結び目射影図の既約度について

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§0. Outline

P: a knot projection

$r(P)$: the reductivity of $P$

Example

\[ r\left(\begin{array}{c}
\includegraphics[width=0.1\textwidth]{knot1}
\end{array}\right) = 0 \quad r\left(\begin{array}{c}
\includegraphics[width=0.1\textwidth]{knot2}
\end{array}\right) = 1 \quad r\left(\begin{array}{c}
\includegraphics[width=0.1\textwidth]{knot3}
\end{array}\right) = 2 \quad r\left(\begin{array}{c}
\includegraphics[width=0.1\textwidth]{knot4}
\end{array}\right) = 3 \]

Theorem (S.)

\[ r(P) \leq 4 \quad (\forall P) \]

Reductivity problem

\[ \exists P \text{ s.t. } r(P) = 4 \]
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§ 1. Knot projections
We consider knot projections which have at least one crossing.
reducible crossing

knot projections

reducible knot projections

reduced knot projections

How reduced are we??
§ 2. Half-twisted splice
Half-twisted splice (HS)

Example
Inverse-half-twisted splice \((HS^{-1})\)

Knot

Example

\[ HS^{-1} \]

Inverse-half-twisted splice

\( HS^{-1} \)

Knot (not link)

cf. smoothing

Knot

link
Remark

We can obtain a reducible knot projection from any knot projection by a finite number of $HS^{-1}$.

8 crossings $\xrightarrow{HS^{-1}}$ 7 crossings $\xrightarrow{HS^{-1}}$ $\ldots$ $\xrightarrow{HS^{-1}}$ 1 crossing

reducible!
§3. Reductivity
**Reductivity** - how much reduced??

**Definition** \[ P: \text{a knot projection} \]

The **reductivity** \( r(P) \) of \( P \) is the minimal number of \( HS^{-1} \) which are needed to obtain a reducible knot projection from \( P \).

**Example**

\[ \begin{array}{cc}
P & \xrightarrow{HS^{-1}} & \xrightarrow{HS^{-1}} & \text{reducible!} \\
\end{array} \]

\( r(P) = 2 \)
Example

\[ r\left( \begin{array}{c}
\text{\includegraphics{example1}}
\end{array}\right) = 0 \quad r\left( \begin{array}{c}
\text{\includegraphics{example2}}
\end{array}\right) = 1 \]

\[ r\left( \begin{array}{c}
\text{\includegraphics{example3}}
\end{array}\right) = 2 \quad r\left( \begin{array}{c}
\text{\includegraphics{example4}}
\end{array}\right) = 3 \]

There exist infinitely many knot projections \( P \) with \( r(P) = 0, 1, 2, \) and \( 3 \).
Reductivity is four or less

Theorem 1 (S)

\[ r(P) \leq 4 \quad (\forall P) \]

Reductivity problem

\[ \exists P \text{ s.t. } r(P) = 4 \]

§ 4. 2-gons & 3-gons
2-gons & 3-gons

There are two types of 2-gons:

- incoherent 2-gon
- coherent 2-gon

There are four types of 3-gons:

- type A
- type B
- type C
- type D
Example

- **incoherent 2-gon**
- **coherent 2-gon**
- **type A**
- **type B**
- **type C**
- **type D**
Lemma 2

If $P$ has an incoherent 2-gon, then $r(P) \leq 1$.

If $P$ has a coherent 2-gon, then $r(P) \leq 2$. 

(coherent bigon) \xrightarrow{HS^{-1}} \text{incoherent bigon at a crossing outside} \xrightarrow{} \text{incoherent bigon}
If $P$ has a 3-gon of type A, then $r(P) \leq 2$.
If $P$ has a 3-gon of type B, then $r(P) \leq 3$.
If $P$ has a 3-gon of type C, then $r(P) \leq 3$.
If $P$ has a 3-gon of type D, then $r(P) \leq 4$. 
Corollary 4

If $P$ has at least one of

- incoherent 2-gon
- coherent 2-gon
- 3-gon of type A
- 3-gon of type B
- 3-gon of type C

then $r(P) \leq 3$. 
§ 5. Unavoidable sets
Definition \( S \): a set consisting of parts of knot projections

\( S \) is an unavoidable set for a knot proj. if every knot projection has at least one of the parts in \( S \).

**Example:**

\[
\{ \text{\rotatebox{90}{\includegraphics[width=1cm]{example1}}}, \text{\rotatebox{90}{\includegraphics[width=1cm]{example2}}}, \text{\rotatebox{90}{\includegraphics[width=1cm]{example3}}}, \text{\rotatebox{90}{\includegraphics[width=1cm]{example4}}}, \text{\rotatebox{90}{\includegraphics[width=1cm]{example5}}}, \text{\rotatebox{90}{\includegraphics[width=1cm]{example6}}} \}
\]

is an unavoidable set for a reduced knot projection.

(prove later)
AST's theorem

Theorem (Adams–Shinjo–Tanaka)

Every reduced knot projection has a 2-gon or 3-gon.

i.e., \{\bigcirc, \times\} is an unavoidable set for a reduced knot projection.

Proof of AST's theorem

P: a reduced knot projection

$C_n$: the number of $n$-gons of P

Euler's characteristic

\[
\sum_k \frac{kC_k}{4} \quad \# \text{ of crossings}
\]

\[
\sum_k \frac{kC_k}{2} \quad \# \text{ of edges}
\]

\[
\sum_k C_k \quad \# \text{ of regions}
\]

\[\nu - e + f = 2\]

\[2C_2 + C_3 = 8 + C_5 + 2C_6 + 3C_7 + \cdots\]

\[C_2 > 0 \text{ or } C_3 > 0\] AST's formula
Proof of Theorem 1

If \( P \) is reducible, then \( r(P) = 0 \).

(by definition)

If \( P \) is reduced, \( P \) has a 2-gon or 3-gon.

(by AST's theorem)

If \( P \) has a 2-gon, then \( r(P) \leq 2 \).

(by Lemma 2)

If \( P \) has a 3-gon, then \( r(P) \leq 4 \).

(by Lemma 3)
Further unavoidable set

Lemma 5

\{ \emptyset, \bigotimes, \bigotimes, \bigcirc, \bigotimes, \bigotimes \}

is an unavoidable set for a reduced knot projection.
Proof of Lemma 5

Use the "discharging method" from graph theory (four-color theorem)!

P: a reduced knot projection

Assume P does not have any part in

\{\begin{align*}
&\includegraphics{symbol1} , \includegraphics{symbol2} , \includegraphics{symbol3} , \includegraphics{symbol4} , \\
&\includegraphics{symbol5} , \includegraphics{symbol6}
\end{align*}\}.

Then, ...
Give “charge” \((4-n)\) to each \(n\)-gon.

3-gon 4-gon 5-gon 6-gon

\[
\begin{array}{c}
1 \\
0 \\
-1 \\
-2 \\
\ldots
\end{array}
\]

Then the total charge is...

\[
c_3 - c_5 - 2c_6 - 3c_7 - \cdots
\]

\[
= 8 \quad \text{AST's formula}
\]

\[
2c_2 + c_3 = 8 + c_5 + 2c_6 + 3c_7 + \cdots
\]

\(c_n\): the number of \(n\)-gons
“Discharging” at every 3-gon to the neighbor six regions by $\frac{1}{6}$.

before

discharging!

after
After discharging...

3-gon  4-gon  5-gon  6-gon

0     0     negative  negative  ...

Contradicts that the total charge is 8.
Hence \{\phi, \phi, \times, \times, \times, \times\} is an un-avoidable set for a reduced knot proj.
§6. 4-gons & 5-gons
There are 13 types of 4-gons:
Lemma 6

If a knot projection $P$ has one of

\[ 2a \quad 2b \quad 3a \quad 4a \]

then $r(P) \leq 3$. 
Unavoidable set for $P$ with $r(P)=4$

Theorem 7 (Onoda–S)

is an unavoidable set for a knot projection with reductivity four.

There are 56 types of 5-gons:

1. abcde, abced, abdec, abedc, acebd, acedb, adbec, aedcb
2. abcde, abced, abdec, abedc, acbd, acedb, adbe, aedcb
3. abcde, abcd, abede, abedc, aceb, aced, adbe, aedcb
4. abcde, abced, abdec, abedc, acbd, aced, adbe, aedcb
Unavoidable set for $P$ with $r(P) = 4$

Theorem 8 (Kashiwabara-S)

\[\{\odot, \bigcirc, \bigstar, \bigtriangledown, \blacklozenge}\]

is an unavoidable set for a knot projection with reductivity four.

§7. 2-gons & 3-gons again
Question

Is \{\text{incoherent 2-gon, coherent 2-gon, 3-gon of type A, 3-gon of type B, 3-gon of type C}\}

an unavoidable set for a reduced knot projection?

(If so, the reductivity problem is to be solved negatively, i.e., \(r(P) \leq 3\) for any \(P\).)
NO!

This does not have \( A, B, C \) or \( D \), and has only \( D \).

However, the reductivity is not four!

to be continued... (?)
Thank you for listening!