『デリバティブ論』
補足: 正規分布確率密度関数の積分

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命題 1. 平均 μ, 分散 σ^2 の正規分布の確率密度関数

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

に対して

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1$$

が成立する.

命題 1 を証明するためにいくつかの補助命題（補題）を証明する.

補題 2. n を自然数として次が成立する.

(1)

$$\int_0^1 (1-x^2)^n \mathrm{d}x < \int_0^\infty \mathrm{e}^{-nx^2}\mathrm{d}x = \frac{1}{\sqrt{n}}\int_0^\infty \mathrm{e}^{-x^2}\mathrm{d}x < \int_0^\infty \frac{1}{(1+x^2)^n}\mathrm{d}x.$$

(2)

$$J_n = \int_0^{\frac{\pi}{2}} \sin^n x \mathrm{d}x$$

として次が成立する.

(a)

$$\int_0^1 (1-x^2)^n \mathrm{d}x = J_{2n+1} = \frac{2}{3}\times\frac{4}{5}\times\cdots\times\frac{2n}{2n+1},$$
$$\int_0^\infty \frac{1}{(1+x^2)^n}\mathrm{d}x = J_{2n-2} = \frac{\pi}{2}\times\frac{1}{2}\times\frac{3}{4}\times\cdots\times\frac{2n-3}{2n-2}.$$

(b)

$$\begin{aligned} 1 < \frac{J_{2n}}{J_{2n+1}} < \frac{J_{2n-1}}{J_{2n+1}} = \frac{2n+1}{2n}, \\ \frac{2n-1}{2n} = \frac{J_{2n}}{J_{2n-2}} < \frac{J_{2n-1}}{J_{2n-2}} < 1. \end{aligned}$$

(c)

$$\begin{aligned} \frac{\sqrt{\pi}}{2} &= \sqrt{\frac{2n+1}{2n}} \sqrt{n} J_{2n+1} \sqrt{\frac{J_{2n}}{J_{2n+1}}} \\ &= \sqrt{\frac{2n-1}{2n}} \sqrt{n} J_{2n-2} \sqrt{\frac{J_{2n-1}}{J_{2n-2}}} \end{aligned}$$

(d)

$$\frac{\sqrt{\pi}}{2} = \lim_{n\to\infty} \sqrt{n} J_{2n+1} = \lim_{n\to\infty} \sqrt{n} J_{2n-2}.$$

証明

(1) e^x のマクローリン展開 $\mathrm{e}^x = 1 + x + \frac{x^2}{2}\mathrm{e}^{\theta x}$, $\theta \in (0,1)$ より，$x \neq 0$ ならば，$\mathrm{e}^x > 1 + x$．したがって，x に $-x^2$ および x^2 を代入すれば,

$$1 - x^2 < \mathrm{e}^{-x^2} < \frac{1}{1+x^2}, \quad x \neq 0.$$

これより,

$$(1-x^2)^n < \mathrm{e}^{-nx^2},\ x \in (0,1), \quad \mathrm{e}^{-nx^2} < \frac{1}{(1+x^2)^n},\ x \in (0,\infty)$$

となるので,

$$\int_0^1 (1-x^2)^n \mathrm{d}x < \int_0^1 \mathrm{e}^{-nx^2} \mathrm{d}x < \int_0^\infty \mathrm{e}^{-nx^2} \mathrm{d}x < \int_0^\infty \frac{1}{(1+x^2)^n} \mathrm{d}x.$$

さらに $x = \frac{y}{\sqrt{n}}$ と変数置換すると，$\mathrm{d}x = \frac{1}{\sqrt{n}}\mathrm{d}y$ であるから，置換積分によって,

$$\int_0^\infty \mathrm{e}^{-nx^2} \mathrm{d}x = \frac{1}{\sqrt{n}} \int_0^\infty \mathrm{e}^{-y^2} \mathrm{d}y.$$

以上により,

$$\int_0^1 (1-x^2)^n \mathrm{d}x < \int_0^\infty \mathrm{e}^{-nx^2} \mathrm{d}x = \frac{1}{\sqrt{n}} \int_0^\infty \mathrm{e}^{-x^2} \mathrm{d}x < \int_0^\infty \frac{1}{(1+x^2)^n} \mathrm{d}x.$$

(2)（a）$x=\cos\theta$ と変数を置換すると，$\mathrm{d}x=-\sin\theta\mathrm{d}\theta,\ x\in(0,1) \iff -\theta\in\left(0,\frac{\pi}{2}\right)$ であるから，

$$\begin{aligned}
\int_0^1(1-x^2)^n\mathrm{d}x &= \int_0^{\frac{\pi}{2}}(1-\cos^2\theta)^n\sin\theta\mathrm{d}\theta\\
&= \int_0^{\frac{\pi}{2}}\sin^{2n+1}\theta\mathrm{d}\theta\\
&\qquad(\because \sin^2\theta+\cos^2\theta=1)\\
&= J_{2n+1}.
\end{aligned}$$

$x=\tan\theta$ と変数を置換すると，$\mathrm{d}x=\frac{1}{\cos^2\theta}\mathrm{d}\theta,\ x\in(0,\infty) \iff \theta\in\left(0,\frac{\pi}{2}\right)$ であるから，

$$\begin{aligned}
\int_0^\infty\frac{1}{(1+x^2)^n}\mathrm{d}x &= \int_0^{\frac{\pi}{2}}\frac{1}{(1+\tan^2\theta)^n\cos^2\theta}\mathrm{d}\theta\\
&= \int_0^{\frac{\pi}{2}}\frac{\cos^{2n-2}\theta}{(\cos^2\theta+\sin^2\theta)^n}\mathrm{d}\theta\\
&= \int_0^{\frac{\pi}{2}}\cos^{2n-2}\theta\mathrm{d}\theta\\
&\quad \text{ここで，}\theta=\frac{\pi}{2}-t\text{ と変数置換を行えば，}\\
&= \int_{\frac{\pi}{2}}^0\cos^{2n-2}\left(\frac{\pi}{2}-t\right)(-1)\mathrm{d}t\\
&= \int_0^{\frac{\pi}{2}}\sin^{2n-2}t\mathrm{d}t\\
&= J_{2n-2}.
\end{aligned}$$

$$\begin{aligned}
\int_0^{\frac{\pi}{2}}\sin^n x\mathrm{d}x &= \int_0^{\frac{\pi}{2}}\sin^{n-1}x\sin x\mathrm{d}x\\
&= \left[-\sin^{n-1}x\cos x\right]_0^{\frac{\pi}{2}}\\
&\quad+(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x\cos^2 x\mathrm{d}x\\
&= (n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x(1-\sin^2x)\mathrm{d}x\\
&= (n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x\mathrm{d}x\\
&\quad-(n-1)\int_0^{\frac{\pi}{2}}\sin^n x\mathrm{d}x\\
\iff J_n &= \frac{n-1}{n}J_{n-2},\ n\geq 2, \qquad (1)
\end{aligned}$$

$$J_1 = \int_0^{\frac{\pi}{2}} \sin x \mathrm{d}x = -\left[\cos x\right]_0^{\frac{\pi}{2}} = 1, \quad J_0 = \int_0^{\frac{\pi}{2}} \mathrm{d}x = \frac{\pi}{2}.$$

したがって,

$$\begin{aligned} J_{2n+1} &= \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} \times \cdots \times \frac{2}{3} \times J_1 \\ &= \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} \times \cdots \times \frac{2}{3}, \\ J_{2n-2} &= \frac{2n-3}{2n-2} \times \frac{2n-5}{2n-4} \times \cdots \times \frac{1}{2} \times J_0 \\ &= \frac{2n-3}{2n-2} \times \frac{2n-5}{2n-4} \times \cdots \times \frac{1}{2} \times \frac{\pi}{2}. \end{aligned}$$

以上により，所与の等式を得る.

（b）$0 < \sin^{2n+1} x < \sin^{2n} x, x \in \left(0, \frac{\pi}{2}\right)$，より，

$$0 < J_{2n+1} < J_{2n} < J_{2n-1} < J_{2n-2}.$$

したがって,

$$\begin{aligned} 1 <& \frac{J_{2n}}{J_{2n+1}} < \frac{J_{2n-1}}{J_{2n+1}}, \\ & \frac{J_{2n}}{J_{2n-2}} < \frac{J_{2n-1}}{J_{2n-2}} < 1. \end{aligned}$$

また (a) から,

$$\frac{J_{2n-1}}{J_{2n+1}} = \frac{2n+1}{2n}, \quad \frac{J_{2n}}{J_{2n-2}} = \frac{2n-1}{2n}$$

を得る.

（c）(a) より,

$$\begin{aligned} J_{2n+1}J_{2n} &= \frac{2n}{2n+1} \times \frac{2n-1}{2n} \times \cdots \times \frac{2}{3} \times \frac{1}{2} \times \frac{\pi}{2} \\ &= \frac{1}{2n+1} \times \frac{\pi}{2}, \\ J_{2n-1}J_{2n-2} &= \frac{1}{2n-1} \times \frac{\pi}{2}. \end{aligned}$$

したがって,

$$
\begin{aligned}
\frac{\pi}{2} &= \frac{2n+1}{1} J_{2n+1} \times J_{2n} \\
&= \frac{2n-1}{1} J_{2n-1} \times J_{2n-2} \\
\iff \qquad \frac{\sqrt{\pi}}{2} &= \sqrt{\frac{2n+1}{2n}} \sqrt{n} J_{2n+1} \sqrt{\frac{J_{2n}}{J_{2n+1}}} \\
&= \sqrt{\frac{2n-1}{2n}} \sqrt{n} J_{2n-2} \sqrt{\frac{J_{2n-1}}{J_{2n-2}}}.
\end{aligned}
$$

(d) (c) より,

$$
\begin{aligned}
\frac{\sqrt{\pi}}{2} &= \lim_{n\to\infty} \frac{\sqrt{\pi}}{2} \\
&= \lim_{n\to\infty} \sqrt{\frac{2n+1}{2n}} \sqrt{n} J_{2n+1} \sqrt{\frac{J_{2n}}{J_{2n+1}}} \\
&= \sqrt{\lim_{n\to\infty} \frac{1+\frac{1}{2n}}{1}} \lim_{n\to\infty} \sqrt{n} J_{2n+1} \sqrt{\lim_{n\to\infty} \frac{J_{2n}}{J_{2n+1}}} \\
&= \lim_{n\to\infty} \sqrt{n} J_{2n+1} \sqrt{\lim_{n\to\infty} \frac{J_{2n}}{J_{2n+1}}}, \\
\frac{\sqrt{\pi}}{2} &= \lim_{n\to\infty} \sqrt{\frac{2n-1}{2n}} \sqrt{n} J_{2n-2} \sqrt{\frac{J_{2n-1}}{J_{2n-2}}} \\
&= \sqrt{\lim_{n\to\infty} \frac{1-\frac{1}{2n}}{1}} \lim_{n\to\infty} \sqrt{n} J_{2n-2} \sqrt{\lim_{n\to\infty} \frac{J_{2n-1}}{J_{2n-2}}} \\
&= \lim_{n\to\infty} \sqrt{n} J_{2n-2} \sqrt{\lim_{n\to\infty} \frac{J_{2n-1}}{J_{2n-2}}}.
\end{aligned}
$$

さらに,

$$
\lim_{n\to\infty} \frac{2n \pm 1}{2n} = \lim_{n\to 0} \frac{2 \pm \frac{1}{n}}{2} = 1
$$

となることに注意すると, (b) より,

$$
\lim_{n\to\infty} \frac{J_{2n}}{J_{2n+1}} = \lim_{n\to\infty} \frac{J_{2n-1}}{J_{2n-2}} = 1
$$

となるので, 結局,

$$
\frac{\sqrt{\pi}}{2} = \lim_{n\to\infty} \sqrt{n} J_{2n+1} = \lim_{n\to\infty} \sqrt{n} J_{2n-2}
$$

を得る．

$\square$

命題 1 の証明　補題 2 より，

$$
\begin{aligned}
&\int_0^1 (1-x^2)^n \mathrm{d}x < \frac{1}{\sqrt{n}} \int_0^\infty \mathrm{e}^{-x^2} \mathrm{d}x < \int_0^\infty \frac{1}{(1+x^2)^n} \mathrm{d}x \\
\iff &\sqrt{n} J_{2n+1} < \int_0^\infty \mathrm{e}^{-x^2} \mathrm{d}x < \sqrt{n} J_{2n-2}, \\
&\lim_{n\to\infty} \sqrt{n} J_{2n+1} = \lim_{n\to\infty} \sqrt{n} J_{2n-2} = \frac{\sqrt{\pi}}{2}.
\end{aligned}
$$

よって，次式を得る．

$$
\frac{\sqrt{\pi}}{2} = \int_0^\infty \mathrm{e}^{-x^2} \mathrm{d}x.
$$

ここで $x = \frac{z}{\sqrt{2}}$ と変数置換すると

$$
\begin{aligned}
&\frac{\sqrt{\pi}}{2} = \int_0^\infty \frac{1}{\sqrt{2}} \mathrm{e}^{-\frac{z^2}{2}} \mathrm{d}z \\
\iff &\int_0^\infty \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \mathrm{d}x = \frac{1}{2}.
\end{aligned}
$$

ここで $x = -y$ と変数置換すると，

$$
\int_0^\infty \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \mathrm{d}x = \int_{-\infty}^0 \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{y^2}{2}} \mathrm{d}y.
$$

したがって，

$$
\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \mathrm{d}x = \int_{-\infty}^0 \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \mathrm{d}x + \int_0^\infty \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \mathrm{d}x = 1.
$$

さらに，$z = \frac{x-\mu}{\sigma}$ と変数置換すると，

$$
\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}} \mathrm{d}x = 1.
$$

$\square$