we define $m = d/2-1$ in the d-dimension. In the following, we consider the 3-dimensional case, i.e., $m = 3/2 - 1 = 1/2$ and omit the upperscript such as $S_p(x) \equiv S_p^{(1/2)}(x)$. The Sonine polynomials satisfy the orthogonal condition

$$\int d\mathbf{c}\phi(c) S_p(c^2) S_q(c^2) = \frac{2(p+1/2)!}{\sqrt{\pi}p!}\delta_{pq} \equiv \mathcal{N}_p \delta_{pq} \ , \tag{3.7}$$

where $p, q = 0$ are also included and we notice the expression

$$\left(p + \frac{1}{2}\right)! = \sqrt{\pi} \prod_{k=0}^{p} \left(k + \frac{1}{2}\right) . \tag{3.8}$$

Thus, from Eq. (3.4), the coefficient a_p is given by

$$a_p = \frac{1}{\mathcal{N}_p} \int d\mathbf{c} S_p(c^2) \tilde{f}(c) . \tag{3.9}$$

The first few terms of the Sonine polynomials are given by

$$S_0(x) = 1 , \tag{3.10}$$

$$S_1(x) = -x + \frac{3}{2} , \tag{3.11}$$

$$S_2(x) = \frac{x^2}{2} - \frac{5x}{2} + \frac{15}{8} , \tag{3.12}$$

thun, the powers of the scaled velocity are written as

$$1 = S_0(c^2) , \tag{3.13}$$

$$c^2 = \frac{3}{2} - S_1(c^2) , \tag{3.14}$$

$$c^4 = 2S_2(c^2) - 5S_1(c^2) + \frac{15}{4} , \tag{3.15}$$

respectively. From Eq. (3.10), we also write $\tilde{f}(c)$ in the form

$$\tilde{f}(c) = \phi(c) \sum_{p=0}^{\infty} a_p S_p(c^2) , \tag{3.16}$$

where we defined $a_0 = 1$. Using Eqs. (3.13)-(3.15), we also find the moments of the scaled velocity

$$\langle c^p \rangle \equiv \int d\mathbf{c} c^p \tilde{f}(x) , \tag{3.17}$$

for instance,

$$\begin{aligned}
\langle c^2 \rangle &= \int d\mathbf{c} c^2 \tilde{f}(c) \\
&= \int d\mathbf{c} \phi(c) \left\{ \frac{3}{2} S_0(c^2) - S_1(c^2) \right\} \sum_{p=0}^{\infty} a_p S_p(c^2) \\
&= \frac{3}{2} a_0 \mathcal{N}_0 - a_1 \mathcal{N}_1 \\
&= \frac{3}{2}(1 - a_1) , && (3.18) \\
\langle c^4 \rangle &= \int d\mathbf{c} c^4 \tilde{f}(c) \\
&= \int d\mathbf{c} \phi(c) \left\{ 2S_2(c^2) - 5S_1(c^2) + \frac{15}{4} \right\} \sum_{p=0}^{\infty} a_p S_p(c^2) \\
&= 2a_2 \mathcal{N}_2 - 5a_1 \mathcal{N}_1 + \frac{15}{4} a_0 \mathcal{N}_0 \\
&= \frac{15}{4}(a_2 - 2a_1 + 1) , && (3.19)
\end{aligned}$$

where we used $\mathcal{N}_0 = 1$, $\mathcal{N}_1 = 3/2$ and $\mathcal{N}_2 = 15/8$. Thus, the coefficients a_1 and a_2 are respectively represented by the moments as

$$a_1 = 1 - \frac{2}{3}\langle c^2 \rangle \,, \tag{3.20}$$

$$a_2 = \frac{4}{15}\langle c^4 \rangle - \frac{4}{3}\langle c^2 \rangle + 1 \,. \tag{3.21}$$

3.3 Dimensionless Boltzmann equation

From Eq. (3.3), the left-hand-side of the Boltzmann equation Eq. (2.31) is written as

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{\partial v_{\mathrm{T}}}{\partial t}\frac{\partial}{\partial v_{\mathrm{T}}}\left(\frac{n}{v_{\mathrm{T}}^3}\tilde{f}(c)\right) \\
&= \left(-\frac{3n}{v_{\mathrm{T}}^4}\tilde{f}(c) + \frac{n}{v_{\mathrm{T}}^3}\frac{\partial c}{\partial v_{\mathrm{T}}}\frac{\partial}{\partial c}\tilde{f}(c)\right)\frac{\partial v_{\mathrm{T}}}{\partial t} \\
&= -\frac{n}{v_{\mathrm{T}}^4}\frac{\partial v_{\mathrm{T}}}{\partial t}\left(3 + c\frac{\partial}{\partial c}\right)\tilde{f}(c) \,,
\end{aligned} \tag{3.22}$$

where we used

$$\frac{\partial c}{\partial v_{\mathrm{T}}} = \frac{\partial}{\partial v_{\mathrm{T}}}\left(\frac{v}{v_{\mathrm{T}}}\right) = -\frac{v}{v_{\mathrm{T}}^2} = -\frac{c}{v_{\mathrm{T}}} \,. \tag{3.23}$$

The right-hand-side of the Boltzmann equation Eq. (2.31) is written as

$$\begin{aligned}
I(f,f) &= \sigma^2 v_{\mathrm{T}}^3 v_{\mathrm{T}} \frac{n^2}{v_{\mathrm{T}}^6}\int d\mathbf{c}_2 \int_{\mathbf{c}_{12}\cdot\mathbf{e}<0} d\Omega |\mathbf{c}_{12}\cdot\mathbf{e}| \left\{\frac{1}{e^2}\tilde{f}(\mathbf{c}_1'')\tilde{f}(\mathbf{c}_2'') - \tilde{f}(\mathbf{c}_1)\tilde{f}(\mathbf{c}_2)\right\} \\
&= \frac{\sigma^2 n^2}{v_{\mathrm{T}}^2}\int d\mathbf{c}_2 \int_{\mathbf{c}_{12}\cdot\mathbf{e}<0} d\Omega |\mathbf{c}_{12}\cdot\mathbf{e}| \left\{\frac{1}{e^2}\tilde{f}(\mathbf{c}_1'')\tilde{f}(\mathbf{c}_2'') - \tilde{f}(\mathbf{c}_1)\tilde{f}(\mathbf{c}_2)\right\} \\
&\equiv \frac{\sigma^2 n^2}{v_{\mathrm{T}}^2}\tilde{I}(\tilde{f},\tilde{f}) \,,
\end{aligned} \tag{3.24}$$

where we defined $\mathbf{c}_{12} \equiv \mathbf{c}_1 - \mathbf{c}_2 = v_{\mathrm{T}}\mathbf{g}$ and used $d\mathbf{v}_2 = dv_x dv_y dv_z = v_{\mathrm{T}}^3 dc_x dc_y dc_z = v_{\mathrm{T}}^3 d\mathbf{c}_2$ and $|\mathbf{g}\cdot\mathbf{e}| = v_{\mathrm{T}}|\mathbf{c}_{12}\cdot\mathbf{e}|$. From Eqs. (3.22) and (3.24), the Boltzmann equation is written as

$$-\frac{1}{v_{\mathrm{T}}^2}\frac{\partial v_{\mathrm{T}}}{\partial t}\left(3 + c\frac{\partial}{\partial c}\right)\tilde{f}(c) = \sigma^2 n \tilde{I}(\tilde{f},\tilde{f}) \,. \tag{3.25}$$

From the definition of the granular temperature, we find

$$\begin{aligned}
\frac{d}{dt}\left(\frac{3}{2}nT\right) &= \int d\mathbf{v}_1 \frac{m}{2}v_1^2 \frac{\partial f}{\partial t} \\
&= \int d\mathbf{v}_1 \frac{m}{2}v_1^2 I(f,f) \\
&= \frac{m}{2}v_{\mathrm{T}}^3 \sigma^2 n^2 \int d\mathbf{c}_1 c_1^2 \tilde{I}(\tilde{f},\tilde{f}) \\
&= \sigma^2 n^2 v_{\mathrm{T}} T \int d\mathbf{c}_1 c_1^2 \tilde{I}(\tilde{f},\tilde{f}) \\
&\equiv -\sigma^2 n^2 v_{\mathrm{T}} T \mu_2 \,,
\end{aligned} \tag{3.26}$$

where we defined the moment of the dimensionless collision integral

$$\mu_p \equiv -\int d\mathbf{c}_1 c_1^p \tilde{I}(\tilde{f}, \tilde{f}) \; . \tag{3.27}$$

Therefore, we find the cooling law as

$$\frac{dT}{dt} = -\frac{2}{3}\sigma^2 n v_{\mathrm{T}} T \mu_2 \; . \tag{3.28}$$

From the definition of the thermal velocity $T = mv_{\mathrm{T}}^2/2$, the time derivative of T is given by $dT/dt = mv_{\mathrm{T}} dv_{\mathrm{T}}/dt$. Then, we find

$$\frac{1}{v_{\mathrm{T}}^2}\frac{\partial v_{\mathrm{T}}}{\partial t} = \frac{m}{2T}\frac{1}{mv_{\mathrm{T}}}\frac{\partial T}{\partial t} = \frac{1}{2v_{\mathrm{T}}T}\frac{\partial T}{\partial t} = -\frac{1}{3}\sigma^2 n\mu_2 \; , \tag{3.29}$$

where we used the cooling law Eq. (3.28). From Eq. (3.25), we find the dimensionless Boltzmann equation

$$\frac{\mu_2}{3}\left(3 + c\frac{\partial}{\partial c}\right)\tilde{f}(c) = \tilde{I}(\tilde{f}, \tilde{f}) \; . \tag{3.30}$$

Eq. (2.42) can be also written in the dimensionless form. The left-hand-side of Eq. (2.42) with changing the function $\psi(\mathbf{v}_1) \to \psi(\mathbf{c}_1)$ gives

$$\int d\mathbf{v}_1 \psi(\mathbf{c}_1) I(f, f) = \sigma^2 n^2 v_{\mathrm{T}} \int d\mathbf{c}_1 \psi(\mathbf{c}_1) \tilde{I}(\tilde{f}, \tilde{f}) \; , \tag{3.31}$$

where we used Eq. (3.24). The right-hand-side of Eq. (2.42) with changing the function $\psi(\mathbf{v}_1), \psi(\mathbf{v}_2) \to \psi(\mathbf{c}_1), \psi(\mathbf{c}_2)$ gives

$$\begin{aligned} & \frac{\sigma^2}{2}\int d\mathbf{v}_1 \int d\mathbf{v}_2 \int_{\mathbf{g}\cdot\mathbf{e}<0} d\Omega |\mathbf{g}\cdot\mathbf{e}| \Delta[\psi(\mathbf{c}_1) + \psi(\mathbf{c}_2)] f_1 f_2 \\ = \; & \frac{1}{2}\sigma^2 n^2 v_{\mathrm{T}} \int d\mathbf{c}_1 \int d\mathbf{c}_2 \int_{\mathbf{c}_{12}\cdot\mathbf{e}<0} d\Omega |\mathbf{c}_{12}\cdot\mathbf{e}| \Delta[\psi(\mathbf{c}_1) + \psi(\mathbf{c}_2)] \tilde{f}(\mathbf{c}_1)\tilde{f}(\mathbf{c}_2) \; . \end{aligned} \tag{3.32}$$

Thus, we find

$$\int d\mathbf{c}_1 \psi(\mathbf{c}_1) \tilde{I}(\tilde{f}, \tilde{f}) = \frac{1}{2}\int d\mathbf{c}_1 \int d\mathbf{c}_2 \int_{\mathbf{c}_{12}\cdot\mathbf{e}<0} d\Omega |\mathbf{c}_{12}\cdot\mathbf{e}| \Delta[\psi(\mathbf{c}_1) + \psi(\mathbf{c}_2)] \tilde{f}(\mathbf{c}_1)\tilde{f}(\mathbf{c}_2) \; . \tag{3.33}$$

3.4 Dimensionless moments

The coefficients of the Sonine polynomials are given by the moments of the scaled velocity as Eqs. (3.20) and (3.21). At first, we find

$$\int d\mathbf{v}\frac{m}{2}v^2 f = n\frac{m}{2}v_{\mathrm{T}}^2 \int d\mathbf{c} c^2 \tilde{f}(\mathbf{c}) = n\frac{m}{2}v_{\mathrm{T}}^2 \langle c^2 \rangle = nT\langle c^2 \rangle \; . \tag{3.34}$$

On the other hand, we also find the following relation from the definition of the granular temperature

$$\int d\mathbf{v}\frac{m}{2}v^2 f = \frac{3}{2}nT \; . \tag{3.35}$$

Eqs. (3.34) and (3.35) are the same equations, thus we find

$$\langle c^2 \rangle = \frac{3}{2} \; . \tag{3.36}$$

From Eq. (3.20),

$$a_1 = 1 - \frac{2}{3}\langle c^2\rangle = 0\,. \tag{3.37}$$

Therefore, a_2 may be the first non-zero coefficient of the expansion Eq. (3.4). To calculate a_2, let us multiply c^p to the dimensionless Boltzmann equation Eq. (3.30) and integrate over $\mathbf{c}_1$. From the definition of the moment μ_p Eq. (3.27), the right-hand-side gives $-\mu_p$. The left-hand-side is written as

$$\frac{\mu_2}{3}\int d\mathbf{c}c^p\left(3 + c\frac{\partial}{\partial c}\right)\tilde{f}(c)\,, \tag{3.38}$$

where the first term is given by

$$\mu_2\int d\mathbf{c}c^p\tilde{f}(c) = \mu_2\langle c^p\rangle\,, \tag{3.39}$$

and the second term is given by

$$\begin{aligned}
\int d\mathbf{c}c^{p+1}\frac{\partial}{\partial c}\tilde{f}(c) &= \int_0^{2\pi}d\varphi\int_0^{\pi}\sin\theta d\theta\int_0^{\infty}c^{p+3}\frac{\partial}{\partial c}\tilde{f}(c)dc \\
&= 4\pi\left\{\left[c^{p+3}\tilde{f}(c)\right]_0^{\infty} - (p+3)\int_0^{\infty}c^{p+2}\tilde{f}(c)dc\right\} \\
&= -4\pi(p+3)\int_0^{\infty}c^{p+2}\tilde{f}(c)dc \\
&= -(p+3)\int d\mathbf{c}c^p\tilde{f}(c) \\
&= -(p+3)\langle c^p\rangle\,,
\end{aligned} \tag{3.40}$$

where we used the spherical coordinate in the integral $\mathbf{c}$ and assume that $\tilde{f}(c)$ decays much faster than any powers of c, i.e., $\lim_{c\to\infty}c^p\tilde{f}(c) = 0$. We also notice the relation

$$4\pi\int dcc^2\cdots = \int d\mathbf{c}\ldots\,. \tag{3.41}$$

Then, from Eq. (3.38), we find

$$\frac{p}{3}\mu_2\langle c^p\rangle = \mu_p\,. \tag{3.42}$$

In Eq. (3.42), $p = 2$ is trivial and $p = 4$ gives

$$\mu_4 = \frac{4}{3}\mu_2\langle c^4\rangle = 5\mu_2(a_2 + 1)\,, \tag{3.43}$$

where we used Eq. (3.21) with $a_1 = 0$. Therefore, the moments μ_2 and μ_4 give the first non-zero coefficient a_2.

3.5 Kinetic integrals

To determine the Sonine coefficients, we need to evaluate the moment μ_p. From Eq. (3.33), μ_p is given by

$$\begin{aligned}
\mu_p &= -\int d\mathbf{c}_1\mathbf{c}_1^p\tilde{I}(\tilde{f},\tilde{f}) \\
&= -\frac{1}{2}\int d\mathbf{c}_1\int d\mathbf{c}_2\int_{\mathbf{c}_{12}\cdot\mathbf{e}<0}d\Omega|\mathbf{c}_{12}\cdot\mathbf{e}|\Delta[\mathbf{c}_1^p + \mathbf{c}_2^p]\tilde{f}(\mathbf{c}_1)\tilde{f}(\mathbf{c}_2)\,.
\end{aligned} \tag{3.44}$$

We assume the coefficient a_2 is small and the distribution function can be well described by the linear approximation

$$\tilde{f}(c) \simeq \phi(c)[1 + a_2 S_2(c^2)] \,. \tag{3.45}$$

Then, if we neglect the second order terms of a_2, Eq. (3.44) can be written as

$$\mu_p \simeq -\frac{1}{2}\int d\mathbf{c}_1 \int d\mathbf{c}_2 \int_{\mathbf{c}_{12}\cdot\mathbf{e}<0} d\Omega|\mathbf{c}_{12}\cdot\mathbf{e}|\phi(c_1)\phi(c_2)\left[1 + a_2\left\{S_2(c_1^2) + S_2(c_2^2)\right\}\right]\Delta[\mathbf{c}_1^p + \mathbf{c}_2^p] \,. \tag{3.46}$$

Let us evaluate Eq. (3.46). At first, we transform the velocity $\mathbf{c}_1$ and $\mathbf{c}_2$ to the velocity of the center of mass $\mathbf{C} \equiv (\mathbf{c}_1 + \mathbf{c}_2)/2$ and the relative velocity $\mathbf{c}_{12} = \mathbf{c}_1 - \mathbf{c}_2$. In the same way with Eq. (2.18), the Jacobian is unity

$$d\mathbf{c}_1 d\mathbf{c}_2 = d\mathbf{C} d\mathbf{c}_{12} \,. \tag{3.47}$$

The Gaussian distribution functions can be rewritten as

$$\phi(c_1)\phi(c_2) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{1}{2}c_{12}^2}\left(\frac{2}{\pi}\right)^{3/2} e^{-2C^2} \equiv \phi(c_{12})\phi(C) \,. \tag{3.48}$$

The Sonine polynomials can be rewritten as

$$\left[1 + a_2\left\{S_2(c_1^2) + S_2(c_2^2)\right\}\right] = C^4 + \frac{1}{2}C^2 c_{12}^2 + \frac{1}{16}c_{12}^4 + (\mathbf{C}\cdot\mathbf{c}_{12})^2 - 5C^2 - \frac{5}{4}c_{12}^2 + \frac{15}{4} \,. \tag{3.49}$$

We also find

$$\Delta[\mathbf{c}_1^2 + \mathbf{c}_2^2] = -\frac{1}{2}(1 - e^2)(\mathbf{c}_{12}\cdot\mathbf{e})^2 \,, \tag{3.50}$$

$$\begin{aligned}\Delta[\mathbf{c}_1^4 + \mathbf{c}_2^4] &= 2(1 + e)^2(\mathbf{c}_{12}\cdot\mathbf{e})^2(\mathbf{C}\cdot\mathbf{e})^2 + \frac{1}{8}(1 - e^2)^2(\mathbf{c}_{12}\cdot\mathbf{e})^4 \\ &\quad -\frac{1}{4}(1 - e^2)(\mathbf{c}_{12}\cdot\mathbf{e})^2 c_{12}^2 - (1 - e^2)C^2(\mathbf{c}_{12}\cdot\mathbf{e})^2 \\ &\quad -4(1 + e)(\mathbf{C}\cdot\mathbf{c}_{12})(\mathbf{C}\cdot\mathbf{e})(\mathbf{c}_{12}\cdot\mathbf{e}) \,,\end{aligned} \tag{3.51}$$

respectively. Then, it is readily found that Eq. (3.46) is given by the combination of the *kinetic integrals*

$$J_{k,l,m,n,q} \equiv \int d\mathbf{C} \int d\mathbf{c}_{12} \int_{\mathbf{c}_{12}\cdot\mathbf{e}<0} d\Omega|\mathbf{c}_{12}\cdot\mathbf{e}|\phi(c_{12})\phi(C)C^k c_{12}^l (\mathbf{C}\cdot\mathbf{c}_{12})^m (\mathbf{C}\cdot\mathbf{e})^n (\mathbf{c}_{12}\cdot\mathbf{e})^q \,. \tag{3.52}$$

For example, $J_{k,l,m,0,q}$, $J_{k,l,m,1,q}$ and $J_{k,l,m,2,q}$ are respectively given by the formulae

$$\begin{aligned}J_{k,l,m,0,q} &= \frac{8(-1)^q 2^{(-k+l+q-1)/2}}{(q+2)(m+1)}[1 - (-1)^{m+1}]\Gamma\left(\frac{k+m+3}{2}\right)\Gamma\left(\frac{l+m+q+4}{2}\right) \,, \\ J_{k,l,m,1,q} &= \frac{4(-1)^{q+1} 2^{(-k+l+q)/2}}{(q+3)(m+2)}[1 - (-1)^{m}]\Gamma\left(\frac{k+m+4}{2}\right)\Gamma\left(\frac{l+m+q+4}{2}\right) \,, \\ J_{k,l,m,2,q} &= \frac{4(-1)^q 2^{(-k+l+q-1)/2}}{(q+4)(p+2)}[1 - (-1)^{m+1}]\Gamma\left(\frac{k+m+5}{2}\right)\Gamma\left(\frac{l+m+q+4}{2}\right)\left(\frac{q+1}{m+3} + \frac{1}{m+1}\right) \,.\end{aligned}$$

Then, μ_2 is evaluated as

$$\begin{aligned}\mu_2 &= \frac{1 - e^2}{4}\Big[J_{0,0,0,0,2} + a_2\Big(J_{4,0,0,0,2} + J_{0,0,2,0,2} + \frac{1}{16}J_{0,4,0,0,2} \\ &\quad + \frac{1}{2}J_{2,2,0,0,2} - 5J_{2,0,0,0,2} - \frac{5}{4}J_{0,2,0,0,2} + \frac{15}{4}J_{0,0,0,0,2}\Big)\Big] \\ &= \sqrt{2\pi}(1 - e^2)\left(1 + \frac{3}{16}a_2 + O(a_2^2)\right) \,.\end{aligned} \tag{3.53}$$

In the same way, μ_4 is evaluated as

$$\mu_4 = 4\sqrt{2\pi}\left[\frac{1-e^2}{4}\left(\frac{9}{2}+e^2\right)+a_2\left\{\frac{3(1-e^2)}{128}(69+10e^2)+\frac{1}{2}(1+e)\right\}+O(a_2^2)\right] . \tag{3.54}$$

Substituting Eqs. (3.53) and (3.54) to Eq. (3.43) and neglecting the second order terms of a_2, we find the second Sonine coefficient

$$a_2 = \frac{16(1-e)(1-2e^2)}{81-17e+30e^2(1-e)} . \tag{3.55}$$

Chapter 4

Inhomogeneous granular gases

4.1 Hydrodynamic equations of granular gases

Although our system of granular gases is discretized into the individual particles, the system includes a number of granular particles and the macroscopic aspects of the granular gases can be described by the continuum description. We adopt the local averages of the collision invariant $\psi(\mathbf{v})$ for the continuum variables, since these averages give the hydrodynamic fields, i.e., the *density field*, the *velocity field* and the *granular temperature*, respectively. From the definition of the velocity distribution function Eq. (2.2), the density field is defined as

$$n(\mathbf{r}, t) = \int f(\mathbf{r}, \mathbf{v}, t) d\mathbf{v} , \tag{4.1}$$

and any macroscopic fields $A(\mathbf{r}, t)$ are given by the average of the function of $\mathbf{v}$ as

$$A(\mathbf{r}, t) = \frac{\int \psi(\mathbf{v}) f(\mathbf{r}, \mathbf{v}, t) d\mathbf{v}}{\int f(\mathbf{r}, \mathbf{v}, t) d\mathbf{v}} = \frac{1}{n(\mathbf{r}, t)} \int \psi(\mathbf{v}) f(\mathbf{r}, \mathbf{v}, t) d\mathbf{v} . \tag{4.2}$$

Then, the velocity field is defined as

$$\mathbf{u}(\mathbf{r}, t) = \frac{1}{n(\mathbf{r}, t)} \int \mathbf{v} f(\mathbf{r}, \mathbf{v}, t) d\mathbf{v} , \tag{4.3}$$

and the granular temperature is defined as

$$\frac{d}{2} T(\mathbf{r}, t) = \frac{1}{n(\mathbf{r}, t)} \int \frac{1}{2} m V^2 f(\mathbf{r}, \mathbf{v}, t) d\mathbf{v} \tag{4.4}$$

in d-dimension, where the local velocity is introduced as $\mathbf{V} \equiv \mathbf{v} - \mathbf{u}(\mathbf{r}, t)$.

4.1.1 Continuum equation

Continuum equation is the result of the mass conservation. Let us multiply 1 (or m) to the Boltzmann equation Eq. (2.31) and integrate $\mathbf{v}$. Then, the right-hand-side is equal to zero because of Eq. (2.44) and the left-hand-side is given by

$$\begin{aligned} \int d\mathbf{v} \frac{\partial}{\partial t} f + \int d\mathbf{v} v_i \nabla_i f &= \frac{\partial}{\partial t} \int d\mathbf{v} f + \nabla_i \int d\mathbf{v} v_i f \\ &= \frac{\partial n}{\partial t} + \nabla_i (n u_i) , \end{aligned} \tag{4.5}$$

where we used the summation rule for the twice appearance of the indices. We notice the time derivative $\partial/\partial t$ and the integral of $\mathbf{v}$ can be exchanged and the gradient ∇_i does not act on v_i. Thus, the continuum equation is derived as

$$\frac{\partial n}{\partial t} + \nabla_i (n u_i) = 0 . \tag{4.6}$$

4.1.2 Equation of motion

Equation of motion is the result of the conservation of momentum. Let us multiply $m v_i$ to the Boltzmann equation and integrate $\mathbf{v}$. Then, the right-hand-side is equal to zero because of Eq. (2.45) and the left-hand-side is given by

$$\int d\mathbf{v} m v_i \frac{\partial}{\partial t} f + \int d\mathbf{v} m v_i v_j \nabla_j f = m \frac{\partial}{\partial t} \int d\mathbf{v} v_i f - m \int d\mathbf{v} \frac{\partial v_i}{\partial t} f + \nabla_j \int d\mathbf{v} m v_i v_j f . \tag{4.7}$$

The first term in the right-hand-side of Eq. (4.7) gives

$$\begin{aligned} m\frac{\partial}{\partial t}\int d\mathbf{v}v_i f &= m\frac{\partial}{\partial t}(nu_i) \\ &= mn\frac{\partial u_i}{\partial t} + mu_i\frac{\partial n}{\partial t} \\ &= mn\frac{\partial u_i}{\partial t} - mu_i\nabla_j(nu_j) , \end{aligned} \tag{4.8}$$

where we used Eqs. (4.3) and (4.6). The second term in the right-hand-side of Eq. (4.7) gives

$$m\int d\mathbf{v}\frac{\partial v_i}{\partial t} f = \int d\mathbf{v}F_i f = 0 , \tag{4.9}$$

becuase the external body force is zero $F_i = 0$. The last term in the right-hand-side of Eq. (4.7) can be written as

$$\begin{aligned} \nabla_j\int d\mathbf{v}mv_iv_j f &= \nabla_j\int d\mathbf{v}m(V_i+u_i)(V_j+u_j)f \\ &= \nabla_j\int d\mathbf{v}mV_iV_j f + \nabla_j(mnu_iu_j) + m\nabla_j u_j\int d\mathbf{v}V_i f + m\nabla_j u_i\int d\mathbf{v}V_j f \\ &= \nabla_j P_{ij} + mnu_j\nabla_j u_i + mu_i\nabla_j(nu_j) , \end{aligned} \tag{4.10}$$

where the average of the local velocity is zero

$$\int d\mathbf{v}V_i f = \int d\mathbf{v}v_i f - u_i\int d\mathbf{v} f = nu_i - nu_i = 0 , \tag{4.11}$$

and we defined the *stress tensor*

$$P_{ij} \equiv \int d\mathbf{v}mV_iV_j f . \tag{4.12}$$

From Eqs. (4.8) and (4.10), we obtain the equation of motion

$$\frac{\partial u_i}{\partial t} + u_j\nabla_j u_i = -\frac{1}{nm}\nabla_j P_{ij} , \tag{4.13}$$

where the term $mu_i\nabla_j(nu_j)$ was canceled and we divided nm. The stress tensor P_{ij} can be also expressed as

$$\begin{aligned} P_{ij} &= \frac{1}{3}\delta_{ij}\int d\mathbf{v}mV^2 f + \int d\mathbf{v}m\left(V_iV_j - \frac{1}{3}\delta_{ij}V^2\right) f \\ &\equiv nT\delta_{ij} + \int d\mathbf{v}D_{ij} f , \end{aligned} \tag{4.14}$$

where we defined the deviatric part of the stress tensor

$$D_{ij} = m\left(V_iV_j - \frac{1}{3}\delta_{ij}V^2\right) , \tag{4.15}$$

and $p \equiv nT$ is the hydrostatic pressure.

4.1.3 Equation of energy

Equation of energy describes the time development of the granular temperature. In the case of molecular gases, the total energy of the system is conserved, however, in the case of the granular gases, the kinetic energy decreases by inelastic collisions and the total energy is not conserved. Let us multiply $mV^2/2$ to the Boltzmann equation and integrate $\mathbf{v}$ so as we can derive the equation of energy.

Right-hand-side

The right-hand-side is given by

$$(\text{r.h.s.}) = \int d\mathbf{v}\frac{m}{2}V^2 I(f,f) \equiv -\frac{3}{2}nT\zeta(\mathbf{r},t)\,, \tag{4.16}$$

where we defined the *cooling rate* as

$$\zeta(\mathbf{r},t) \equiv -\frac{m}{3nT}\int d\mathbf{v}V^2 I(f,f)\,. \tag{4.17}$$

Because of $V^2 = v^2 - 2v_i u_i + u^2$, we find

$$\begin{aligned}\int d\mathbf{v}V^2 I(f,f) &= \int d\mathbf{v}v^2 I(f,f) - 2u_i\int d\mathbf{v}v_i I(f,f) + u^2\int d\mathbf{v}I(f,f)\\ &= \int d\mathbf{v}v^2 I(f,f)\,,\end{aligned} \tag{4.18}$$

where we used Eqs. (2.42), (2.44) and (2.45). Then, the cooling rate is rewritten as

$$\begin{aligned}\zeta(\mathbf{r},t) &= -\frac{m}{3nT}\int d\mathbf{v}v^2 I(f,f)\\ &= (1-e^2)\frac{m\sigma^2}{12nT}\int d\mathbf{v}_1\int d\mathbf{v}_2\int_{\mathbf{g}\cdot\mathbf{e}<0} d\Omega|\mathbf{g}\cdot\mathbf{e}|(\mathbf{g}\cdot\mathbf{e})^2 f_1 f_2\\ &= (1-e^2)\frac{\pi m\sigma^2}{24nT}\int d\mathbf{v}_1\int d\mathbf{v}_2 g^3 f_1 f_2\,,\end{aligned} \tag{4.19}$$

where we used Eqs. (2.42) and (2.46), and the integral of Ω was calculated as follows. The inner product is given by $\mathbf{g}\cdot\mathbf{e} = -g\cos\theta < 0$ $(0 \le \theta \le \pi/2)$. Then, the integral over Ω becomes

$$\begin{aligned}\int_{\mathbf{g}\cdot\mathbf{e}<0} d\Omega|\mathbf{g}\cdot\mathbf{e}|(\mathbf{g}\cdot\mathbf{e})^2 &= g^3\int_0^{2\pi}d\varphi\int_0^{\pi/2}d\theta\sin\theta\cos^3\theta\\ &= 2\pi g^3\int_{\pi/2}^{0}\cos^3\theta(-\sin\theta d\theta)\\ &= 2\pi g^3\int_0^1\chi^3 d\chi\\ &= \frac{\pi}{2}g^3\,,\end{aligned} \tag{4.20}$$

where we defined $\chi = \cos\theta$ which gives $d\chi = -\sin\theta d\theta$.

Left-hand-side

The left-hand-side is given by

$$(\text{l.h.s.}) = \int d\mathbf{v}\frac{m}{2}V^2\frac{\partial}{\partial t}f + \int d\mathbf{v}\frac{m}{2}V^2 v_i\nabla_i f\,. \tag{4.21}$$

The first term of Eq. (4.21) is written as

$$\begin{aligned}\int d\mathbf{v}\frac{m}{2}V^2\frac{\partial}{\partial t}f &= \frac{\partial}{\partial t}\left(\int d\mathbf{v}\frac{m}{2}V^2 f\right) - \int d\mathbf{v}\frac{m}{2}\left(\frac{\partial}{\partial t}V^2\right)f\\ &= \frac{\partial}{\partial t}\left(\frac{3}{2}nT\right) - \int d\mathbf{v}mV_i\frac{\partial V_i}{\partial t}f\\ &= \frac{\partial}{\partial t}\left(\frac{3}{2}nT\right)\,,\end{aligned} \tag{4.22}$$

where the second term was vanished as

$$\int d\mathbf{v} m V_i \frac{\partial V_i}{\partial t} f = \int d\mathbf{v} m V_i \left\{ \frac{\partial}{\partial t}(v_i - u_i) \right\} f = \int d\mathbf{v} V_i F_i f - m\frac{\partial u_i}{\partial t} \int d\mathbf{v} V_i f = 0 \,, \tag{4.23}$$

because of $F_i = 0$ and Eq. (4.11). The second term of Eq. (4.21) is written as

$$\begin{aligned} \int d\mathbf{v} \frac{m}{2} V^2 v_i \nabla_i f &= \int d\mathbf{v} \frac{m}{2} V^2 V_i \nabla_i f + \int d\mathbf{v} \frac{m}{2} V^2 u_i \nabla_i f \\ &= \nabla_i \left(\int d\mathbf{v} \frac{m}{2} V^2 V_i f \right) - \int d\mathbf{v} \frac{m}{2} \left(\nabla_i V^2 V_i \right) f + \int d\mathbf{v} \frac{m}{2} V^2 u_i \nabla_i f \\ &= \nabla_i q_i - \int d\mathbf{v} \frac{m}{2} \left(\nabla_i V^2 V_i \right) f + \int d\mathbf{v} \frac{m}{2} V^2 u_i \nabla_i f \,, \end{aligned} \tag{4.24}$$

where we defined the *heat flux* as

$$q_i \equiv \int d\mathbf{v} \frac{m}{2} V^2 V_i f \,. \tag{4.25}$$

If we notice the relation

$$\begin{aligned} \nabla_i V^2 V_i &= \nabla_i (V_x^2 V_i + V_y^2 V_i + V_z^2 V_i) \\ &= 2V_x V_i \nabla_i V_x + 2V_y V_i \nabla_i V_y + 2V_z V_i \nabla_i V_z + V_x^2 \nabla_i V_i + V_y^2 \nabla_i V_i + V_z^2 \nabla_i V_i \\ &= 2V_i V_j \nabla_i V_j + V^2 \nabla_i V_i \,, \end{aligned} \tag{4.26}$$

Eq. (4.24) is written as

$$\begin{aligned} & \nabla_i q_i - \int d\mathbf{v} m V_i V_j (\nabla_i V_j) f - \int d\mathbf{v} \frac{m}{2} V^2 (\nabla_i V_i) f + \int d\mathbf{v} \frac{m}{2} V^2 u_i \nabla_i f \\ =\; & \nabla_i q_i + (\nabla_i u_j) \int d\mathbf{v} m V_i V_j f + (\nabla_i u_i) \int d\mathbf{v} \frac{m}{2} V^2 f + \int d\mathbf{v} \frac{m}{2} V^2 u_i \nabla_i f \\ =\; & \nabla_i q_i + P_{ij} \nabla_i u_j + \frac{3}{2} nT \nabla_i u_i + \int d\mathbf{v} \frac{m}{2} V^2 u_i \nabla_i f \,, \end{aligned} \tag{4.27}$$

where we notice the relations $\nabla_i V_j = -\nabla_i u_j$ and $\nabla_i V_i = -\nabla_i u_i$, and used the definition of the stress tensor Eq. (4.12). The last term of Eq. (4.27) is rewritten as

$$\begin{aligned} \int d\mathbf{v} \frac{m}{2} V^2 u_i \nabla_i f &= u_i \nabla_i \left(\int d\mathbf{v} \frac{m}{2} V^2 f \right) - u_i \int d\mathbf{v} \frac{m}{2} (\nabla_i V^2) f \\ &= u_i \nabla_i \left(\frac{3}{2} nT \right) - \frac{m}{2} u_i \int d\mathbf{v} (\nabla_i V^2) f \,. \end{aligned} \tag{4.28}$$

If we notice the relation

$$\begin{aligned} \nabla_i V^2 &= \nabla_i (V_x^2 + V_y^2 + V_z^2) \\ &= 2V_x \nabla_i V_x + 2V_y \nabla_i V_y + 2V_z \nabla_i V_z \\ &= 2V_j \nabla_i V_j \,, \end{aligned} \tag{4.29}$$

the second term of Eq. (4.28) is vanihsed as

$$\frac{m}{2} u_i \int d\mathbf{v} (\nabla_i V^2) f = m u_i \int d\mathbf{v} V_j (\nabla_i V_j) f = -m u_i (\nabla_i u_j) \int d\mathbf{v} V_j f = 0 \,, \tag{4.30}$$

where we used $\nabla_i V_j = -\nabla_i u_j$ and Eq. (4.11). From Eqs. (4.22), (4.27) and (4.30), we find that the left-hand-side Eq. (4.21) is given by

$$\begin{aligned} (\text{l.h.s.}) &= \frac{\partial}{\partial t} \left(\frac{3}{2} nT \right) + \nabla_i q_i + P_{ij} \nabla_i u_j + \frac{3}{2} nT \nabla_i u_i + u_i \nabla_i \left(\frac{3}{2} nT \right) \\ &= \frac{\partial}{\partial t} \left(\frac{3}{2} nT \right) + \nabla_i q_i + P_{ij} \nabla_i u_j + \nabla_i \left(\frac{3}{2} nT u_i \right) \,. \end{aligned} \tag{4.31}$$

If we notice

$$\frac{\partial}{\partial t}\left(\frac{3}{2}nT\right) = \frac{3}{2}n\frac{\partial T}{\partial t} + \frac{3}{2}T\frac{\partial n}{\partial t}$$
$$= \frac{3}{2}n\frac{\partial T}{\partial t} - \frac{3}{2}T\nabla_i(nu_i)\ , \tag{4.32}$$
$$\nabla_i\left(\frac{3}{2}nTu_i\right) = \frac{3}{2}nu_i\nabla_i T + \frac{3}{2}T\nabla_i(nu_i)\ , \tag{4.33}$$

where we used the continuum equation Eq. (4.6), Eq. (4.31) is rewritten as

$$\text{(l.h.s.)} = \frac{3}{2}n\frac{\partial T}{\partial t} + \frac{3}{2}nu_i\nabla_i T + \nabla_i q_i + P_{ij}\nabla_i u_j\ . \tag{4.34}$$

From Eqs. (4.16) and (4.34), the equation of energy is obtained as

$$\frac{\partial T}{\partial t} + u_i\nabla_i T = -\frac{2}{3n}\left(\nabla_i q_i + P_{ij}\nabla_i u_j\right) - \zeta T\ . \tag{4.35}$$

It should be noticed, if we use Eq. (4.11), the heat flux q_i can be also written as

$$q_i = \int d\mathbf{v} S_i f\ , \tag{4.36}$$

where we defined

$$S_i \equiv \left(\frac{m}{2}V^2 - \frac{5}{2}T\right)V_i\ , \tag{4.37}$$

and the second term of S_i is vanished because of $(5T/2)\int \mathbf{v}V_i f = 0$.

4.1.4 Phenomenological transport coefficients

In linear approximation with respect to the gradient ∇, the stress tensor and the heat flux in d-dimension are given in the phenomenological expressions as

$$P_{ij} = p\delta_{ij} - \eta\left(\nabla_i u_j + \nabla_j u_i - \frac{2}{d}\delta_{ij}\nabla_k u_k\right)\ , \tag{4.38}$$
$$q_i = -\kappa\nabla_i T - \mu\nabla_i n\ , \tag{4.39}$$

respectively, where p, η, κ, and μ are the *hydrostatic pressure*, the *shear viscosity*, the *thermal conductivity*, and the coefficient of the density gradient, respectively. The coefficient μ does not have an analogue in the usual hydrodynamics, becuase the heat flux $-\mu\nabla_i n$ is the results of the inelastic collisions of the granular particles, where the collision frequency in dense region is higher than that in dilute region and the heat flows from the dilute region to the dense region. By using the expressions Eqs. (4.38) and (4.39), the gradient of the stress tensor is given by

$$\nabla_j P_{ij} = \delta_{ij}\nabla_j p - \eta\left(\nabla_j\nabla_i u_j + \nabla_j\nabla_j u_i - \frac{2}{3}\delta_{ij}\nabla_j\nabla_k u_k\right)$$
$$= \nabla_i p - \eta\left(\nabla_i\nabla_j u_j + \nabla^2 u_i - \frac{2}{3}\nabla_i\nabla_k u_k\right)$$
$$= \nabla_i p - \eta\left\{\nabla^2 u_i + \frac{1}{3}\nabla_i(\nabla_j u_j)\right\}\ , \tag{4.40}$$

where we introduced the Laplacian ∇^2, and the gradient of the heat flux is given by

$$\nabla_i q_i = -\kappa\nabla^2 T - \mu\nabla^2 n\ , \tag{4.41}$$

and the second term in the right-hand-side of Eq. (4.35) is given by

$$\begin{aligned} P_{ij}\nabla_i u_j &= p\delta_{ij}\nabla_i u_j - \eta\left\{(\nabla_i u_j)(\nabla_i u_j) + (\nabla_j u_i)(\nabla_i u_j) - \frac{2}{3}\delta_{ij}(\nabla_k u_k)(\nabla_i u_j)\right\} \\ &= p\nabla_i u_i - \eta\left\{(\nabla_i u_j)(\nabla_i u_j) + (\nabla_j u_i)(\nabla_i u_j) - \frac{2}{3}(\nabla_i u_i)^2\right\} . \end{aligned} \quad (4.42)$$

4.2 Chapman-Enskog theory

We develop the Boltzmann theory explained in the previous sections to slightly non-uniform gases. The aim of the Chapman-Enskog theory is to construct a perturbative expansion of the velocity distribution function and gives the microscopic expressions of the phenomenological transport coefficients η, κ and μ. The first step of the Chapman-Enskog theory is to assume the velocity distribution $f(\mathbf{r}, \mathbf{v}, t)$ develops in the long time scale and spatially changes in the long wave length, which means the time development and the spatial changes of the velocity distribution function happen through the hydrodynamic fields $n(\mathbf{r}, t)$, $\mathbf{u}(\mathbf{r}, t)$ and $T(\mathbf{r}, t)$. Therefore, the time derivative and the gradient of $f(\mathbf{r}, \mathbf{v}, t)$ are given by

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial n}\frac{\partial n}{\partial t} + \frac{\partial f}{\partial u_i}\frac{\partial u_i}{\partial t} + \frac{\partial f}{\partial T}\frac{\partial T}{\partial t} , \quad (4.43)$$

$$\nabla f = \frac{\partial f}{\partial n}\nabla n + \frac{\partial f}{\partial u_i}\nabla u_i + \frac{\partial f}{\partial T}\nabla T , \quad (4.44)$$

respectively. Such dependence of the velocity distribution function on the hydrodynamic fields can be justified if the spatial gradient of non-uniformity is small enough. Then, we introduce the small parameter λ as the measure of the gradient

$$\lambda \sim O(k) \ll 1 , \quad (4.45)$$

where k is the wave number which is the same order with the gradient $\nabla = i\mathbf{k}$. Thus, we scale the gradient as

$$\nabla \longrightarrow \lambda\nabla . \quad (4.46)$$

Generally, the dispersion relation gives the frequency as a function of the wave number $\omega(k)$, and if $k \ll 1$, we can expand $\omega(k)$ into the series of k as

$$\begin{aligned} \omega(k) &= \omega_0 + \omega_1 k + \omega_2 k^2 + \dots \\ &\sim \omega_0 + \lambda\omega_1 + \lambda^2\omega_2 + \dots \end{aligned} \quad (4.47)$$

Therefore, from the relation $\partial/\partial t = i\omega$, the time derivative can be also expanded as

$$\frac{\partial}{\partial t} = \frac{\partial^{(0)}}{\partial t} + \lambda\frac{\partial^{(1)}}{\partial t} + \lambda^2\frac{\partial^{(2)}}{\partial t} + \dots . \quad (4.48)$$

4.2.1 Expansion of the Boltzmann equation

The velocity distribution function is also expanded into the series of the gradient

$$f = f^{(0)} + \lambda f^{(1)} + \lambda^2 f^{(2)} + \dots , \quad (4.49)$$

and the Boltzmann equation Eq. (2.31) is expanded as

$$\left(\frac{\partial^{(0)}}{\partial t} + \lambda\frac{\partial^{(1)}}{\partial t} + \dots + \lambda\mathbf{v}\cdot\nabla\right)\left(f^{(0)} + \lambda f^{(1)} + \dots\right) = I(f^{(0)} + \lambda f^{(1)} + \dots, f^{(0)} + \lambda f^{(1)} + \dots) . \quad (4.50)$$

From Eq. (2.34), the zero-th and the first orders of λ are respectively given by

$$\frac{\partial^{(0)}}{\partial t} f^{(0)} = I(f^{(0)}, f^{(0)}) \, , \tag{4.51}$$

$$\frac{\partial^{(0)}}{\partial t} f^{(1)} + \left(\frac{\partial^{(1)}}{\partial t} + \mathbf{v} \cdot \nabla \right) f^{(0)} = I(f^{(0)}, f^{(1)}) + I(f^{(1)}, f^{(0)}) \, . \tag{4.52}$$

Corresponding to the expansion Eq. (4.49), the cooling rate Eq. (4.19) is also expanded into the series of λ as

$$\zeta = \zeta^{(0)} + \lambda \zeta^{(1)} + \lambda^2 \zeta^{(2)} + \dots \, , \tag{4.53}$$

where the zero-th order term is given by

$$\zeta^{(0)} = (1 - e^2) \frac{\pi m \sigma^2}{24 n T} \int d\mathbf{v}_1 \int d\mathbf{v}_2 g^3 f_1^{(0)} f_2^{(0)} \, , \tag{4.54}$$

and the first order term is given by

$$\begin{aligned} \zeta^{(1)} &= (1 - e^2) \frac{\pi m \sigma^2}{24 n T} \left(\int d\mathbf{v}_1 \int d\mathbf{v}_2 g^3 f_1^{(0)} f_2^{(1)} + \int d\mathbf{v}_1 \int d\mathbf{v}_2 g^3 f_1^{(1)} f_2^{(0)} \right) \\ &= (1 - e^2) \frac{\pi m \sigma^2}{12 n T} \int d\mathbf{v}_1 \int d\mathbf{v}_2 g^3 f_1^{(0)} f_2^{(1)} \, . \end{aligned} \tag{4.55}$$

We notice that we exchanged the indices 1 and 2 in the second term of the first line of Eq. (4.55), which does not change the absolute value g.

4.2.2 Expansion of the hydrodynamic equations

Because we scale the gradient as $\lambda \nabla$, the hydrodynamic equations with the phenomenological transport coefficients are also scaled as

$$\frac{\partial n}{\partial t} = -\lambda \nabla_i (n u_i) \, , \tag{4.56}$$

$$\frac{\partial u_i}{\partial t} = -\lambda \left(u_j \nabla_j u_i + \frac{1}{nm} \nabla_i p \right) + \lambda^2 \frac{\eta}{nm} \left\{ \nabla^2 u_i + \frac{1}{3} \nabla_i (\nabla_j u_j) \right\} \, , \tag{4.57}$$

$$\frac{\partial T}{\partial t} = -\zeta^{(0)} T - \lambda \left(u_i \nabla_i T + \frac{2}{3n} p \nabla_i u_i + \zeta^{(1)} T \right) + \lambda^2 Q + O(\lambda^3) \, , \tag{4.58}$$

where we used Eqs.(4.40), (4.41) and (4.42), and defined

$$Q \equiv \frac{2}{3n} (\kappa \nabla^2 T + \mu \nabla^2 n) + \frac{2\eta}{3n} \left\{ (\nabla_i u_j)(\nabla_i u_j) + (\nabla_j u_i)(\nabla_i u_j) - \frac{2}{3} (\nabla_i u_i)^2 \right\} - \zeta^{(2)} T \, . \tag{4.59}$$

Becuase of the expansion of the time derivative Eq. (4.48), the zero-th order hydrodynamic equations is given by

$$\frac{\partial^{(0)}}{\partial t} n = 0 \, , \tag{4.60}$$

$$\frac{\partial^{(0)}}{\partial t} u_i = 0 \, , \tag{4.61}$$

$$\frac{\partial^{(0)}}{\partial t} T = -\zeta^{(0)} T \, , \tag{4.62}$$

and the first order hydrodynamic equations is given by

$$\begin{aligned}
\frac{\partial^{(1)}}{\partial t} n &= -\nabla_i (n u_i) , && (4.63)\\
\frac{\partial^{(1)}}{\partial t} u_i &= -u_j \nabla_j u_i - \frac{1}{nm} \nabla_i p , && (4.64)\\
\frac{\partial^{(1)}}{\partial t} T &= -u_i \nabla_i T - \frac{2}{3} T \nabla_i u_i - \zeta^{(1)} T , && (4.65)
\end{aligned}$$

where we used the equation of state $p = nT$ in Eq. (4.65). We note that the zero-th order hydrodynamic equations Eqs. (4.60)-(4.62) represent the *homogeneous cooling state* and the first order hydrodynamic equations Eqs. (4.63)-(4.65) are the *Euler equations*. We also give another expression of Eqs. (4.63)-(4.65):

$$\begin{aligned}
\frac{D^{(1)}}{Dt} n &= -n \nabla_i u_i , && (4.66)\\
\frac{D^{(1)}}{Dt} u_i &= -\frac{1}{nm} \nabla_i p , && (4.67)\\
\frac{D^{(1)}}{Dt} T &= -\frac{2}{3} T \nabla_i u_i - \zeta^{(1)} T , && (4.68)
\end{aligned}$$

where we introduced the *material derivative*

$$\frac{D^{(1)}}{Dt} \equiv \frac{\partial^{(1)}}{\partial t} + u_i \nabla_i . \tag{4.69}$$

4.2.3 Zero-th order equation

The zero-th order Boltzmann equation Eq. (4.51)

$$\frac{\partial^{(0)}}{\partial t} f^{(0)} = I(f^{(0)}, f^{(0)}) \tag{4.70}$$

is the same form with the Boltzmann equation in the homogeneous cooling state Eq. (3.2) and $f^{(0)}$ corresponds to the distribution function of the homogeneous cooling. Thus, we also scale $f^{(0)}$ by the thermal velocity

$$f^{(0)} = \frac{n}{v_{\mathrm{T}}} \tilde{f}(c) , \tag{4.71}$$

where we note that $c \equiv V/v_{\mathrm{T}}$ and both n and v_{T} depend on the space and time, i.e., $n(\mathbf{r}, t)$ and $v_{\mathrm{T}}(\mathbf{r}, t)$, which is different from the previous definition Eq. (3.3). Since the distribution function in the homogeneous state can be written by the Sonine polynomials expansion, $\zeta^{(0)}$ can be calculated. From Eq. (4.17), $\zeta^{(0)}$ is given by

$$\zeta^{(0)} = -\frac{m}{3nT} \int d\mathbf{V} V^2 I(f^{(0)}, f^{(0)}) , \tag{4.72}$$

where we changed the integration variable from $\mathbf{v}$ to $\mathbf{V}$, since $d\mathbf{v} = d\mathbf{V}$ and both variables cover the range $-\infty < \mathbf{v}, \mathbf{V} < \infty$. From Eq. (3.24), Eq. (4.72) is scaled as

$$\begin{aligned}
\zeta^{(0)} &= -\frac{m}{3nT} v_{\mathrm{T}}^3 v_{\mathrm{T}}^2 \frac{\sigma^2 n^2}{v_{\mathrm{T}}^2} \int d\mathbf{c} c^2 \tilde{I}(\tilde{f}, \tilde{f}) \\
&= -\frac{m\sigma^2 n}{3T} v_{\mathrm{T}}^3 \int d\mathbf{c} c^2 \tilde{I}(\tilde{f}, \tilde{f}) \\
&= -\frac{2}{3} \sigma^2 n \sqrt{\frac{2T}{m}} \int d\mathbf{c} c^2 \tilde{I}(\tilde{f}, \tilde{f}) \\
&= \frac{2}{3} \sigma^2 n \sqrt{\frac{2T}{m}} \mu_2 , && (4.73)
\end{aligned}$$

where we used $v_\mathrm{T} = \sqrt{2T/m}$ and μ_2 is given by Eq. (3.53). Therefore, $\zeta^{(0)} \propto nT^{1/2}$ and we find the derivatives

$$\frac{\partial \zeta^{(0)}}{\partial n} = \frac{\zeta^{(0)}}{n} , \qquad \frac{\partial \zeta^{(0)}}{\partial T} = \frac{\zeta^{(0)}}{2T} . \tag{4.74}$$

4.2.4 First order equation

The first order Boltzmann equation is given by Eq. (4.52)

$$\frac{\partial^{(0)}}{\partial t} f^{(1)} + \left(\frac{\partial^{(1)}}{\partial t} + \mathbf{v} \cdot \nabla \right) f^{(0)} = I(f^{(0)}, f^{(1)}) + I(f^{(1)}, f^{(0)}) . \tag{4.75}$$

Since the zero-th order distribution function $f^{(0)}$ is known, the second term in the left-hand-side

$$\left(\frac{\partial^{(1)}}{\partial t} + \mathbf{v} \cdot \nabla \right) f^{(0)} = \left(\frac{D^{(1)}}{Dt} + \mathbf{V} \cdot \nabla \right) f^{(0)} \tag{4.76}$$

can be calculated. Because of Eqs. (4.43) and (4.44), Eq. (4.76) is written as

$$\begin{aligned}
\left(\frac{\partial^{(1)}}{\partial t} + \mathbf{v} \cdot \nabla \right) f^{(0)} &= \left(\frac{D^{(1)} n}{Dt} + V_j \nabla_j n \right) \frac{\partial f^{(0)}}{\partial n} + \left(\frac{D^{(1)} u_i}{Dt} + V_j \nabla_j u_i \right) \frac{\partial f^{(0)}}{\partial u_i} + \left(\frac{D^{(1)} T}{Dt} + V_j \nabla_j T \right) \frac{\partial f^{(0)}}{\partial T} \\
&= (V_j \nabla_j n - n \nabla_i u_i) \frac{\partial f^{(0)}}{\partial n} + \left(V_j \nabla_j u_i - \frac{1}{nm} \nabla_i p \right) \frac{\partial f^{(0)}}{\partial u_i} \\
&\quad + \left(V_j \nabla_j T - \frac{2}{3} T \nabla_i u_i - \zeta^{(1)} T \right) \frac{\partial f^{(0)}}{\partial T} ,
\end{aligned} \tag{4.77}$$

where we used Eqs. (4.66), (4.67) and (4.68). Then, the derivatives $\partial f^{(0)}/\partial n$, $\partial f^{(0)}/\partial u_i$ and $\partial f^{(0)}/\partial T$ are calculated as

$$\begin{aligned}
\frac{\partial f^{(0)}}{\partial n} &= \frac{1}{v_\mathrm{T}^3} \tilde{f}^{(0)}(c) \\
&= \frac{1}{n} f^{(0)} ,
\end{aligned} \tag{4.78}$$

$$\begin{aligned}
\frac{\partial f^{(0)}}{\partial u_i} &= \frac{\partial V}{\partial u_i} \frac{\partial}{\partial V} f^{(0)} \\
&= -\frac{V_i}{V} \frac{\partial f^{(0)}}{\partial V} ,
\end{aligned} \tag{4.79}$$

$$\begin{aligned}
\frac{\partial f^{(0)}}{\partial T} &= \frac{\partial v_\mathrm{T}}{\partial T} \frac{\partial}{\partial v_\mathrm{T}} \left\{ \frac{n}{v_\mathrm{T}^3} \tilde{f}^{(0)}(c) \right\} \\
&= \frac{1}{m v_\mathrm{T}} \left\{ -\frac{3n}{v_\mathrm{T}^4} \tilde{f}^{(0)}(c) + \frac{n}{v_\mathrm{T}^3} \frac{\partial c}{\partial v_\mathrm{T}} \frac{\partial}{\partial c} \tilde{f}^{(0)}(c) \right\} \\
&= -\frac{n}{m v_\mathrm{T}^5} \left(3 + c \frac{\partial}{\partial c} \right) \tilde{f}^{(0)}(c) \\
&= -\frac{1}{m v_\mathrm{T}^2} \left(3 + V \frac{\partial}{\partial V} \right) f^{(0)} \\
&= -\frac{1}{2T} \left(3 + V \frac{\partial}{\partial V} \right) f^{(0)} ,
\end{aligned} \tag{4.80}$$

respectively, where we used the relations

$$\frac{\partial V}{\partial u_i} = -\frac{V_i}{V} , \qquad \frac{\partial v_\mathrm{T}}{\partial T} = \frac{1}{m v_\mathrm{T}} , \qquad \frac{\partial c}{\partial v_\mathrm{T}} = -\frac{c}{v_\mathrm{T}} , \qquad c \frac{\partial}{\partial c} = V \frac{\partial}{\partial V} . \tag{4.81}$$

Therefore, Eq. (4.77) is given by

$$\begin{aligned}\left(\frac{\partial^{(1)}}{\partial t}+\mathbf{v}\cdot\nabla\right)f^{(0)} &= \left(V_j\frac{1}{n}\nabla_j n-\nabla_i u_i\right)f^{(0)}-\left(V_j\nabla_j u_i-\frac{1}{nm}\nabla_i p\right)\frac{V_i}{V}\frac{\partial f^{(0)}}{\partial V}\\ &\quad -\left(\frac{V_i}{2}\frac{1}{T}\nabla_i T-\frac{1}{3}\nabla_i u_i-\frac{1}{2}\zeta^{(1)}\right)\left(3+V\frac{\partial}{\partial V}\right)f^{(0)}\\ &= \left(V_i\nabla_i\log n-\nabla_i u_i\right)f^{(0)}-\left(V_j\nabla_j u_i-\frac{1}{nm}\nabla_i p\right)\frac{V_i}{V}\frac{\partial f^{(0)}}{\partial V}\\ &\quad -\left(\frac{V_i}{2}\nabla_i\log T-\frac{1}{3}\nabla_i u_i-\frac{1}{2}\zeta^{(1)}\right)\left(3+V\frac{\partial}{\partial V}\right)f^{(0)}\,. \end{aligned} \tag{4.82}$$

From the equation of state $p=nT$, we find

$$\begin{aligned}\frac{1}{nm}\nabla_i p &= \frac{1}{nm}\nabla_i(nT)\\ &= \frac{1}{nm}T\nabla_i n+\frac{1}{m}\nabla_i T\\ &= \frac{T}{m}\left(\nabla_i\log n+\nabla_i\log T\right)\,, \end{aligned} \tag{4.83}$$

and Eq. (4.82) is written as

$$\begin{aligned}\left(\frac{\partial^{(1)}}{\partial t}+\mathbf{v}\cdot\nabla\right)f^{(0)} &= \left\{\frac{T}{m}\frac{V_i}{V}\frac{\partial f^{(0)}}{\partial V}-\frac{V_i}{2}\left(3+V\frac{\partial}{\partial V}\right)f^{(0)}\right\}\nabla_i\log T+\left(\frac{T}{m}\frac{V_i}{V}\frac{\partial f^{(0)}}{\partial V}+V_i f^{(0)}\right)\nabla_i\log n\\ &\quad +\left(\frac{1}{3}V\frac{\partial f^{(0)}}{\partial V}\nabla_i u_i-\frac{V_iV_j}{V}\frac{\partial f^{(0)}}{\partial V}\nabla_j u_i\right)+\frac{1}{2}\zeta^{(1)}\left(3+V\frac{\partial}{\partial V}\right)f^{(0)}\\ &= -V_i\left\{\frac{T}{m}\left(\frac{mV^2}{2T}-1\right)\frac{1}{V}\frac{\partial}{\partial V}+\frac{3}{2}\right\}f^{(0)}\nabla_i\log T+V_i\left(\frac{T}{m}\frac{1}{V}\frac{\partial}{\partial V}+1\right)f^{(0)}\nabla_i\log n\\ &\quad -\left(V_iV_j-\frac{1}{3}\delta_{ij}V^2\right)\frac{1}{V}\frac{\partial f^{(0)}}{\partial V}\nabla_j u_i+\frac{1}{2}\zeta^{(1)}\left(3+V\frac{\partial}{\partial V}\right)f^{(0)}\,. \end{aligned} \tag{4.84}$$

If we define the prefactors of $\nabla_i\log T$, $\nabla_i\log n$ and $\nabla_j u_i$ as

$$A_i \equiv V_i\left\{\frac{T}{m}\left(\frac{mV^2}{2T}-1\right)\frac{1}{V}\frac{\partial}{\partial V}+\frac{3}{2}\right\}f^{(0)}\,, \tag{4.85}$$

$$B_i \equiv -V_i\left(\frac{T}{m}\frac{1}{V}\frac{\partial}{\partial V}+1\right)f^{(0)}\,, \tag{4.86}$$

$$C_{ij} \equiv \left(V_iV_j-\frac{1}{3}\delta_{ij}V^2\right)\frac{1}{V}\frac{\partial f^{(0)}}{\partial V}\,, \tag{4.87}$$

respectively, the first order Boltzmann equation Eq. (4.52) is given by

$$\frac{\partial^{(0)}}{\partial t}f^{(1)}+J\left(f^{(0)},f^{(1)}\right)+\frac{1}{2}\zeta^{(1)}\left(3+V\frac{\partial}{\partial V}\right)f^{(0)}=A_i\nabla_i\log T+B_i\nabla_i\log n+C_{ij}\nabla_j u_i\,, \tag{4.88}$$

where we defined

$$J\left(f^{(0)},f^{(1)}\right)\equiv -I\left(f^{(0)},f^{(1)}\right)-I\left(f^{(1)},f^{(0)}\right)\,. \tag{4.89}$$

From Eqs. (4.85)-(4.87), we note both A_i and B_i are proportional to V_i

$$A_i\propto V_i\,,\qquad B_i\propto V_i\,, \tag{4.90}$$

and C_{ij} is traceless

$$C_{ii}=\left(V_iV_i-\frac{1}{3}3V^2\right)\frac{1}{V}\frac{\partial f^{(0)}}{\partial V}=\left(V^2-V^2\right)\frac{1}{V}\frac{\partial f^{(0)}}{\partial V}=0\,. \tag{4.91}$$

4.2.5 Correction of the distribution function

The deviation of the distribution function $f^{(1)}$ can be obtained by solving Eq. (4.88). Except for the term related to $\zeta^{(0)}$, Eq. (4.88) is proportional to $f^{(1)}$. On the other hand, the right-hand-side contains the terms proportional to $\log T$, $\log n$ and $\nabla_j u_i$, respectively. Thus, we can assume $f^{(1)}$ can be written in the form

$$f^{(1)} = \alpha_i \nabla_i \log T + \beta_i \nabla_i \log n + \gamma_{ij} \nabla_j u_i \,, \tag{4.92}$$

where the coefficients α_i, β_i and γ_{ij} are the functions of V_i and the hydrodynamic fields. If we adopt Eq. (4.92) for $f^{(1)}$, $\zeta^{(0)}$ is vanished as follows. Substituting Eq. (4.92) to Eq. (4.88), we compare the prefactors of $\log T$, $\log n$ and $\nabla_j u_i$ in both sides. Then, we find α_i, β_i and γ_{ij} are proportional to A_i, B_i and C_{ij}, respectively, and from Eqs. (4.90) and (4.91), both α_i and β_i are also proportional to V_i

$$\alpha_i \propto V_i \,, \qquad \beta_i \propto V_i \,, \tag{4.93}$$

and γ_{ij} also satisfies

$$\gamma_{ij} \propto V_i V_j - \frac{1}{3}\delta_{ij} V^2 \,, \tag{4.94}$$

and traceless $\gamma_{ii} = 0$. From Eq. (4.55)

$$\zeta^{(1)} = (1-e^2)\frac{\pi m \sigma^2}{12nT}\int d\mathbf{g}\int d\mathbf{V}_2 g^3 f^{(0)}(\mathbf{V}_2+\mathbf{g}) f^{(1)}(\mathbf{V}_2) \,, \tag{4.95}$$

where we changed the integration variables as $(\mathbf{v}_1, \mathbf{v}_2) \to (\mathbf{g}, \mathbf{V}_2)$ since $d\mathbf{v}_1 d\mathbf{v}_2 = d\mathbf{g} d\mathbf{V}_2$. Because the zero-th order distribution function is homogeneous, $f^{(0)}(\mathbf{V}_2+\mathbf{g})$ is symmetric about $\mathbf{V}_2 = -\mathbf{g}$ and we can always find the counterpart $f^{(0)}(\mathbf{V}_2 - \mathbf{g})$ by the integral $\int d\mathbf{g}$. From Eqs. (4.93) and Eq. (4.94), $f^{(1)}(\mathbf{V}_2)$ is always odd function of $\mathbf{V}_2$. Of course, the absolute value g^3 is even function. Therefore, the integral in Eq. (4.95) is vanished and $\zeta^{(1)} = 0$.

Becuase of Eqs. (4.60)-(4.62), the time derivatives of α_i, β_i and γ_{ij} in the zero-th order are given by

$$\frac{\partial^{(0)}\alpha_i}{\partial t} = \frac{\partial \alpha_i}{\partial T}\frac{\partial^{(0)} T}{\partial t} = -\zeta^{(0)} T \frac{\partial \alpha_i}{\partial T} \,, \tag{4.96}$$

$$\frac{\partial^{(0)}\beta_i}{\partial t} = \frac{\partial \beta_i}{\partial T}\frac{\partial^{(0)} T}{\partial t} = -\zeta^{(0)} T \frac{\partial \beta_i}{\partial T} \,, \tag{4.97}$$

$$\frac{\partial^{(0)}\gamma_{ij}}{\partial t} = \frac{\partial \gamma_{ij}}{\partial T}\frac{\partial^{(0)} T}{\partial t} = -\zeta^{(0)} T \frac{\partial \gamma_{ij}}{\partial T} \,, \tag{4.98}$$

respectively, and we also find

$$\begin{aligned}
\frac{\partial^{(0)}}{\partial t}\nabla_i \log T &= \nabla_i\left(\frac{1}{T}\frac{\partial^{(0)}T}{\partial t}\right) \\
&= -\nabla_i \zeta^{(0)} \\
&= -\frac{\partial \zeta^{(0)}}{\partial n}\nabla_i n - \frac{\partial \zeta^{(0)}}{\partial T}\nabla_i T \\
&= -\zeta^{(0)}\left(\nabla_i \log n + \frac{1}{2}\nabla_i \log T\right) \,,
\end{aligned} \tag{4.99}$$

where we used Eq. (4.74). Then, we find the time derivative of $f^{(1)}$ as

$$\begin{aligned}
\frac{\partial^{(0)} f^{(1)}}{\partial t} &= \left(\frac{\partial \alpha_i}{\partial t}\right)\nabla_i \log T + \alpha_i\left(\frac{\partial}{\partial t}\nabla_i \log T\right) + \left(\frac{\partial \beta_i}{\partial t}\right)\nabla_i \log n + \left(\frac{\partial \gamma_{ij}}{\partial t}\right)\nabla_j u_i \\
&= -\left(\zeta^{(0)} T\frac{\partial \alpha_i}{\partial T} + \frac{1}{2}\zeta^{(0)}\alpha_i\right)\nabla_i \log T - \left(\zeta^{(0)} T\frac{\partial \beta_i}{\partial T} + \zeta^{(0)}\alpha_i\right)\nabla_i \log n \\
&\quad -\zeta^{(0)} T \frac{\partial \gamma_{ij}}{\partial T}\nabla_j u_i \,.
\end{aligned} \tag{4.100}$$

If we substitute Eq. (4.100) to the first order Boltzmann equation Eq. (4.88) and compare the prefactors of $\log T$, $\log n$ and $\nabla_j u_i$ in both sides, we find

$$
\begin{aligned}
-\zeta^{(0)}\left(T\frac{\partial}{\partial T}+\frac{1}{2}\right)\alpha_i + J\left(f^{(0)},\alpha_i\right) &= A_i\ , && (4.101)\\
-\zeta^{(0)}\left(T\frac{\partial\beta_i}{\partial T}+\alpha_i\right) + J\left(f^{(0)},\beta_i\right) &= B_i\ , && (4.102)\\
-\zeta^{(0)}T\frac{\partial\gamma_{ij}}{\partial T} + J\left(f^{(0)},\gamma_{ij}\right) &= C_{ij}\ . && (4.103)
\end{aligned}
$$

The coefficients A_i, B_i and C_{ij} are calculated by Eqs. (4.85)-(4.87), and the coefficients α_i, β_i and γ_{ij} can be determined from Eqs. (4.101)-(4.103). Then, we can obtain the first correction to the distribution function $f^{(1)}$ from Eq. (4.92).

4.3 Transport coefficients

Comparing the definition of the stress tensor Eq. (4.14) with the phenomenological expression of the stress tensor Eq. (4.38), we find $p = nT$ and

$$\int d\mathbf{v}D_{ij}f = -\eta\left(\nabla_i u_j + \nabla_j u_i - \frac{2}{3}\delta_{ij}\nabla_k u_k\right)\ . \tag{4.104}$$

Substituting $f = f^{(0)} + \lambda f^{(1)}$ to Eq. (4.104), we find the zero-th order gradient gives

$$\int d\mathbf{v}D_{ij}f^{(0)} = 0\ . \tag{4.105}$$

Substituting $f^{(1)} = \alpha_i\nabla_i \log T + \beta_i\nabla_i \log n + \gamma_{ij}\nabla_j u_i$, we find the first order gradient gives

$$\int d\mathbf{v}D_{ij}\alpha_k = \int d\mathbf{v}D_{ij}\beta_k = 0\ , \tag{4.106}$$

because the right-hand-side of Eq. (4.104) does not include the corresponding terms of $\nabla_i T$ and $\nabla_i n$, and

$$
\begin{aligned}
\int d\mathbf{v}D_{ij}\gamma_{kl}\nabla_l u_k &= -\eta\left(\nabla_i u_j + \nabla_j u_i - \frac{2}{3}\delta_{ij}\nabla_k u_k\right)\\
&= -\eta\left(\delta_{li}\delta_{kj} + \delta_{lj}\delta_{ki} - \frac{2}{3}\delta_{ij}\delta_{lk}\right)\nabla_l u_k\ . && (4.107)
\end{aligned}
$$

Therefore, we find

$$\int d\mathbf{v}D_{ij}\gamma_{kl} = -\eta\left(\delta_{li}\delta_{kj} + \delta_{lj}\delta_{ki} - \frac{2}{3}\delta_{ij}\delta_{lk}\right)\ . \tag{4.108}$$

If we use $k = j$ and $l = i$ in Eq. (4.108),

$$
\begin{aligned}
\int d\mathbf{v}D_{ij}\gamma_{ji} &= -\eta\left(\delta_{ii}\delta_{jj} + \delta_{ij}\delta_{ji} - \frac{2}{3}\delta_{ij}\delta_{ij}\right)\\
&= -10\eta\ , && (4.109)
\end{aligned}
$$

where we notice the relations $\delta_{ii} = \delta_{jj} = 3$ and $\delta_{ij}\delta_{ji} = \delta_{ij}\delta_{ij} = 3$. Then, we find the formal expression of the shear viscosity as

$$\eta = -\frac{1}{10}\int d\mathbf{v}D_{ij}\gamma_{ji}\ . \tag{4.110}$$

In the same way, we compare the definition of the heat flux Eq. (4.36) with the phenomenological expression of the heat flux Eq. (4.39) and find

$$\int d\mathbf{v} S_i f = -\kappa \nabla_i T - \mu \nabla_i n \,. \tag{4.111}$$

Substituting $f = f^{(0)} + \lambda f^{(1)}$ to Eq. (4.111), we find the zero-th order gradient gives

$$\int d\mathbf{v} S_i f^{(0)} = 0 \,. \tag{4.112}$$

Substituting $f^{(1)} = \alpha_i \nabla_i \log T + \beta_i \nabla_i \log n + \gamma_{ij} \nabla_j u_i$, we find the first order gradient gives

$$\int d\mathbf{v} S_i \gamma_{kl} = 0 \,, \tag{4.113}$$

because the right-hand-side of Eq. (4.111) does not include the corresponding term of $\nabla_l u_k$, and

$$\int d\mathbf{v} S_i \alpha_j \nabla_j \log T = -\kappa \nabla_i T \,, \tag{4.114}$$

$$\int d\mathbf{v} S_i \beta_j \nabla_j \log n = -\mu \nabla_i n \,. \tag{4.115}$$

Eqs. (4.114) and (4.115) reduce to

$$\frac{1}{T} \int d\mathbf{v} S_i \alpha_j \nabla_j T = -\kappa \nabla_i T \,, \tag{4.116}$$

$$\frac{1}{n} \int d\mathbf{v} S_i \beta_j \nabla_j n = -\mu \nabla_i n \,, \tag{4.117}$$

or

$$\frac{1}{T} \int d\mathbf{v} S_i \alpha_j = -\kappa \delta_{ij} \,, \tag{4.118}$$

$$\frac{1}{n} \int d\mathbf{v} S_i \beta_j = -\mu \delta_{ij} \,. \tag{4.119}$$

Then, we use $i = j$ in Eqs. (4.118) and (4.119) and find

$$\frac{1}{T} \int d\mathbf{v} S_i \alpha_i = -3\kappa \,, \tag{4.120}$$

$$\frac{1}{n} \int d\mathbf{v} S_i \beta_i = -3\mu \,, \tag{4.121}$$

where we notice $\delta_{ii} = 3$. Therefore, the formal expressions of the thermal conductivity and the coefficient of the density gradient are respectively given by

$$\kappa = -\frac{1}{3T} \int d\mathbf{v} S_i \alpha_i \,, \tag{4.122}$$

$$\mu = -\frac{1}{3n} \int d\mathbf{v} S_i \beta_i \,. \tag{4.123}$$

Now, we have the formal expressions of η, κ and μ which are the functions of γ_{ji}, α_i and β_i, respectively. The coefficients α_i, β_i and γ_{ij} can be obtained by solving Eqs. (4.101)-(4.103), where the coefficients A_i, B_i and C_{ij} defined as Eqs. (4.85)-(4.87) can be calculated by using the zero-th order distribution function $f^{(0)}$. To evaluate the transport coefficients, we truncate $f^{(0)}$ at the second Sonine polynomials

$$f^{(0)} \simeq \frac{n}{v_{\mathrm{T}}(t)^3}\phi(c)\left\{1 + a_2 S_2(c^2)\right\} . \tag{4.124}$$

Then, we will see the kinetic integrals Eq. (3.52) in Eqs. (4.101)-(4.103) and η, κ and μ can be written by the combination of the kinetic integrals. The evaluations of the transport coefficients are straightforward, thus we refer the details in the reference and only show the final results

$$\eta = \frac{15}{2(1+e)(13-e)\sigma^2}\sqrt{\frac{mT}{\pi}}\left(1 + \frac{3(4-3e)}{8(13-e)}a_2\right) , \tag{4.125}$$

$$\kappa = \frac{75}{2(1+e)(9+7e)\sigma^2}\sqrt{\frac{T}{\pi m}}\left(1 + \frac{797+211e}{32(9+7e)}a_2\right) , \tag{4.126}$$

$$\mu = \frac{750(1-e)}{(1+e)(9+7e)(19-3e)n\sigma^2}\sqrt{\frac{T^3}{\pi m}}\left(1 + h(e)a_2\right) , \tag{4.127}$$

where we defined

$$h(e) \equiv \frac{50201 - 30971e - 7253e^2 + 4407e^3}{80(1-e)(19-3e)(9+7e)} . \tag{4.128}$$

Chapter 5

Collisional transfer

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