Invariants of graphs for the existence of cycles

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1. Definition and Invariants

2. Degree conditions of two vertices (on the order)

3. Degree conditions of $k$ vertices (on the order, the independence number and the connectivity)
The order of a graph and a hamiltonian cycle

\[ G = (V, E) : \text{a simple graph} \]

**Invariant 1.**

The order (the number of vertices) : \( n := |V(G)| \)

**Definition 2.**

hamiltonian cycle

\[ \iff \text{cycle passing thru all vertices} \]

\[ \iff \text{cycle of length } n \text{ (order } n) \]
The connectivity of a graph

Notation 3.

Let \( x, y \in V(G) \) s.t. \( x \neq y \).

\( \kappa(x, y) \) : the max. number of vertex-disjoint paths joining \( x \) & \( y \)

Invariant 4.

The connectivity : \( \kappa := \min\{\kappa(x, y) : x, y \in V(G), x \neq y\} \)
Definition 6.

$S$ : an independent set (indep. set)

$\iff \forall x, \forall y \in S \rightarrow xy \notin E(G)$

Invariant 7.

The independence number : $\alpha := \max \{|S| : S \text{ is an indep. set}\}$
The degree sum condition

Notation 8.
The degree $d_G(x)$ of $x \in V(G)$: # of neighbors of $x$ in $G$

Invariant 9.
The degree sum of $t$ vertices:

$$\sigma_t := \min \left\{ \sum_{x \in X} d_G(x) : X \text{ is an independ. set of order } t \right\}$$

if $\alpha \geq t$; otherwise $\sigma_t := +\infty$.

Remark.

$$\frac{\sigma_s}{s} \geq \frac{\sigma_t}{t} \quad \text{for } s \geq t$$

\[\begin{array}{ll}
\sigma_2 &= 4 \\
\sigma_3 &= 8 \\
\sigma_4 &= +\infty
\end{array}\]
Conditions for the existence of a hamiltonian cycle

Ore (1960)

\[ \sigma_2 \geq n \]

\[ \Rightarrow \exists \text{hamiltonian cycle} \]

Chvátal & Erdős (1972)

\[ \alpha \leq \kappa \]

\[ \Rightarrow \exists \text{hamiltonian cycle} \]
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\alpha \leq \kappa \\
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These conditions are sharp.

The graph $\overline{K}_\kappa + \overline{K}_{\kappa+1}$ has no hamiltonian cycle.

The degree sum $\sigma_2 = n - 1$, the connectivity $\kappa$, and the indep. number $\alpha = \kappa + 1$. 

$k + 1$ 頂点

$\ldots$

$k$ 頂点
Comparison between Ore’s Thm and Chvátal & Erdős’s Thm

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Ore (1960)
\[ \sigma_2 \geq n \]
\[ \Rightarrow \exists \text{hamiltonian cycle} \]

Chvátal & Erődős (1972)
\[ \alpha \leq \kappa \]
\[ \Rightarrow \exists \text{hamiltonian cycle} \]

Bondy (1978)
\[ \sigma_2 \geq n \Rightarrow \alpha \leq \kappa \]

Chvátal&Erődős’s Thm is stronger than Ore’s Thm.
Bondy’s Thm

Ore (1960)

\[ \sigma_2 \geq n \]

\[ \Rightarrow \exists \text{hamiltonian cycle} \]

Chvátal & Erdős (1972)

\[ \alpha \leq \kappa \]

\[ \Rightarrow \exists \text{hamiltonian cycle} \]
By considering Chvátal & Erdős’s Thm, we should consider degree conditions for the existence of hamiltonian cycles for graphs satisfying $\alpha \geq \kappa + 1$, that is, graphs having an indep. set of order $\kappa + 1$. 

Ore (1960)

\[
\sigma_2 \geq n \\
\Rightarrow \exists \text{hamiltonian cycle}
\]

Chvátal & Erdős (1972)

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\alpha \leq \kappa \\
\Rightarrow \exists \text{hamiltonian cycle}
\]
Bondy’s Thm

Ore (1960)
\[ \sigma_2 \geq n \]
\[ \Rightarrow \exists \text{hamiltonian cycle} \]

Chvátal & Erdős (1972)
\[ \alpha \leq \kappa \]
\[ \Rightarrow \exists \text{hamiltonian cycle} \]

Bondy (1980)
\[ \sigma_{\kappa+1} > \frac{(\kappa+1)(n-1)}{2} \]
\[ \Rightarrow \exists \text{hamiltonian cycle} \]

Note that if \( \alpha \leq \kappa \) then \( \sigma_{\kappa+1} = +\infty \).

Bondy’s Thm is a common generalization of Ore’s Thm and Chvátal & Erdős’s Thm.
Bondy’s Thm is a generalization of Ore’s Thm

\[
\begin{align*}
\text{Ore (1960)} & : \sigma_2 \geq n \\
& \Rightarrow \exists \text{hamiltonian cycle}
\end{align*}
\]

\[
\begin{align*}
\text{Bondy (1980)} & : \sigma_{\kappa+1} > \frac{(\kappa+1)(n-1)}{2} \\
& \Rightarrow \exists \text{hamiltonian cycle}
\end{align*}
\]

Note that Bondy’s condition can be written by \( 2 \cdot \frac{\sigma_{\kappa+1}}{\kappa+1} > n - 1 \).
Bondy’s Thm is a generalization of Ore’s Thm

Ore (1960)
\[ \sigma_2 \geq n \]
\[ \Rightarrow \exists \text{hamiltonian cycle} \]

Bondy (1980)
\[ \sigma_{\kappa+1} > \frac{(\kappa+1)(n-1)}{2} \]
\[ \Rightarrow \exists \text{hamiltonian cycle} \]

Note that Bondy’s condition can be written by
\[ 2 \cdot \frac{\sigma_{\kappa+1}}{\kappa+1} > n - 1. \]

Since \[ 2 \cdot \frac{\sigma_{\kappa+1}}{\kappa+1} \geq \sigma_2, \]

**Bondy’s Thm is a generalization of Ore’s Thm.**
A further generalization

Ore (1960)
\[
\sigma_2 \geq n \\
\Rightarrow \exists \text{hamiltonian cycle}
\]

Chvátal & Erdős (1972)
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\alpha \leq \kappa \\
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A further generalization

Ore (1960)
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Chvátal & Erdős (1972)
\[ \alpha \leq \kappa \]
\[ \Rightarrow \exists \text{hamiltonian cycle} \]

Thm (Yamashita (2007))
For \( \forall S \) : indep. set of order \( \kappa + 1 \),
\[ \max \{ d_G(x) + d_G(y) : x, y \in S \} \geq n \]
\[ \Rightarrow \exists \text{hamiltonian cycle} \]
A further generalization

Ore (1960)
\[ \sigma_2 \geq n \]
\[ \Rightarrow \exists \text{hamiltonian cycle} \]

Chvátal & Erdős (1972)
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Thm (Yamashita (2007))
For \( \forall S \) : indep. set of order \( \kappa + 1 \),
\[ \max\{d_G(x) + d_G(y) : x, y \in S\} \geq n \]
\[ \Rightarrow \exists \text{hamiltonian cycle} \]

This theorem is stronger than Bondy’s Thm.
For $\forall S$ : indep. set of order $\kappa + 1$

\[ 2 \cdot \text{ave}\{d_G(x) : x \in S\} \geq n \quad \text{(by Bondy)} \]

\[ \max\{d_G(x) + d_G(y) : x, y \in S\} \geq n \quad \text{(by Y)} \]
Comparison of Ore’s Thm, Bondy’s Thm and Yamashita’s Thm

For $\forall S :$ indep. set of order $\kappa + 1$

$$2 \cdot \text{ave}\{d_G(x) : x \in S\} \geq n$$  \hspace{1cm} \text{(by Bondy)}

$$\max\{d_G(x) + d_G(y) : x, y \in S\} \geq n$$  \hspace{1cm} \text{(by Y)}

Since $\max\{d_G(x) + d_G(y) : x, y \in S\} \geq 2 \cdot \text{ave}\{d_G(x) : x \in S\}$,

Yamashita’s Thm is a generalization of Bondy’s Thm,

and hence is a generalization of Ore’s Thm.
Yamashita’s degree condition

Notation 11.

For $S \subseteq V(G)$ satisfying $|S| \geq t$

$$\Delta_t(S) = \max \left\{ \sum_{x \in X} d_G(x) : X \subseteq S, |X| = t \right\}$$

Invariant 12.

For $t \leq s$,

$$\sigma^s_t := \min \{ \Delta_t(S) : S \text{ is an indep. set of order } s \}$$

if $\alpha \geq s$; otherwise $\sigma^s_t = +\infty$.

Remark. $\sigma_2 = \sigma^2_2$, $\frac{\sigma^s_m}{m} \geq \frac{\sigma^s_t}{t}$ for $m \leq t$
Yamashita’s degree condition is stronger than Bondy’s condition

There exists a graph such that

Bondy’s Thm does not guarantee the existence of a hamiltonian cycle, but Yamashita’s Thm does.

\[ G = K_n + (K_1 \cup P_3) \]

Since \( n + 1 = 2(n + 1) = n \times 2 \),

\( G \) satisfies Yamashita’s condition.

Since \( n + 1 = 2 + n + 2 < (n + 1) \times 2 \),

\( G \) does not satisfy Bondy’s.
Yamashita’s degree condition is stronger than Bondy’s condition

There exists a graph such that

Bondy’s Thm does not guarantee the existence of a hamiltonian cycle, but Yamashita’s Thm does.

The graph $G = \overline{K}_\kappa + (\overline{K}_{\kappa-1} \cup P_3)$ has a hamiltonian cycle.
Yamashita’s degree condition is stronger than Bondy’s condition

There exists a graph such that Bondy’s Thm does not guarantee the existence of a hamiltonian cycle, but Yamashita’s Thm does.

The graph \( G = \overline{K}_\kappa + (\overline{K}_{\kappa-1} \cup P_3) \) has a hamiltonian cycle.

Since \( \sigma^{\kappa+1}_2 = 2(\kappa + 1) = n \), \( G \) satisfies Yamashita’s condition.
Yamashita’s degree condition is stronger than Bondy’s condition

There exists a graph such that

Bondy’s Thm does not guarantee the existence of a hamiltonian cycle, but Yamashita’s Thm does.

The graph $G = \overline{K}_\kappa + (\overline{K}_{\kappa-1} \cup P_3)$ has a hamiltonian cycle.

Since $\sigma_{2}^{\kappa+1} = 2(\kappa + 1) = n$, $G$ satisfies Yamashita’s condition.

Since $\sigma_{\kappa+1} = \kappa^2 + \kappa + 2 < \frac{(\kappa+1)n}{2}$, $G$ does not satisfy Bondy’s.
Thm (Yamashita (2007))

\[ \sigma_2^{k+1} \geq n \Rightarrow \exists \text{hamiltonian cycle} \]
Thm (Yamashita (2007))

\[ \sigma_{2}^{\kappa+1} \geq n \Rightarrow \exists \text{hamiltonian cycle} \]

Is the degree sum of \textbf{two} vertices important?
Let’s consider degree sum conditions other than two vertices.
\[ \sigma_t^{k+1} \] degree condition \((t \neq 2)\)

Thm (Yamashita (2007))

\[ \sigma_2^{k+1} \geq n \Rightarrow \exists \text{hamiltonian cycle} \]

Is the degree sum of two vertices important?
Let’s consider degree sum conditions other than two vertices.

Problem. Find best possible function \(f(n, t)\) and \(g(n)\).

1. \(\sigma_t^{k+1} > f(n, t)\) for \(t \geq 3 \Rightarrow \exists \text{hamiltonian cycle}\)
2. \(\sigma_1^{k+1} \geq g(n) \Rightarrow \exists \text{hamiltonian cycle}\)
The $\sigma_t^{\kappa+1}$ degree condition ($t \geq 3$)

The graph $\overline{K}_\kappa + \overline{K}_{\kappa+1}$ has no hamiltonian cycle.
The $\sigma_{t}^{\kappa+1}$ degree condition ($t \geq 3$)

The graph $\overline{K}_{\kappa} + \overline{K}_{\kappa+1}$ has no hamiltonian cycle.

Then $\sigma_{t}^{\kappa+1} = t\kappa = \frac{t(n-1)}{2}$ for $t \geq 3$
The degree condition \( (t \geq 3) \)

The graph \( \overline{K}_\kappa + \overline{K}_{\kappa+1} \) has no hamiltonian cycle.

Then \( \sigma_t^{\kappa+1} = t\kappa = \frac{t(n-1)}{2} \) for \( t \geq 3 \)

Therefore, the degree condition for hamiltonian cycles must be

\[
\sigma_t^{\kappa+1} > \frac{t(n-1)}{2}
\]
The $\sigma_t^{k+1}$ degree condition ($t \geq 3$)

**Problem**

$$\sigma_t^{k+1} > \frac{t(n-1)}{2} \text{ for } t \geq 3 \implies \exists \text{ hamiltonian cycle}$$
The $\sigma_t^{k+1}$ degree condition ($t \geq 3$)

**Problem**

\[
\sigma_t^{k+1} > \frac{t(n-1)}{2} \quad \text{for } t \geq 3 \Rightarrow \exists \text{hamiltonian cycle}
\]

\[
\frac{\sigma_2^{k+1}}{2} \geq \frac{\sigma_t^{k+1}}{t} \quad \text{for } t \geq 3, \text{ and so } \sigma_2^{k+1} \geq 2 \times \frac{\sigma_t^{k+1}}{t} > n - 1.
\]

Since $\sigma_2^{k+1}$ is an integer, $\sigma_2^{k+1} \geq n$. 
The $\sigma_{t}^{\kappa+1}$ degree condition ($t \geq 3$)

Problem Corollary

$$\sigma_{t}^{\kappa+1} > \frac{t(n - 1)}{2} \quad \text{for} \ t \geq 3 \Rightarrow \exists \text{hamiltonian cycle}$$

$$\frac{\sigma_{2}^{\kappa+1}}{2} \geq \frac{\sigma_{t}^{\kappa+1}}{t} \quad \text{for} \ t \geq 3, \ \text{and so} \ \sigma_{2}^{\kappa+1} \geq 2 \times \frac{\sigma_{t}^{\kappa+1}}{t} > n - 1.$$

Since $\sigma_{2}^{\kappa+1}$ is an integer, $\sigma_{2}^{\kappa+1} \geq n$.

Thus, the above problem is a corollary of Yamashita’s Thm.

Thm (Yamashita (2007))

$$\sigma_{2}^{\kappa+1} \geq n \Rightarrow \exists \text{hamiltonian cycle}$$
The $\sigma_1^{\kappa+1}$ condition

The graph $\overline{K}_\kappa + (\overline{K}_\kappa \cup K_m)$ has no hamiltonian cycle.
The $\sigma_1^{\kappa+1}$ condition

The graph $\overline{K}_\kappa + (\overline{K}_\kappa \cup K_m)$ has no hamiltonian cycle.

Then $\sigma_1^{\kappa+1} = \kappa + m - 1 = n - \kappa - 1$. 
The $\sigma_1^{\kappa+1}$ condition

The graph $\overline{K}_\kappa + (\overline{K}_\kappa \cup K_m)$ has no hamiltonian cycle.

Then $\sigma_1^{\kappa+1} = \kappa + m - 1 = n - \kappa - 1$.

Therefore, if $n \gg \kappa$, that is, $m$ is sufficient large,

(as far as we consider degree condition on order $n$)

the degree condition for hamiltonian cycles must be

$$\sigma_1^{\kappa+1} \geq n$$
Comparison between degree conditions

**Thm (Yamashita (2007))**

\[\sigma_2^{k+1} \geq n \Rightarrow \exists \text{hamiltonian cycle}\]

**Corollary**

\[\sigma_1^{k+1} \geq n \Rightarrow \exists \text{hamiltonian cycle}\]

**Corollary**

\[\sigma_t^{k+1} > \frac{t(n-1)}{2} \text{ for } t \geq 3 \Rightarrow \exists \text{hamiltonian cycle}\]
Comparison between degree conditions

**Thm (Yamashita (2007))**

\[ \sigma_2^{k+1} \geq n \Rightarrow \exists \text{hamiltonian cycle} \]

**Corollary**

\[ \sigma_1^{k+1} \geq n \Rightarrow \exists \text{hamiltonian cycle} \]

**Corollary**

\[ \sigma_t^{k+1} > \frac{t(n-1)}{2} \quad \text{for} \quad t \geq 3 \Rightarrow \exists \text{hamiltonian cycle} \]

Hence when we consider a \( \sigma_t^{k+1} \) condition, the case \( t = 2 \) is best.
A corollary of Yamashita’s Thm

**Thm (Yamashita (2007))**

For $\forall S$: index. set of order $\kappa + 1$,

$$\max\{d_G(x) + d_G(y) : x, y \in S\} \geq n$$

$\Rightarrow \exists$ hamiltonian cycle

**Flandrin, Li, Marczyk & Wozniak (2005)**

Let $V(G) = X_1 \cup X_2 \cup \cdots \cup X_\kappa$.

for $\forall i$, for $\forall x, \forall y \in X_i$ s.t. $xy \notin E(G)$,

$$d_G(x) + d_G(y) \geq n$$

$\Rightarrow \exists$ hamiltonian cycle
Flandrin, Li, Marczyk & Wozniak (2005)

Let $V(G) = X_1 \cup X_2 \cup \cdots \cup X_\kappa$.

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Let $\mathcal{S}$ be an indep. set of order $\kappa + 1$. 
A corollary of Yamashita’s Thm

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for $\forall i$, for $\forall x, \forall y \in X_i$ s.t. $xy \not\in E(G)$,

$$d_G(x) + d_G(y) \geq n$$

$\Rightarrow \exists$ hamiltonian cycle

Let $S$ be an indep. set of order $\kappa + 1$.

By the pigeonhole principal,

there exits an integer $i$ s.t. $x, y \in S \cap X_i$. 
A corollary of Yamashita’s Thm

Flandrin, Li, Marczyk & Wozniak (2005)

Let $V(G) = X_1 \cup X_2 \cup \cdots \cup X_\kappa$.

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$\Rightarrow$ hamiltonian cycle

Let $S$ be an indep. set of order $\kappa + 1$.

By the pigeonhole principal, there exits an integer $i$ s.t. $x, y \in S \cap X_i$.

From the above condition, $d_G(x) + d_G(y) \geq n$
A corollary of Yamashita’s Thm

Flandrin, Li, Marczyk & Wozniak (2005)

Let $V(G) = X_1 \cup X_2 \cup \cdots \cup X_\kappa$.
for $\forall i$, for $\forall x$, $\forall y \in X_i$ s.t. $xy \notin E(G)$,
$$d_G(x) + d_G(y) \geq n$$
$\Rightarrow \exists$ hamiltonian cycle

Let $S$ be an indep. set of order $\kappa + 1$.
By the pigeonhole principal, there exits an integer $i$ s.t. $x, y \in S \cap X_i$.
From the above condition, $d_G(x) + d_G(y) \geq n$
Therefore the graph satisfies Yamashita’s condition.
A corollary of Yamashita’s Thm

Thm (Yamashita (2007))

For \( \forall S: \) index. set of order \( \kappa + 1 \),

\[
\max\{d_G(x) + d_G(y) : x, y \in S\} \geq n
\]

\( \Rightarrow \exists \) hamiltonian cycle

Let \( S \) be an indep. set of order \( \kappa + 1 \).
By the pigeonhole principal,
there exits an integer \( i \) s.t. \( x, y \in S \cap X_i \).
From the above condition, \( d_G(x) + d_G(y) \geq n \)
Therefore the graph satisfies Yamashita’s condition.
Two generalizations of a hamiltonian cycles.
Two generalizations of Hamiltonian cycle

\[ C : \text{Hamiltonian cycle} \]
\[ \iff V(C) = V(G) \text{ or } |V(C)| = |V(G)| \]
Two generalizations of hamiltonian cycle

$$C : \text{hamiltonian cycle}$$

$$\iff V(C) = V(G) \text{ or } |V(C)| = |V(G)|$$

\[\text{generalizations}\]

For $S \subseteq V(G)$

$C : \text{cycle s.t. } S \subseteq V(C)$

cycle passing thru $S$

For $\ell \in \mathbb{Z}$

$C : \text{cycle s.t. } \ell \leq |V(C)|$

cycle of length at least $\ell$
Let $S \subseteq V(G)$.

**Invariant 13.**

$\alpha(S) := \max\{|X| : X \text{ is an indep. set of } G[S]\}$,

where $G[S]$ is a subgraph of $G$ induced by $S$. 
Let $S \subseteq V(G)$.

**Invariant 13.**

$\alpha(S) := \max\{|X| : X \text{ is an indep. set of } G[S]\}$,

where $G[S]$ is a subgraph of $G$ induced by $S$.

**Invariant 14.**

$\kappa(S) := \min\{\kappa(x, y) : x, y \in S, \ x \neq y\}$
Let \(S \subseteq V(G)\).

Invariant 13.

\[
\alpha(S) := \max \{ |X| : X \text{ is an indep. set of } G[S] \},
\]

where \(G[S]\) is a subgraph of \(G\) induced by \(S\).

Invariant 14.

\[
\kappa(S) := \min \{ \kappa(x, y) : x, y \in S, x \neq y \}
\]

Invariant 15.

\[
\sigma_t(S) := \min \left\{ \sum_{x \in X} d_G(x) : X \text{ is an indep. set of } G[S] \text{ of order } t \right\}
\]

if \(\alpha(S) \geq t\); otherwise \(\sigma_t(S) = +\infty\)
The degree condition for cycles passing through specified vertices

\[ G : \text{connected graph, } S \subseteq V(G), \kappa(S) \geq 2 \]

\[
\begin{align*}
\text{Shi (1991)} & : \sigma_2(S) \geq n \\
\Rightarrow & \exists \text{ cycle passing thru } S
\end{align*}
\]

\[
\begin{align*}
\text{Broersma et al. (1997)} & : \alpha(S) \leq \kappa(S) \\
\Rightarrow & \exists \text{ cycle passing thru } S
\end{align*}
\]
The degree condition for cycles passing thru specified vertices

\[G: \text{connected graph, } S \subseteq V(G), \ \kappa(S) \geq 2\]

Shi (1991)\[\sigma_2(S) \geq n\] \[\Rightarrow \exists \text{cycle passing thru } S\]

Broersma et al. (1997)\[\alpha(S) \leq \kappa(S)\] \[\Rightarrow \exists \text{cycle passing thru } S\]

Ore (1960)\[\sigma_2 \geq n\] \[\Rightarrow \exists \text{hamiltonian cycle}\]

Chvátal & Erdős (1972)\[\alpha \leq \kappa\] \[\Rightarrow \exists \text{hamiltonian cycle}\]
The degree condition for cycles passing thru specified vertices

\( G : \) connected graph, \( S \subseteq V(G), \kappa(S) \geq 2 \)

- **Shi (1991)**
  \[ \sigma_2(S) \geq n \]
  \[ \Rightarrow \exists \text{ cycle passing thru } S \]

- **Broersma et al. (1997)**
  \[ \alpha(S) \leq \kappa(S) \]
  \[ \Rightarrow \exists \text{ cycle passing thru } S \]

**Thm (Yamashita (2007))**

\[ \sigma_2^{k+1}(S') \geq n \]

\[ \Rightarrow \exists \text{ cycle passing through } S \]
The degree condition for long cycle

**Definition 13.**

\[ G : m\text{-connected graph} \iff m \leq \kappa \]

\[ G : 2\text{-connected graph, } \sigma_2 \geq \ell \]

\[ \Rightarrow \exists \text{cycle of length at least } \ell, \text{ or } \exists \text{hamiltonian cycle} \]

Bermond (1976), Linial (1976)
Definition 13.

$G : m$-connected graph $\iff m \leq \kappa$

Bermond (1976), Linial (1976)

$G : 2$-connected graph, $\sigma_2 \geq \ell$

$\Rightarrow \exists$ cycle of length at least $\ell$, or $\exists$ hamiltonian cycle

Later, we omit “or $\exists$ hamiltonian cycle” as below.

Bermond (1976), Linial (1976)

$G : 2$-connected graph $\Rightarrow \exists$ cycle of length $\geq \sigma_2$
Definition 13.
$G : m$-connected graph $\iff m \leq \kappa$

Bermond (1976), Linial (1976)
$G : 2$-connected graph $\Rightarrow \exists \text{cycle of length} \geq \sigma_2$
Definition 13.

$G : m$-connected graph $\iff m \leq \kappa$

Bermond (1976), Linial (1976)

$G : 2$-connected graph $\Rightarrow \exists$ cycle of length $\geq \sigma_2$

Thm (Yamashita (2007))

$G : 2$-connected graph $\Rightarrow \exists$ cycle of length $\geq \sigma_2^{\kappa+1}$
The degree condition for long cycle

Definition 13.

\[ G : m\text{-connected graph} \iff m \leq \kappa \]

Thm (Yamashita (2007))

\[ G : 2\text{-connected graph} \Rightarrow \exists \text{cycle of length} \geq \sigma_{2}^{\kappa+1} \]

Fournier & Fraisse (1985)

\[ G : 2\text{-connected graph} \Rightarrow \exists \text{cycle of length at least} \geq 2 \cdot \frac{\sigma_{\kappa+1}}{\kappa+1} \]
The combination of two generalizations
The combination of two generalizations

For $S \subseteq V(G)$, cycle passing thru $S$

For $\ell \in \mathbb{Z}$, cycle of length $\geq \ell$

combine

For $S \subseteq V(G)$, $\ell \in \mathbb{Z}$, cycle passing thru $S$ of length $\geq \ell$
Dirac’s results for cycles

Dirac (1951)

\[ G : \text{2-connected graph} \]
\[ \Rightarrow \exists \text{cycle of length at least } 2\delta \]

Invariant 14.

\[ \delta := \min \{ d_G(x) : x \in V(G) \} \]

Dirac (1960)

\[ G : \text{2-connected graph, } S \subseteq V(G), |S| \leq \kappa \]
\[ \Rightarrow \exists \text{cycle passing through } S \]
The sharpness of Dirac’s results

Dirac (1951)

\[ G : 2\text{-connected graph} \]

\[ \Rightarrow \exists \text{ cycle of length } \geq 2\delta \]

Dirac (1960)

\[ G : 2\text{-connected graph} \]

\[ S \subseteq V(G), |S| \leq \kappa \]

\[ \Rightarrow \exists \text{ cycle passing thru } S \]

In the graph \( \overline{K}_\kappa + \overline{K}_{\kappa+1} \)

and \( S = V(\overline{K}_{\kappa+1}) \),

\( \not\exists \text{ cycle of length } \geq 2\delta + 1 \)

\( \not\exists \text{ cycle passing thru } S \)
Egawa, Glas & Lock’s result

Dirac (1951)  
\[ G : 2\text{-connected graph} \]
\[ \Rightarrow \exists \text{cycle of length} \geq 2\delta \]

Dirac (1960)
\[ G : 2\text{-connected graph} \]
\[ S \subseteq V(G), |S| \leq \kappa \]
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Egawa, Glas & Lock’s result

Dirac (1951)

\[ G : \text{2-connected graph} \]
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Dirac (1960)

\[ G : \text{2-connected graph} \]
\[ S \subseteq V(G), \ |S| \leq \kappa \]
\[ \Rightarrow \exists \text{cycle passing thru } S \]

Egawa, Glas & Lock (1991)

\[ G : \text{2-connected graph, } S \subseteq V(G), \ |S| \leq \kappa \]
\[ \Rightarrow \exists \text{cycle passing through } S \text{ of length } \geq 2\delta \]
Fujisawa & Yamashita’s result

\[ \text{Dirac (1951)} \]
\[ G : \text{2-connected graph} \]
\[ \Rightarrow \exists \text{cycle of length } \geq 2\delta \]

\[ \text{Dirac (1960)} \]
\[ G : \text{2-connected graph} \]
\[ S \subseteq V(G), |S| \leq \kappa \]
\[ \Rightarrow \exists \text{cycle passing thru } S \]
Fujisawa & Yamashita’s result

Thm (Yamashita (2007))
\[ G : 2\text{-connected graph} \]
\[ \Rightarrow \exists \text{cycle of length} \geq \sigma_2^{\kappa+1} \]

Dirac (1960)
\[ G : 2\text{-connected graph} \]
\[ S \subseteq V(G), |S| \leq \kappa \]
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Fujisawa & Yamashita’s result

**Thm (Yamashita (2007))**

\[ G : 2\text{-}connected \text{ graph} \]
\[ \Rightarrow \exists \text{ cycle of length } \geq \sigma_2^{\kappa+1} \]

**Dirac (1960)**

\[ G : 2\text{-}connected \text{ graph} \]
\[ S \subseteq V(G), \quad |S| \leq \kappa \]
\[ \Rightarrow \exists \text{ cycle passing thru } S \]

**Thm (Fujisawa & Yamashita (2007))**

\[ G : 2\text{-}connected \text{ graph, } S \subseteq V(G), \quad |S| \leq \kappa - 2 \]
\[ \Rightarrow \exists \text{ cycle passing thru } S \text{ of length } \geq \sigma_2^{\kappa+1} \]
The sharpness of Fujisawa & Yamashita’s result

The graph $K_\kappa + (\overline{K}_{\kappa-1} \cup 2K_m)$ has no hamiltonian cycle, and

$$\sigma_{2}^{\kappa+1} = 2(\kappa + m - 1)$$

$\kappa$ 頂点

$\bullet \bullet \bullet \bullet \bullet$

$K_m$

$K_m$

$+ \bullet \bullet \bullet \bullet \bullet$

$\kappa$ 頂点
The sharpness of Fujisawa & Yamashita’s result

The graph $K_\kappa + (\overline{K}_{\kappa-1} \cup 2K_m)$ has no hamiltonian cycle, and

$$\sigma_{2}^{\kappa+1} = 2(\kappa + m - 1)$$

If $S \subset V(\overline{K}_{\kappa-1})$ s.t. $|S| = \kappa - 2$, then

the max. length of a cycle passing thru $S$ is

$$2\kappa + 2m - 2 = \sigma_{2}^{\kappa+1}$$
The sharpness of Fujisawa & Yamashita’s result

The graph $K_\kappa + (\overline{K}_{\kappa-1} \cup 2K_m)$ has no hamiltonian cycle, and

$$\sigma_{2}^{\kappa+1} = 2(\kappa + m - 1)$$

If $S \subset V(\overline{K}_{\kappa-1})$ s.t. $|S| = \kappa - 2$, then

the max. length of a cycle passing thru $S$ is $2\kappa + 2m - 2 = \sigma_{2}^{\kappa+1}$

If $S = V(\overline{K}_{\kappa-1})$ then,

the max. length of a cycle passing thru $S$ is $2\kappa + m - 1 < \sigma_{2}^{\kappa+1}$
The sharpness of Fujisawa & Yamashita’s result

The graph $K_\kappa + (\overline{K}_{\kappa-1} \cup 2K_m)$ has no hamiltonian cycle, and

$$\sigma_{2}^{\kappa+1} = 2(\kappa + m - 1)$$

If $S \subseteq V(\overline{K}_{\kappa-1})$ s.t. $|S| = \kappa - 2$, then

the max. length of a cycle passing thru $S$ is $2\kappa + 2m - 2 = \sigma_{2}^{\kappa+1}$

If $S \subseteq V(\overline{K}_{\kappa-1} \cup K_m)$ s.t. $|S| = \kappa$, then

the max. length of a cycle passing thru $S$ is $2\kappa + m - 1$
The sharpness of Fujisawa & Yamashita’s result

The graph \( K_\kappa + (\overline{K}_{\kappa-1} \cup 2K_m) \) has no hamiltonian cycle, and
\[
\sigma_2^{\kappa+1} = 2(\kappa + m - 1)
\]

If \( S \subset V(\overline{K}_{\kappa-1}) \) s.t. \(|S| = \kappa - 2\), then
the max. length of a cycle passing thru \( S \) is \( 2\kappa + 2m - 2 = \sigma_2^{\kappa+1} \)

If \( S \subset V(\overline{K}_{\kappa-1} \cup K_m) \) s.t. \(|S| = \kappa\), then
the max. length of a cycle passing thru \( S \) is \( 2\kappa + m - 1 = \sigma_2^\kappa \)
Fujisawa & Yamashita’s Conjecture

**Thm (Yamashita (2007))**

\[ G : 2\text{-connected graph} \]

\[ \Rightarrow \exists \text{ cycle of length } \sigma_2^{\kappa+1} \]

**Dirac (1960)**

\[ G : 2\text{-connected graph} \]

\[ S \subseteq V(G), \ |S| \leq \kappa \]

\[ \Rightarrow \exists \text{ cycle passing thru } S \]

**Conjecture (Fujisawa & Yamashita (2007))**

\[ G : 2\text{-connected graph, } S \subseteq V(G), \ |S| \leq \kappa \]

\[ \Rightarrow \exists \text{ cycle passing thru } S \text{ of length } \geq \sigma_2^{\kappa} \]
**Fujisawa & Yamashita’s Theorem and Conjecture**

**Thm (Yamashita (2007))**

\[ G : 2\text{-connected graph} \]
\[ \Rightarrow \exists \text{ cycle of length } \sigma_{2}^{\kappa+1} \]

**Dirac (1960)**

\[ G : 2\text{-connected graph} \]
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**Thm (Fujisawa & Yamashita (2007))**

\[ G : 2\text{-connected graph}, \ S \subseteq V(G), \ |S| \leq \kappa - 2 \]
\[ \Rightarrow \exists \text{ cycle passing through } S \text{ of length } \geq \sigma_{2}^{\kappa+1} \]

**Conjecture (Fujisawa & Yamashita (2007))**

\[ G : 2\text{-connected graph}, \ S \subseteq V(G), \ |S| \leq \kappa \]
\[ \Rightarrow \exists \text{ cycle passing through } S \text{ of length } \geq \sigma_{2}^{\kappa} \]
Degree condition of $k$ vertices
Chvátal & Erdős (1972)

\[ \forall x \leq \kappa \Rightarrow \exists \text{hamiltonian cycle} \]

By considering Chvátal&Erdős’s Thm, we should consider degree conditions for graphs having indep. sets of order \( \kappa + 1 \).
Degree conditions containing other invariants than order $n$

Chvátal & Erdős (1972)

$\alpha \leq \kappa \Rightarrow \exists \text{hamiltonian cycle}$

By considering Chvátal & Erdős’s Thm, we should consider degree conditions for graphs having indep. sets of order $\kappa + 1$.

Thm (Yamashita (2007))

$\sigma_{2^{\kappa+1}} \geq n \Rightarrow \exists \text{hamiltonian cycle}$
Degree conditions containing other invariants than order $n$

Chvátal & Erdős (1972)

$$\alpha \leq \kappa \Rightarrow \exists \text{hamiltonian cycle}$$

By considering Chvátal & Erdős’s Thm, we should consider degree conditions for graphs having indep. sets of order $\kappa + 1$.

Thm (Yamashita (2007))

$$\sigma_{2}^{\kappa+1} \geq n \Rightarrow \exists \text{hamiltonian cycle}$$

Yamashita’s Thm says that we should consider degree sum of two vertices.
Degree conditions containing other invariants than order $n$

Chvátal & Erdős (1972)

\[ \alpha \leq \kappa \Rightarrow \exists \text{hamiltonian cycle} \]

By considering Chvátal & Erdős’s Thm, we should consider degree conditions for graphs having indep. sets of order $\kappa + 1$.

Thm (Yamashita (2007))

\[ \sigma_{2}^{\kappa+1} \geq n \Rightarrow \exists \text{hamiltonian cycle} \]

Yamashita’s Thm says that we should consider degree sum of two vertices.

But ...
It is the case where the lower bound is a function of the order $n$.

“If $\sigma_m \geq f(n)$ then $\exists$ hamiltonian cycle.”
It is the case where the lower bound is a function of the order $n$.

“If $\sigma_m \geq f(n)$ then $\exists$ hamiltonian cycle.”

How about if we consider also another invariant $\varphi$?

“If $\sigma_m \geq f(n) + g(\varphi)$ then $\exists$ hamiltonian cycle.”
$\sigma_3$ for Hamiltonian cycles

It is the case where the lower bound is a function of the order $n$.

“If $\sigma_m \geq f(n)$ then $\exists$ Hamiltonian cycle.”

How about if we consider also another invariant $\varphi$?

“If $\sigma_m \geq f(n) + g(\varphi)$ then $\exists$ Hamiltonian cycle.”

Bauer, Broersma, Li & Veldman (1989)

$G$ : 2-connected graph

$\sigma_3 \geq n + \kappa$

$\Rightarrow \exists$ Hamiltonian cycle

$\exists \sigma_3$ condition on the order $n$ and the connectivity $\kappa$. 
The $\sigma_3$ condition is sharp.

Let $2 \leq \kappa < m$ and $\frac{n+\kappa-1}{3} < m \leq \frac{n-1}{2}$.

The graph $K_{n-2m} + \overline{K}_\kappa + \overline{K}_m + \overline{K}_{m-\kappa}$ has no hamiltonian cycle.

\[
\begin{align*}
\text{d}_G(v) &= m \text{ for } \forall v \in V(\overline{K}_m \cup \overline{K}_{m-\kappa}) \\
\text{d}_G(v) &= n - 2m + \kappa - 1 \leq m \text{ for } \forall v \in V(K_{n-2m}) \\
\text{d}_G(v) &= n - m \geq n - 2m + \kappa - 1 \text{ for } \forall v \in V(\overline{K}_\kappa)
\end{align*}
\]

Hence $\sigma_3 = n + \kappa - 1$.

Bauer et al. (1989)

$G: 2$-connected graph

$\sigma_3 \geq n + \kappa$

$\Rightarrow \exists$ hamiltonian cycle
If $\sigma_3^{\kappa+1} \geq n + \kappa$, then the graph has a hamiltonian cycle?

Problem

$G$ : 2-connected graph

$\sigma_3^{\kappa+1} \geq n + \kappa \Rightarrow \exists$ hamiltonian cycle
If $\sigma_3^{k+1} \geq n + \kappa$, then the graph has a hamiltonian cycle?

**Problem**  False statement

$G: 2$-connected graph

$\sigma_3^{k+1} \geq n + \kappa \Rightarrow \exists$ hamiltonian cycle

Let $2 \leq \kappa < m$ and $\frac{n+\kappa-1}{3} < m \leq \frac{n-1}{2}$.

The graph $K_{n-2m} + \overline{K}_\kappa + \overline{K}_m + \overline{K}_{m-\kappa}$ has no hamiltonian cycle.

Then $\sigma_3^{k+1} = 3m \geq n + \kappa$. 

\[ 
\begin{tikzpicture}
  \node (m) at (0,0) {\text{m 頂点}};
  \node (kappa) at (0,-2) {\text{\kappa 頂点}};
  \node (m2) at (2,-1) {\text{\kappa 頂点}};
  \node (K) at (4,0) {\text{\text{K}}_{n-2m}};
  \node (n) at (-2,0) {\text{\text{K}}_\kappa + \text{\overline{K}}_m + \text{\overline{K}}_{m-\kappa}};
  \draw (m) -- (kappa) -- (m2) -- (K) -- (n);
  \end{tikzpicture} 
\]
We now compare between the $\sigma_2$ condition and the $\sigma_3$ condition.

**Ore (1960)\)**

\[
\sigma_2 \geq n \\
\Rightarrow \exists \text{hamiltonian cycle}
\]

**Bauer, Broersma, Li & Veldman (1989)**

$G$ : 2-connected graph

\[
\sigma_3 \geq n + \kappa \\
\Rightarrow \exists \text{hamiltonian cycle}
\]
Problem

\[ \sigma_2 \geq n \Rightarrow \exists \text{hamiltonian cycle } \text{(Ore (1960))} \]
\[ \downarrow + \kappa \]

\[ \sigma_3 \geq n + \kappa \Rightarrow \exists \text{hamiltonian cycle } \text{(Bauer et al. (1989))} \]
Problem

\[ \sigma_2 \geq n \Rightarrow \exists \text{hamiltonian cycle} \quad \text{(Ore (1960))} \]
\[ \downarrow \quad + \kappa \]

\[ \sigma_3 \geq n + \kappa \Rightarrow \exists \text{hamiltonian cycle} \quad \text{(Bauer et al. (1989))} \]
\[ \downarrow \quad + \kappa \quad ??? \]

\[ \sigma_4 \geq n + 2\kappa \Rightarrow \exists \text{hamiltonian cycle} \quad ??? \]
Relation degree conditions for hamiltonian cycle

Problem

\[ \sigma_2 \geq n \Rightarrow \exists \text{hamiltonian cycle} \quad \text{(Ore (1960))} \]
\[ \downarrow + \kappa \]

\[ \sigma_3 \geq n + \kappa \Rightarrow \exists \text{hamiltonian cycle} \quad \text{(Bauer et al. (1989))} \]
\[ \downarrow + \kappa \quad ??? \]

\[ \sigma_4 \geq n + 2\kappa \Rightarrow \exists \text{hamiltonian cycle} \quad ??? \]

\[ \uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow \]

This is not true.
If $\sigma_4 \geq n + 2\kappa$, then the graph has hamiltonian cycle?

**False statement**

$G$: 3-connected graph

$\sigma_4 \geq n + 2\kappa \Rightarrow \exists$ hamiltonian cycle

Let $2 \leq \kappa < m \leq \frac{n-1}{2}$.

The graph $K_{n-2m} + \overline{K}_\kappa + \overline{K}_m + \overline{K}_{m-\kappa}$ has no hamiltonian cycle.

$\sigma_4 = n + \kappa + m - 1 \geq n + 2\kappa$. 
Relation of degree conditions for hamiltonian cycle

Problem

$$\sigma_2 \geq n \Rightarrow \exists \text{hamiltonian cycle}$$
$$\downarrow \quad + \kappa$$
$$\sigma_3 \geq n + \kappa \Rightarrow \exists \text{hamiltonian cycle}$$
$$\downarrow$$
$$\sigma_4 \geq n + ??? \Rightarrow \exists \text{hamiltonian cycle}$$

What kind of invariant is necessary?
A $\sigma_4$ condition

(Chvátal & Erdős (1972))

$\alpha \leq \kappa \Rightarrow \exists \text{hamiltonian cycle}$

We should consider degree conditions for graphs satisfying $\alpha \geq \kappa + 1$. 
A $\sigma_4$ condition

Chvátal & Erdős (1972)

\[ \alpha \leq \kappa \Rightarrow \exists \text{hamiltonian cycle} \]

We should consider degree conditions for graphs satisfying $\alpha \geq \kappa + 1$.

For such graphs, the following inequality holds:

\[ n + 2\alpha - 2 \geq n + 2\kappa \]
A $\sigma_4$ condition

Chvátal & Erdős (1972)

$\alpha \leq \kappa \Rightarrow \exists \text{hamiltonian cycle}$

We should consider degree conditions for graphs satisfying $\alpha \geq \kappa + 1$.
For such graphs, the following inequality holds:

$$n + 2\alpha - 2 \geq n + 2\kappa$$

False statement

$G : 3$-connected graph

$\sigma_4 \geq n + 2\kappa$

$\Rightarrow \exists \text{hamiltonian cycle}$
A $\sigma_4$ condition

Chvátal & Erdős (1972)

\[ \alpha \leq \kappa \Rightarrow \exists \text{hamiltonian cycle} \]

We should consider degree conditions for graphs satisfying $\alpha \geq \kappa + 1$.

For such graphs, the following inequality holds:

\[ n + 2\alpha - 2 \geq n + 2\kappa \]

Harkat, Tian & Li (2000)

$G$: 3-connected graph

\[ \sigma_4 \geq n + 2\alpha - 2 \]

$\Rightarrow \exists \text{hamiltonian cycle}$

False statement

$G$: 3-connected graph

\[ \sigma_4 \geq n + 2\kappa \]

$\Rightarrow \exists \text{hamiltonian cycle}$
Three $\sigma_4$ conditions

For graphs satisfying $\alpha \geq \kappa + 1$,

$$n + 2\alpha - 2 \geq n + 2\kappa$$

Harkat, Tian & Li (2000)

$G$ : 3-connected graph

$\sigma_4 \geq n + 2\alpha - 2$

$\Rightarrow \exists$ hamiltonian cycle

False statement

$G$ : 3-connected graph

$\sigma_4 \geq n + 2\kappa$

$\Rightarrow \exists$ hamiltonian cycle
Three $\sigma_4$ conditions

For graphs satisfying $\alpha \geq \kappa + 1$,

$$n + 2\alpha - 2 \geq n + \kappa + \alpha - 1 \geq n + 2\kappa$$

Harkat, Tian & Li (2000)

$G : 3$-connected graph

$\sigma_4 \geq n + 2\alpha - 2$

$\Rightarrow \exists$ hamiltonian cycle

False statement

$G : 3$-connected graph

$\sigma_4 \geq n + 2\kappa$

$\Rightarrow \exists$ hamiltonian cycle
Three $\sigma_4$ conditions

For graphs satisfying $\alpha \geq \kappa + 1$,

$$n + 2\alpha - 2 \geq n + \kappa + \alpha - 1 \geq n + 2\kappa$$

**Thm (Ozeki & Yamashita (2008))**

- $G$ : 3-connected graph
- $\sigma_4 \geq n + \kappa + \alpha - 1$
- $\Rightarrow \exists$ hamiltonian cycle

**False statement**

- $G$ : 3-connected graph
- $\sigma_4 \geq n + 2\kappa$
- $\Rightarrow \exists$ hamiltonian cycle

Harkat, Tian & Li (2000)

- $G$ : 3-connected graph
- $\sigma_4 \geq n + 2\alpha - 2$
- $\Rightarrow \exists$ hamiltonian cycle

Tomoki Yamashita (Kinki University)
The $\sigma_4$ condition is sharp.

Let $2 \leq \kappa < m$ and $\frac{n+\kappa-1}{3} < m \leq \frac{n-1}{2}$.

The graph $K_{n-2m} + \overline{K}_\kappa + \overline{K}_m + \overline{K}_{m-\kappa}$ has no hamiltonian cycle.

\[
\sigma_4 = n - 2m + \kappa - 1 + 3m \\
= n + \kappa + m - 1 \\
= n + \kappa + \alpha - 2
\]

**Thm (Ozeki & Y (2008))**

$G$: 3-connected graph

$\sigma_4 \geq n + \kappa + \alpha - 1$

$\Rightarrow \exists$ hamiltonian cycle
Problem

\[
\begin{align*}
\sigma_2 & \geq n \\
\downarrow & + \kappa \\
\sigma_3 & \geq n + \kappa \\
\downarrow & + (\alpha - 1) \\
\sigma_4 & \geq n + \kappa + \alpha - 1
\end{align*}
\]
Relation of degree conditions for a hamiltonian cycle

Problem

\[
\begin{align*}
\sigma_2 & \geq n \\
\downarrow & \quad + \kappa \\
\sigma_3 & \geq n + \kappa \\
\downarrow & \quad + (\alpha - 1) \\
\sigma_4 & \geq n + \kappa + \alpha - 1 \\
\downarrow & \quad + ??? \\
\sigma_5 & \geq n + \kappa + \alpha - 1 + ??? \\
\uparrow \uparrow \uparrow \\
\text{Which invariant do we need?}
\end{align*}
\]
Another $\sigma_2$ condition than Ore’s condition

Ore (1960)

\[ \sigma_2 \geq n \]
\[ \Rightarrow \exists \text{hamiltonian cycle} \]

Chvátal & Erdős (1972)

\[ \alpha \leq \kappa \]
\[ \Rightarrow \exists \text{hamiltonian cycle} \]
Another $\sigma_2$ condition than Ore’s condition

Ore (1960)

\[
\sigma_2 \geq n \\
\Rightarrow \exists \text{hamiltonian cycle}
\]

Chvátal & Erdős (1972)

\[
\alpha \leq \kappa \\
\Rightarrow \exists \text{hamiltonian cycle}
\]

For graphs satisfying $\alpha \geq \kappa + 1$, the following holds:

\[
n \geq n + \kappa - \alpha + 1
\]
Another $\sigma_2$ condition than Ore’s condition

Ore (1960)

$\sigma_2 \geq n$

$\Rightarrow \exists$ hamiltonian cycle

Chvátal & Erdős (1972)

$\alpha \leq \kappa$

$\Rightarrow \exists$ hamiltonian cycle

For graphs satisfying $\alpha \geq \kappa + 1$, the following holds:

$n \geq n + \kappa - \alpha + 1$

Fraisse & Jung (1989)

$\sigma_2 \geq n + \kappa - \alpha + 1$

$\Rightarrow \exists$ hamiltonian cycle
The $\sigma_2$ condition is sharp.

Let $2 \leq \kappa < m$ and $\frac{n+\kappa-1}{3} < m \leq \frac{n-1}{2}$.

The graph $K_{n-2m} + \overline{K}_\kappa + \overline{K}_m + \overline{K}_{m-\kappa}$ has no hamiltonian cycle.

\[
\sigma_2 = n - 2m + \kappa - 1 + m \\
= n + \kappa - m - 1 \\
= n + \kappa - \alpha - 2
\]

Fraisse & Jung (1989)

$\sigma_2 \geq n + \kappa - \alpha + 1$

$\therefore$ hamiltonian cycle
Problem

\[ \sigma_2 \geq n + \kappa - (\alpha - 1) \]
\[ \downarrow \quad + (\alpha - 1) \]
\[ \sigma_3 \geq n + \kappa \]
\[ \downarrow \quad + (\alpha - 1) \]
\[ \sigma_4 \geq n + \kappa + (\alpha - 1) \]
Relation of degree conditions for a hamiltonian cycle

Problem

\[ \sigma_2 \geq n + \kappa - (\alpha - 1) \]
\[ \downarrow \quad + (\alpha - 1) \]
\[ \sigma_3 \geq n + \kappa \]
\[ \downarrow \quad + (\alpha - 1) \]
\[ \sigma_4 \geq n + \kappa + (\alpha - 1) \]
\[ \downarrow \quad + (\alpha - 1) \]
\[ \sigma_5 \geq n + \kappa + 2(\alpha - 1) \]
Conjecture (Ozeki & Y (2008))

\[ G : m \text{-connected graph} \]

\[ \sigma_{m+1} \geq n + \kappa + (m - 3)(\alpha - 1) \]

\[ \implies \exists \text{hamiltonian cycle} \]

The case \( m = 1 \) is Fraisse&Jung’s Thm

The case \( m = 2 \) is Bauer et al.’s Thm

The case \( m = 3 \) is Ozeki&Y’s Thm
How about two generalizations of a hamiltonian cycle?
Two generalizations of Hamiltonian cycle

- Hamiltonian cycle
- Cycle passing through $V(G)$
- Cycle of length $n$

Generalizations

For $S \subseteq V(G)$,
- Cycle passing through $S$

For $\ell \in \mathbb{Z}$,
- Cycle of length $\geq \ell$
Broersma, H. Li, J. Li, Tian & Veldman (1997)

\( G : \) connected graph, \( S \subseteq V(G), \kappa'(S) \geq 2 \)

\[ \sigma_3(S) \geq n + \min\{\kappa'(S), \delta(S)\} \]

\Rightarrow \exists \text{ cycle passing through } S

**Invariant 15.**

\( \kappa'(S) : \) the mini. order of a set which separates two vertices of \( S \)

**Invariant 16.**

\( \delta(S) := \min\{d_G(x) : x \in S\} \)
Relation between $\kappa'(S')$, $\delta(S')$ and $\kappa(S')$

**Invariant 15.**

$\kappa'(S')$ : the mini. order of a set which separates two vertices of $S'$

**Invariant 16.**  

$\delta(S') := \min \{ d_G(x) : x \in S' \}$

**Invariant 13.**  

$\kappa(S') = \min \{ \kappa(x, y) : x, y \in S, \ x \neq y \}$,

where $\kappa(x, y)$ : the max. $\#$ of vertex-disjoint paths joining $x$ & $y$. 
Relation between $\kappa'(S)$, $\delta(S)$ and $\kappa(S)$

**Invariant 15.**

$\kappa'(S)$ : the mini. order of a set which separates two vertices of $S$

**Invariant 16.**

$\delta(S) := \min\{d_G(x) : x \in S\}$

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$\kappa(S) = \min\{\kappa(x, y) : x, y \in S, \ x \neq y\}$,

where $\kappa(x, y)$ : the max. # of vertex-disjoint paths joining $x \& y$.

We consider the graph $\overline{K}_2 + \overline{K}_m$. 

![Diagram of graph $\overline{K}_2 + \overline{K}_m$.]
Relation between $\kappa'(S)$, $\delta(S)$ and $\kappa(S)$

**Invariant 15.**

$\kappa'(S)$: the mini. order of a set which separates two vertices of $S$

**Invariant 16.**

$\delta(S) := \min\{d_G(x) : x \in S\}$

**Invariant 13.**

$\kappa(S) = \min\{\kappa(x, y) : x, y \in S, \ x \neq y\}$, where $\kappa(x, y)$: the max. $\#$ of vertex-disjoint paths joining $x$ & $y$.

We consider the graph $\overline{K_2} + \overline{K_m}$.

$\kappa'(S) = m$, $\delta(S) = 2$
Relation between $\kappa'(S')$, $\delta(S')$ and $\kappa(S')$

**Invariant 15.**

$\kappa'(S')$: the mini. order of a set which separates two vertices of $S$

**Invariant 16.**

$\delta(S') := \min\{d_G(x) : x \in S\}$

**Invariant 13.**

$\kappa(S') = \min\{\kappa(x, y) : x, y \in S, x \neq y\}$,

where $\kappa(x, y)$: the max. # of vertex-disjoint paths joining $x$ & $y$.

We consider the graph $\overline{K}_2 + \overline{K}_m$.

$\kappa'(S') = m$, $\delta(S') = 2$

$\min\{\kappa'(S'), \delta(S')\} = 2$
Relation between $\kappa'(S)$, $\delta(S)$ and $\kappa(S)$

Invariant 15. $\kappa'(S)$ : the mini. order of a set which separates two vertices of $S$

Invariant 16. $\delta(S) := \min\{d_G(x) : x \in S\}$

Invariant 13. $\kappa(S) = \min\{\kappa(x, y) : x, y \in S, \ x \neq y\}$,
where $\kappa(x, y)$ : the max. # of vertex-disjoint paths joining $x \& y$.

We consider the graph $\overline{K}_2 + \overline{K}_m$.

$\kappa'(S) = m$, $\delta(S) = 2$

$\min\{\kappa'(S), \delta(S)\} = 2$

$\kappa(S) = 2$
Improvement of Broersma et al.’s result

Broersma, H.Li, J.Li, Tian & Veldman (1997)

$G$: connected graph, $S \subseteq V(G)$, $\kappa'(S) \geq 2$

$\sigma_3(S) \geq n + \min\{\kappa'(S), \delta(S)\}$

$\Rightarrow \exists$ cycle passing through $S$

By considering the relation between $\kappa'(S)$, $\delta(S)$ and $\kappa(S)$, we rewrite the proof, and can obtain the following theorem.

Broersma, H.Li, J.Li, Tian & Veldman (1997)

$G$: connected graph, $S \subseteq V(G)$, $\kappa(S) \geq 2$

$\sigma_3(S) \geq n + \kappa(S)$

$\Rightarrow \exists$ cycle passing thru $S$
Cycle passing through specified vertices

Thm (Ozeki & Y (2008))
\[ \kappa(S') \geq 2, \ \sigma_2(S') \geq n + \kappa(S') - \alpha(S') + 1 \]
\[ \Rightarrow \exists \text{ cycle passing through } S \]

Broersma et al. (1997)
\[ \kappa(S') \geq 2, \ \sigma_3(S') \geq n + \kappa(S') \]
\[ \Rightarrow \exists \text{ cycle passing through } S \]

Thm (Ozeki & Y (2008))
\[ \kappa(S') \geq 3, \ \sigma_4(S') \geq n + \kappa(S') + \alpha(S') - 1 \]
\[ \Rightarrow \exists \text{ cycle passing through } S \]
Problem on degree conditions for cycles passing through $S$

\[
\sigma_2(S) \geq n + \kappa(S) - \alpha(S) + 1 \\
\downarrow + (\alpha(S) - 1)
\]

\[
\sigma_3(S) \geq n + \kappa(S) \\
\downarrow + (\alpha(S) - 1)
\]

\[
\sigma_4(S) \geq n + \kappa(S) + \alpha(S) - 1 \\
\downarrow + (\alpha(S) - 1)
\]

\[
\sigma_5(S) \geq n + \kappa(S) + 2(\alpha(S) - 1) 
\]
Problem on degree conditions for cycle of length at least $\ell$

\[
\sigma_2 \geq \ell + \kappa - \alpha + 1
\]

\[
\downarrow + (\alpha - 1)
\]

\[
\sigma_3 \geq \ell + \kappa
\]

\[
\downarrow + (\alpha - 1)
\]

\[
\sigma_4 \geq \ell + \kappa + \alpha - 1
\]
False statement
$G: 2$-connected graph
$\sigma_2 \geq \ell + \kappa - \alpha + 1 \Rightarrow \exists$ cycle of length at least $\ell$

$G = \overline{K}_\kappa + \overline{K}_\alpha \quad (\alpha \geq \kappa + 2)$

$\sigma_2 = 2\kappa \geq \frac{2\kappa + 1}{\kappa} + \kappa - \alpha + 1$

But, $\nexists$ cycle of length $\geq 2\kappa + 1$
False statement

$G : 2$-connected graph

$\sigma_2 \geq \ell + \kappa - \alpha + 1 \Rightarrow \exists$ cycle of length at least $\ell$

Bermond (1976), Linial (1976)

$G : 2$-connected graph

$\sigma_2 \geq \ell \Rightarrow \exists$ cycle of length at least $\ell$
Relation of degree conditions for longest cycles

Problem

$$\sigma_2 \geq \ell$$

\[ \downarrow \quad + \kappa \]

$$\sigma_3 \geq \ell + \kappa$$

\[ \downarrow \quad + (\alpha - 1) \]

$$\sigma_4 \geq \ell + \kappa + \alpha - 1$$
The $\sigma_3$ condition and the $\sigma_4$ condition

**Thm (Yamashita (2007))**

$G$: 3-connected graph

$\sigma_3 \geq \ell + \kappa \Rightarrow \exists \text{cycle of length at least } \ell$
The $\sigma_3$ condition and the $\sigma_4$ condition

Thm (Yamashita (2007))

$G$ : 3-connected graph

$\sigma_3 \geq \ell + \kappa \Rightarrow \exists \text{cycle of length at least } \ell$

Thm (Chiba, Tsugaki & Yamashita (2013+))

$G$ : 4-connected graph

$\sigma_4 \geq \ell + \kappa + \alpha - 1 \Rightarrow \exists \text{cycle of length at least } \ell$
## Conclusion

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<td>$\sigma_3(S) \geq n + \min{\kappa'(S), \delta(S)}$ (Broersma et al.)</td>
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<td>$\sigma_3 \geq n + \kappa$ (Bauer et al.)</td>
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