Flow equation, conformal symmetry and AdS geometries

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Ref.

Ref.	S.Aoki-SY	ArXiv:1707.03982	PTEP (2018) 031B01
	S.Aoki-SY	ArXiv:1709.07281	NPB 933 (2018)
	S.Aoki-J.Balog-SY	ArXiv:1804.04636	

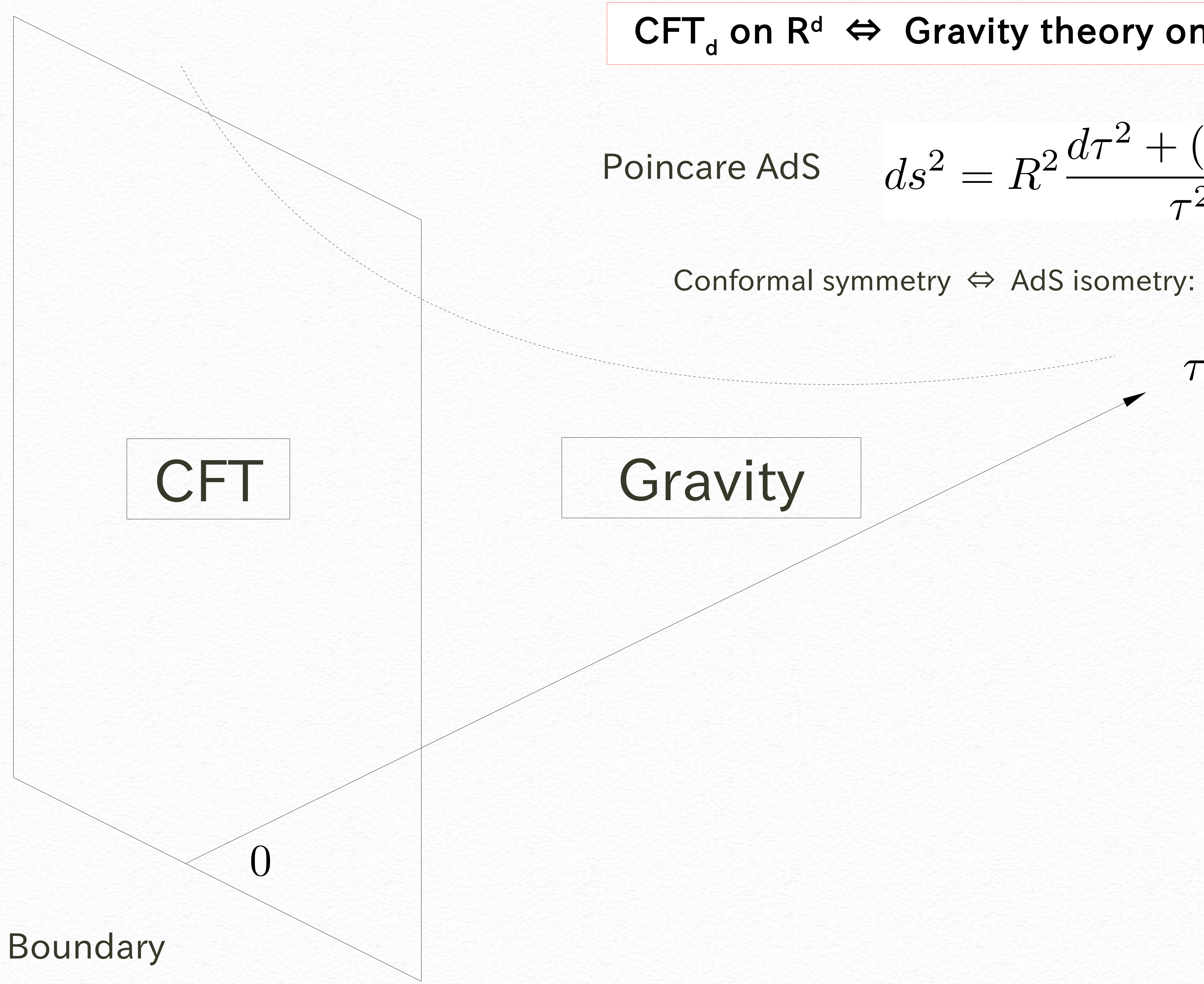
Holography and AdS/CFT

[Maldacena '97]

CFT$_d$ on R^d ⇔ Gravity theory on AdS$_{d+1}$

Poincare AdS $$ds^2 = R^2 \frac{d\tau^2 + (dx^\mu)^2}{\tau^2}$$

Conformal symmetry ⇔ AdS isometry: SO(1,d+1)



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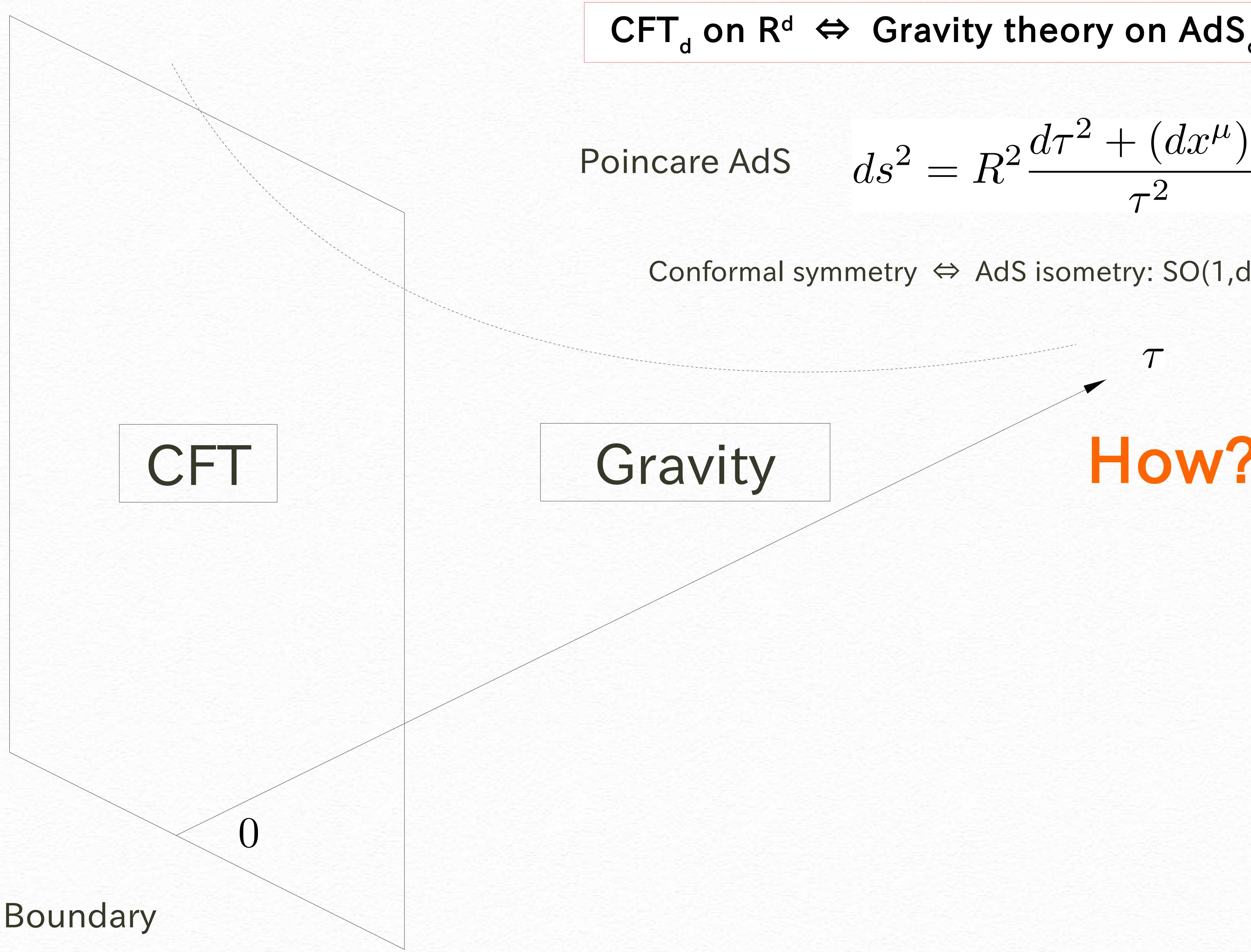

AdS radial direction from CFT

1. Relevant RG flow

Construction of gravity solutions corresponding to UV and IR CFTs in the asymptotic regions.

[Girardello-Petrini-Porrati-Zaffaroni ’98] [Distler-Zamora ’98] [de-Boer-Verlinde-Verlinde ’99] [Skenderis '00]

2. Wilsonian RG flow

The Wilsonian cut-off will correspond to sharp cut-off at the AdS radial direction

[Heemskerk-Polchinski ’10]

3. Stochastic quantization

Euclidean path integral ≡ Equilibrium limit of statistical mechanical system coupled to a heat bath.

[Lifshytz-Periwal ’00]

4. Entanglement entropy

continuous multi-scale entanglement renormalization ansatz (cMERA) [Swingle ’09]...

5. bilocal field

Relative coordinate of bi-local field in vector models [Das-Jevicki ’03]

6. Flow equation

Smearing operators so as to resolve a UV singularity in the coincidence limit

[Aoki-Kikuchi-Onogi ’15] [Aoki-Balog-Onogi-Weisz ’16,'17]

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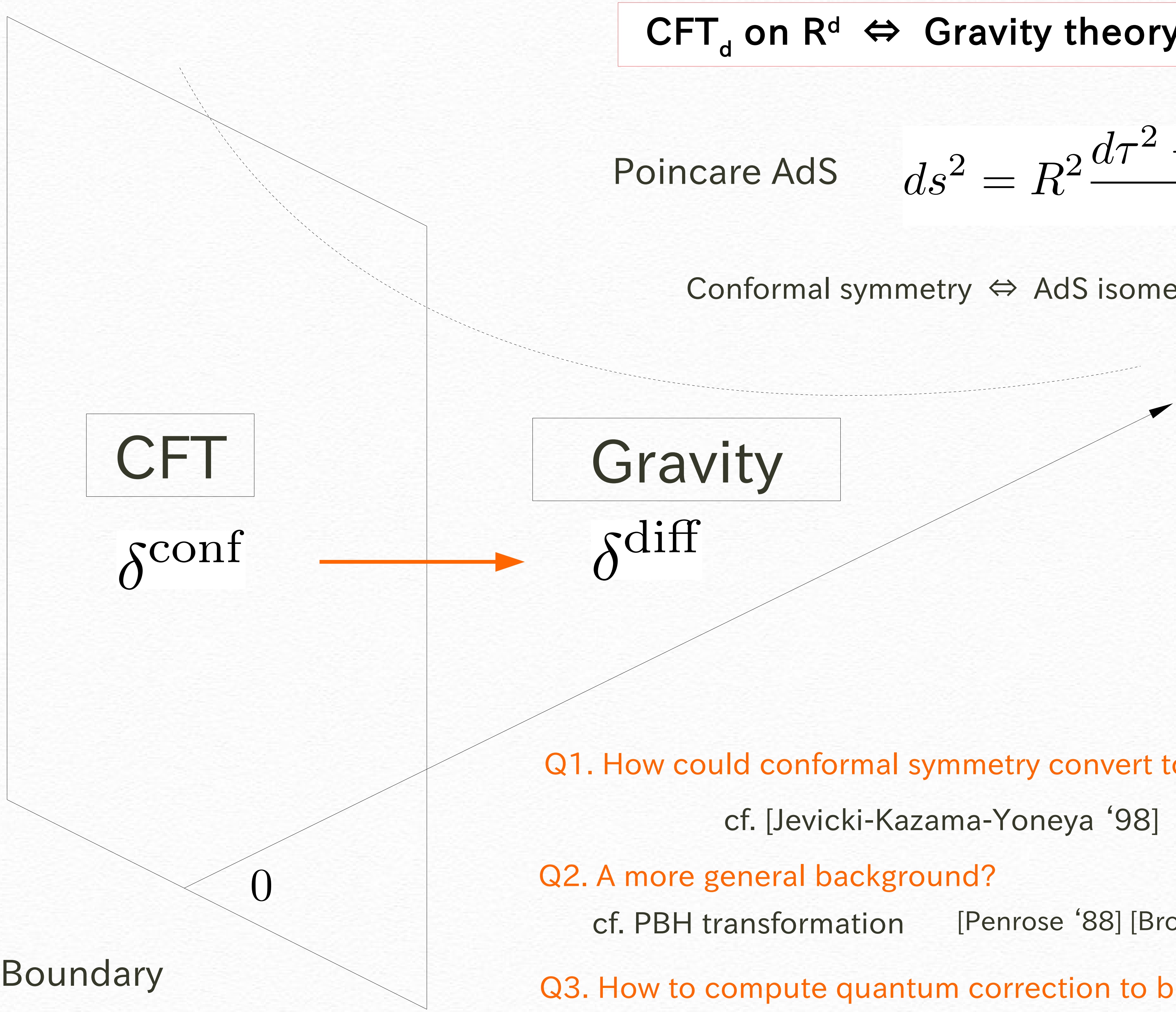

Q1. How could conformal symmetry convert to AdS isometry?

cf. [Jevicki-Kazama-Yoneya '98] [Das-Jevicki '03]

Q2. A more general background?

cf. PBH transformation [Penrose '88] [Brown-Henneaux '86]

Q3. How to compute quantum correction to bulk observables?

Plan

✓ 1. Introduction

2. Flow equation & induced metric

   Induced metric = information metric

3. Conformal symmetry ➡ AdS isometry

   ⬅ Answer for Q1

4. Generalization to conformally flat manifolds

   4.1. Primary flow equation

   ⬅ Answer for Q2

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Flow equation

(Gradient) flow equation

Equation to smear operators to resolve a UV singularity in the coincidence limit

[Albanese et al. (APE) '87] [Narayanan-Neuberger '06] [Luscher '10,'13]

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Consider QFT with n real component scalar field in d dimensions

Flow equation

$$\boxed{\frac{\partial \phi^a(x;t)}{\partial t} = -\left.\frac{\delta S_f(\varphi)}{\delta \varphi^a(x)}\right|_{\varphi(x)\to\phi(x;t)} \qquad \phi(x;0) = \varphi(x)}$$

t: flow time, ϕ (x;t): flowed field

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Comments

(I) If S_f coincide with the action of the theory S
➡ the flow equation is called the gradient one.

(II) If S_f coincide with the free theory
➡ the flow equation becomes the heat equation.
➡ the operator ϕ is smeared similarly as in finite temperature t.

(III) General solution ➡ "flow kernel method".

“Flow kernel method”

1. Rewrite the flow equation in the heat equation form.

$$\frac{\partial O(x;t)}{\partial t} = -\hat{H}O(x;t) \qquad O(x;0) = O(x)$$

2. Introduce the flow kernel (density).

$$\frac{\partial \rho(x,y;t)}{\partial t} = -\hat{H}\rho(x,y;t) \qquad \rho(x,y;0) = \delta^d(x-y)$$

3. A general solution is given by

$$O(x;t) = \int d^d y \rho(x,y;t) O(y)$$

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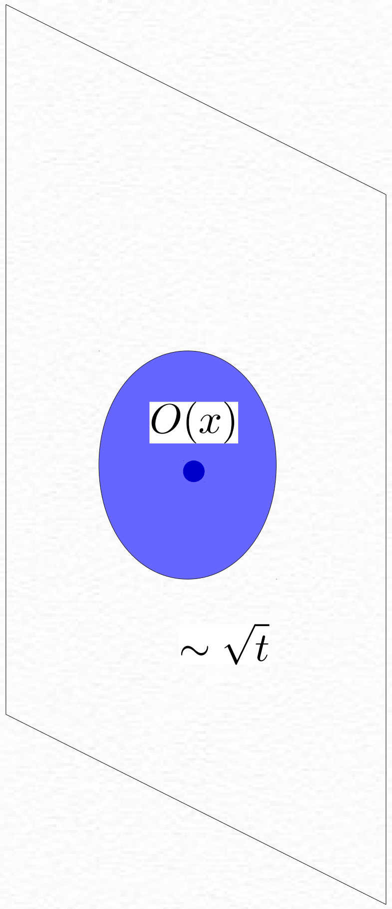
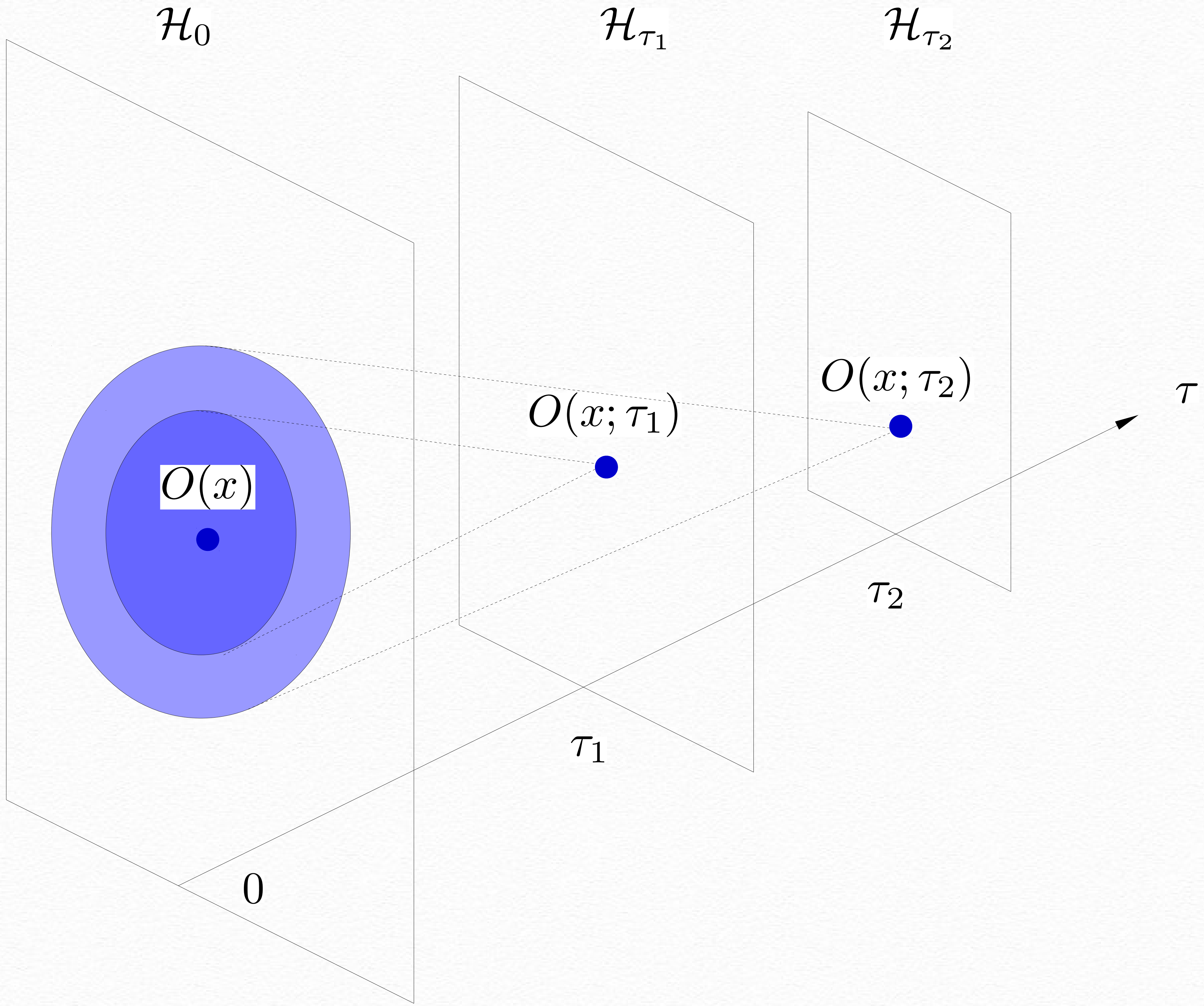


Ex. Free flow

$$\rho(x,y;t) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{(x-y)^2}{4t}}$$

Sketch of smearing and extra direction



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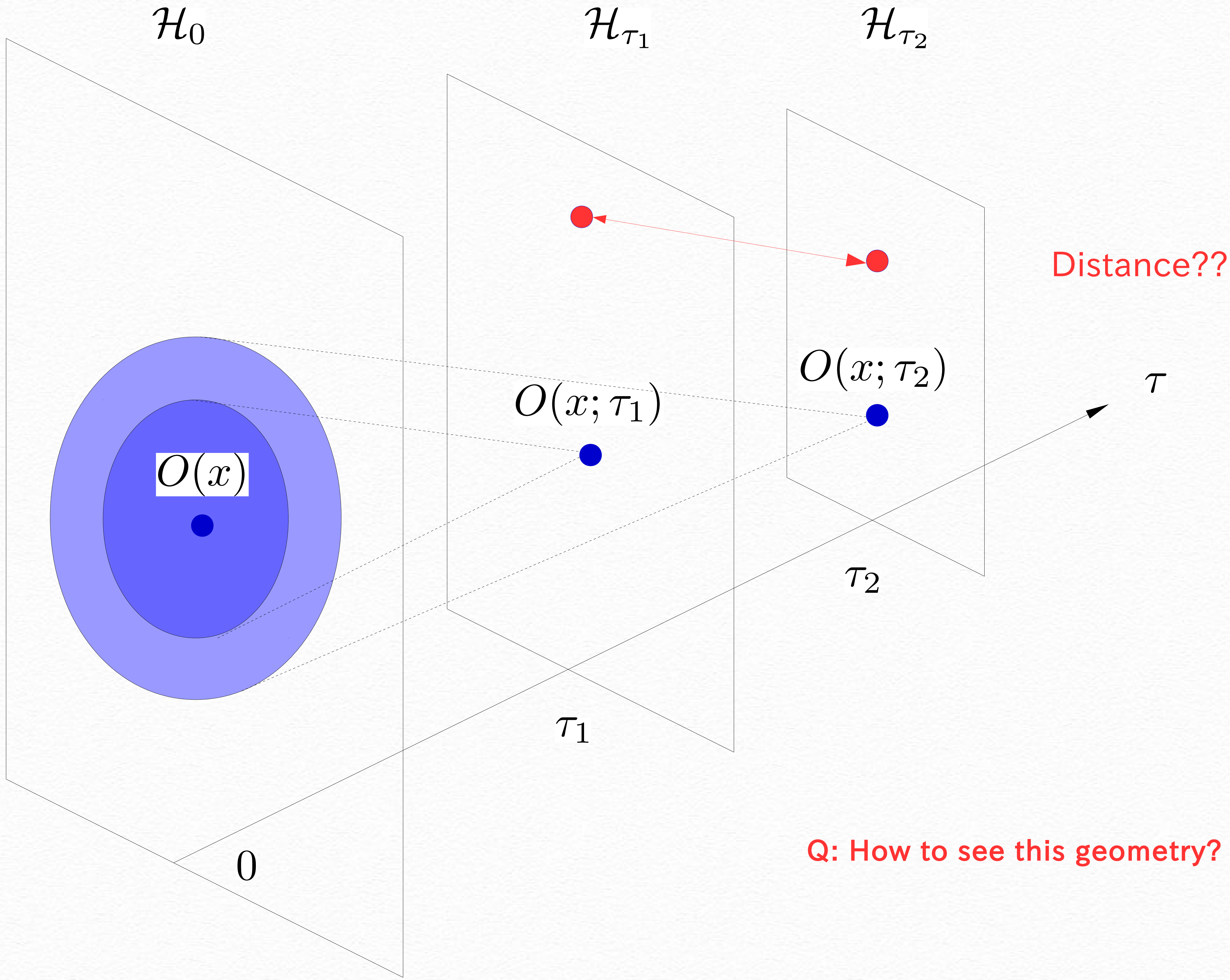

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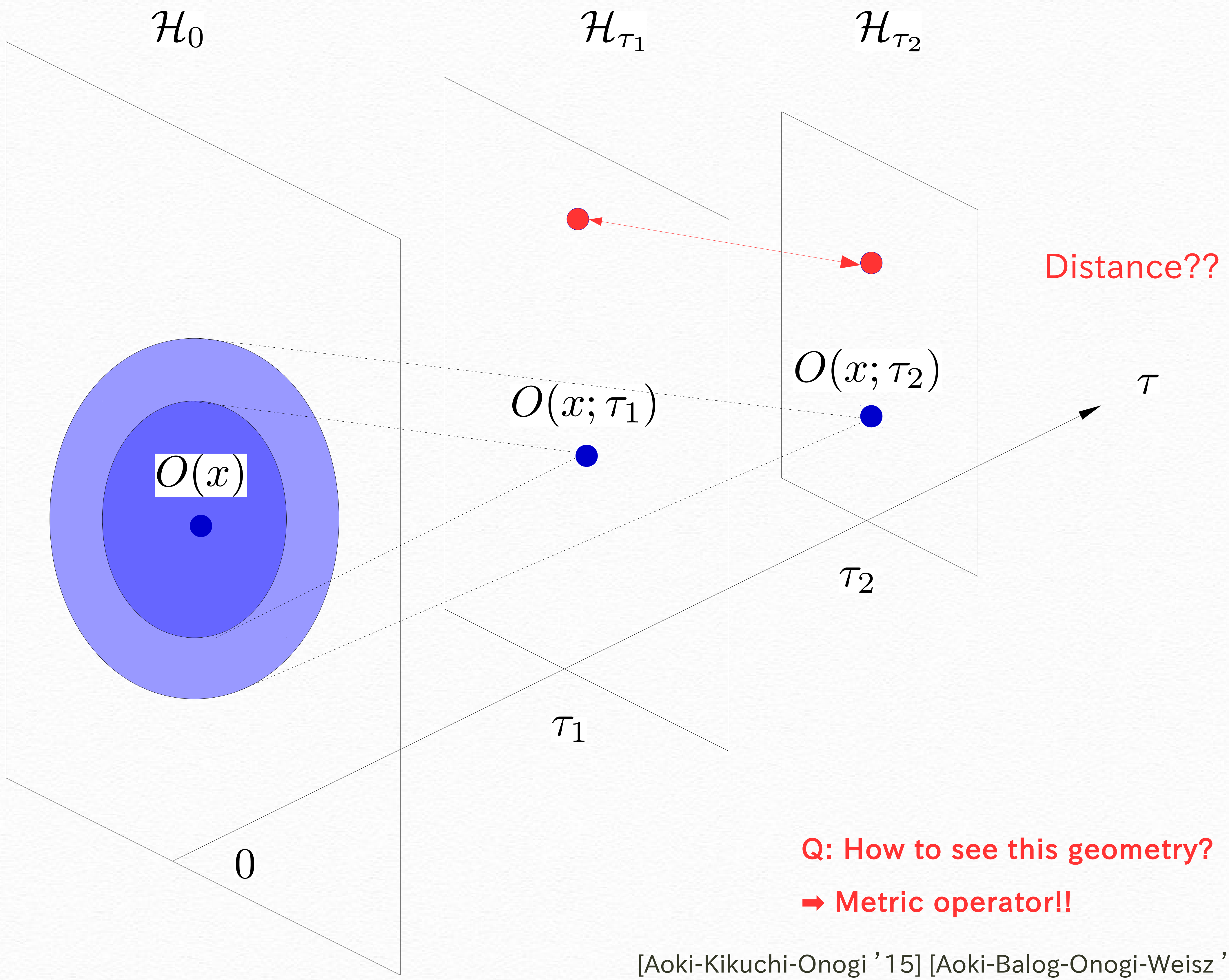

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➡ Metric operator!!

[Aoki-Kikuchi-Onogi '15] [Aoki-Balog-Onogi-Weisz '16,'17]

Metric operator & induced metric

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Def. (Dimensionless normalized operator)

$$\sigma(x;t) := \frac{O(x;t)}{\sqrt{\langle O(x;t)^2 \rangle_{CFT}}}$$

"Operator renormalization"

NOTE: $\langle \sigma(x;t)\sigma(x;t)\rangle_{CFT} = 1$

Def. (Metric operator and induced metric)

$$\hat{g}_{MN}(x;t) := R^2 \frac{\partial \sigma(x;t)}{\partial z^M} \frac{\partial \sigma(x;t)}{\partial z^N}$$

$$g_{MN}(z) := \langle \hat{g}_{MN}(x;t) \rangle_{CFT}$$

R: constant of length dimension $z^M = (x^\mu, \tau)$ with $\tau = \sqrt{2dt}$

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Result 1 Induced metrics with the free flow equation become the AdS metric for a couple of vector models in the free or critical limit.

[Aoki-Kikuchi-Onogi ’15] [Aoki-Balog-Onogi-Weisz ’16,'17]

Result 2 An induced metric becomes quantum information metric for a general QFT.

[Aoki-SY '17]

Result 3 An induced metric defined in this way becomes AdS for a general CFT.

[Aoki-SY '17]

Induced metric = information metric

[S.Aoki-SY '17] PTEP (2018) 031B01

Induced metric = information metric

Def. (Bure metric for a density matrix)

$$D(\rho, \rho + d\rho)^2 = \frac{1}{2}\mathrm{tr}(d\rho\, G)$$

ρ : density matrix

G: hermitian 1 form operator satisfying $\rho\, G + G\, \rho = d\rho$

For a pure state, G is given by $G = d\rho$

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For $\forall$z = (x, τ), we can assign a density matrix such that

$$\rho_z = |\sigma(x;t)\rangle\langle\sigma(x;t)|, \quad |\sigma(x;t)\rangle := \sigma(x;t)|0\rangle, \quad \langle\sigma(x;t)| := \langle 0|\sigma(x;t)$$

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Def. (Inner product)

$$\langle\sigma(x;t)|\sigma(w;s)\rangle := \langle\sigma(x;t)\sigma(w;s)\rangle_{CFT}$$

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We measure the infinitesimal distance between 2pt in the bulk manifold by the information metric for the density matrices of associated pure states.

$$ds_z^2 := D(\rho_z, (\rho + d\rho)_z)^2 = \frac{1}{2}\mathrm{tr}(d\rho_z d\rho_z) = g_{MN}(z) dz^M dz^N,$$

cf. Fischer information metric $\quad |\langle\Psi_\lambda|\Psi_{\lambda+\delta\lambda}\rangle|^2 = 1 - 2G_{\lambda\lambda}\delta\lambda^2 + \cdots$

$|\Psi_\lambda\rangle$: Vacuum state for $H_0 + \lambda V$

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Correlator of free flowed field

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The 2pt function is $$G_0(x;t|y;s) := \langle O(x;t) O(y;s) \rangle_{\rm CFT} = e^{(t\partial_x^2 + s\partial_y^2)} \langle O(x) O(y) \rangle_{\rm CFT}$$

Poincare symmetry: $$G_0(x;t|y;s) = {}^{\exists}G_0((x-y)^2, t+s)$$

Scaling property: $$G_0(\lambda x; \lambda^2 t | \lambda y; \lambda^2 s) = \lambda^{-2\Delta} G_0(x;t|y;s)$$

➡ $$G_0(x;t|y;s) = \frac{1}{(t+s)^\Delta} F_0\left(\frac{(x-y)^2}{t+s}\right)$$

F_0: a smooth function

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➡ $G_0(x;t|y;s) = \frac{1}{(t+s)^\Delta} F_0\left(\frac{(x-y)^2}{t+s}\right)$ $\quad$ F_0: a smooth function

The 2pt function of normalized field $G(x;t|y;s) := \langle \sigma(x;t)\sigma(y;s)\rangle_{\rm CFT} = \left(\frac{2\sqrt{ts}}{t+s}\right)^\Delta F\left(\frac{(x-y)^2}{t+s}\right)$

$$F(x) \equiv F_0(x)/F_0(0),\ F(0) = 1,\ 2dF'(0) = -\Delta$$

Explicit computation: $F(x) = \frac{\Gamma(d/2)}{\Gamma(\Delta)\Gamma(d/2-\Delta)} \int_0^1 dv\, v^{\Delta-1}(1-v)^{d/2-\Delta-1} e^{-xv/4}.$

NOTE: UV singularity is resolved (regularized) ⇔ F is a smooth function for x>0.

Q1:Conformal symmetry ➡ AdS isometry?

[S.Aoki-SY '17]　PTEP (2018) 031B01

cf. [Jevicki-Kazama-Yoneya '98]

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Conformal transformation:

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$$\delta^{\rm conf} O(x) = -\,\delta x^\mu \partial_\mu O(x) - \frac{\Delta}{d}(\partial_\mu \delta x^\mu) O(x)$$

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$$\delta^{\rm conf}\sigma(x;t) = -\,\{2\lambda - 4(b_\mu x^\mu)\} t\partial_t \sigma(x;t) - \{\delta x^\mu + 2t(d-2-\Delta) b^\mu\}\partial_\mu \sigma(x;t) + 4t^2 b^\mu \partial_\mu \partial_t \sigma(x;t)$$

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$$\delta^{\mathrm{conf}} \hat{g}_{MN}(x;t) = \delta^{\mathrm{diff}} \hat{g}_{MN}(x;t) + R^2 \lim_{(y;s)\to(x;t)} \frac{\partial}{\partial z^M} \frac{\partial}{\partial w^N} \left\{ \delta^{\mathrm{extra}} \sigma(x;t) \sigma(y;s) + \sigma(x;t) \delta^{\mathrm{extra}} \sigma(y;s) \right\}.$$

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Vanish by taking VEV!!

⬇

$$\boxed{\left\langle \delta^{\rm conf}\hat g_{MN}(x;t)\right\rangle = \left\langle \delta^{\rm diff}\hat g_{MN}(x;t)\right\rangle .}$$

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We need to construct a flow equation associated with the curved manifold. But how?

$$\frac{\partial O(x;t)}{\partial t} = \partial^2 O(x;t) \quad \rightarrow \quad ???$$

Setup

$\mathbf{R}^d$

x

$$\Omega : \text{conformal map}$$
$$\longrightarrow$$

M_d

Ω_x

$$ds^2 = \delta_{\mu\nu} dx^\mu dx^\nu$$

$$(ds^2)_{M_d} = g^{\frac{1}{d}}(x) \delta_{\mu\nu} dx^\mu dx^\nu$$

the conformal factor

$$O(x)$$

$$\Omega * : \text{pullbuck}$$
$$\longleftarrow$$

$$O(\Omega_x)$$

$$\Omega * O(\Omega_x) = U_\Omega^{-1} O(\Omega_x) U_\Omega = g^{-\frac{\Delta}{2d}}(x) O(x)$$

Primary flow

$\mathbf{R}^d$

x

$$\xrightarrow{\Omega\ :\ \text{conformal map}}$$

M_d

Ω_x

$$ds^2 = \delta_{\mu\nu} dx^\mu dx^\nu$$

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flow $\downarrow$ $\downarrow$ flow

$O(x;t)$???

$$\frac{\partial O(x;t)}{\partial t} = \partial^2 O(x;t)$$

Primary flow equation

How can we generalize the result of flat space boundary to a more general curved boundary?

A curved manifold need to admit CFT to live. ➡ Restrict ourselves to a conforamlly flat manifold.

We need to construct a flow equation associated with the curved manifold. But how?

$$\frac{\partial O(x;t)}{\partial t} = \partial^2 O(x;t) \quad \to \quad ???$$

We request the following 2 conditions:

① There exists a flow time ˜t associated with M_d corresponding to the flow time t on R^d such that the flowed operator inserted at Ω_x is related to the flowed one at x by the pullback of a conformal map Ω:

$$\Omega * O(\Omega_x;\tilde{t}) = U_\Omega^{-1} O(\Omega_x;\tilde{t}) U_\Omega = g^{-\frac{\Delta}{2d}}(x) O(x;t)$$

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Primary flow equation

⬇

$$\tilde{t} = g^{\frac{1}{d}}(x) t$$

$$\frac{\partial}{\partial \tilde{t}} O(\Omega_x;\tilde{t}) = g^{-\frac{1}{d}}(x) g^{-\frac{\Delta}{2d}}(x) \partial^2 \cdot g^{\frac{\Delta}{2d}}(x) O(\Omega_x;\tilde{t}), \qquad O(\Omega_x;0) = O(\Omega_x)$$

Primary flow

$\mathbf{R}^d$

x

Ω : conformal map $\longrightarrow$

M_d

Ω_x

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Primary flow equation

Q2: Non-trivial curved boundary?

[S.Aoki-SY '17] NPB 933 (2018)

Generalization to curved background

A curved manifold needs to admit CFT to live. ➡ Restrict ourselves to a conformally flat manifold.

We need to construct a flow equation associated with the curved manifold.

"Canonical" free flow equation **(Primary flow equation)**

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The induced metric:

$$\begin{aligned}
\tilde{g}_{\tilde{\tau}\tilde{\tau}}(z) =& R^2 \frac{\Delta}{\tilde{\tau}^2}, \\
\tilde{g}_{\tilde{\tau}\mu}(z) =& g_{\mu\tilde{\tau}}(z) = -R^2 \frac{\Delta}{\tilde{\tau}} \frac{\partial}{\partial x^\mu} \log\{g^{\frac{1}{2d}}(x)\}, \\
\tilde{g}_{\mu\nu}(z) =& R^2 \Delta \left[\frac{\partial}{\partial x^\mu} \log\{g^{\frac{1}{2d}}(x)\} \frac{\partial}{\partial x^\nu} \log\{g^{\frac{1}{2d}}(x)\} + \frac{\delta_{\mu\nu} g^{\frac{1}{d}}(x)}{\tilde{\tau}^2} \right],
\end{aligned}$$

This is the (local) AdS metric whose radius is $R\sqrt{\Delta}$ with the boundary M_d!!

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\end{aligned}$$

This is the (local) AdS metric whose radius is $R\sqrt{\Delta}$ with the boundary M_d!!

This AdS metric can be obtained from the usual Poincare AdS by a bulk diffeomorphism

$$ds^2(\tilde{t}, x) = ds^2_{\mathrm{PAdS}}(t, x)\big|_{t = g^{-1/d}(x)\tilde{t}}$$

cf. PBH transformation [Penrose '88] [Brown-Henneaux '86]

Plan

✓ 1. Introduction

✓ 2. Flow equation & induced metric

✓ 3. Conformal symmetry ➡ AdS isometry

✓ 4. Generalization to conformally flat manifolds

4.1. Primary flow equation

4.2. AdS metric with conformally flat boundary

5. Quantum corrections

6. Summary

Q3: How to compute quantum correction?

[S.Aoki-J.Balog-SY '18]

Pre-geometric operators

Def. Pre-geometric operators are defined by replacing the metric which appears in the definition of the corresponding (classical) geometric object with the metric operator.

$$\boxed{\mathcal{O}[g] \to \mathcal{O}[\hat{g}] =: \hat{\mathcal{O}}}$$

Ex.

$$\hat{\Gamma}^M_{LN}(x;t) = \frac{1}{2}\hat{g}^{MP}(x;t)(\hat{g}_{P\{N,L\}}(x;t) - \hat{g}_{NL,P}(x;t))$$

$$\hat{R}_{LP}{}^M{}_N(x;t) = \partial_{[L}\hat{\Gamma}^M_{P]N}(x;t) + \hat{\Gamma}^M_{[LQ}(x;t)\hat{\Gamma}^Q_{P]N}(x;t)$$

$$\hat{R}_{PN}(x;t) = \hat{R}_{MP}{}^M{}_N(x;t),$$

$$\hat{R}(x;t) = \hat{g}^{PN}(x;t)\hat{R}_{PN}(x;t),$$

$$\hat{G}_{MN}(x;t) = \hat{R}_{MN}(x;t) - \frac{1}{2}\hat{g}_{MN}(x;t)\hat{R}(x;t).$$

Bulk interpretation

Taking the expectation value of these operators for a CFT state gives the observable in the bulk evaluated at its corresponding geometry, which obeys Einstein equation:

$$\boxed{\langle \hat{G}_{AB} \rangle_\psi = T^{\text{bulk}}_{AB}|.}$$

$$\langle \hat{G}_{AB} \rangle_\psi := \langle \psi | \hat{G}_{AB} | \psi \rangle$$

<u>Ex.</u> CFT vacuum ➡ Only cosmological constant.

$$T^{\text{bulk}}_{AB}| = -\Lambda g_{AB}$$

<u>CLAIM</u>: This induced Einstein tensor is expected to describe **that of dual quantum gravity**.

In particular, let us compute LHS in the **1/n expansion**:

$$\langle G_{AB}[\hat{g}] \rangle = G_{AB}[\langle \hat{g} \rangle] + \langle G_{AB}[\hat{g}] \rangle_c$$

Leading Order (LO) ↕ Classical geometry

Next to Leading Order (NLO) ↕ Quantum correction

Quantum correction to bulk CC

Ex. CFT vacuum, O(n) free vector model ➡ Quantum correction to cosmological constant.

① Expand the operator around the vacuum:

$$\hat{G}_{AB} = G_{AB} + \dot{G}_{AB} + \ddot{G}_{AB} + \cdots .$$

$$\mathcal{O}(h^0) \quad \mathcal{O}(h^1) \quad \mathcal{O}(h^2) \qquad h_{AB} = \hat{g}_{AB} - g_{AB}$$

$$G_{AB} = G_{AB}[\langle \hat{g} \rangle] = \frac{d(d-1)}{2L^2\Delta} g_{AB}$$

$$\Lambda = -\frac{d(d-1)}{2L^2\Delta} + \mathcal{O}\left(\frac{1}{n}\right),$$

$$\ddot{G}_{AB} = R_{AB} - \frac{1}{2} h_{AB} \dot{R} - \frac{1}{2} g_{AB} \ddot{R}$$

② Taking the VEV:

$$\left\langle \hat{G}_{AB} \right\rangle = G_{AB} + \left\langle \ddot{G}_{AB} \right\rangle + \cdots .$$

$$\langle \ddot{G}_{AB} \rangle = \frac{(d-1)d(d+4)}{2nL^2\Delta} g_{AB}.$$

$$\Lambda = -\frac{d(d-1)}{2L^2\Delta}\left(1 + \frac{d+4}{2n}\right) + \mathcal{O}\left(\frac{1}{n^2}\right), \quad L^2_{\mathrm{AdS}} = L^2\Delta\left(1 - \frac{d+4}{2n}\right) + \mathcal{O}\left(\frac{1}{n^2}\right).$$

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Summary

- An induced metric defined from the normalized flowed field generally corresponds to the **quantum information metric**.

- For a general CFT, we explicitly showed that the induced metric with the free flow equation always becomes the **AdS** metric.

- Conformal symmetry converts to AdS isometry **after quantum averaging**.

- We have constructed a canonical flow equation for a primary scalar operator on a conformally flat manifold, called the **primary flow equation**.

- The induced metric associated with the primary flow equation becomes AdS with **the conformally flat boundary**.

- AdS with the conformally flat boundary is obtained from the usual Poincare AdS **by a simple bulk diffeomorphism transformation**.

- Demonstrate how to compute **quantum corrections to bulk observables** and compute it explicitly for the **cosmological constant**.

Future works

- How to encode dynamics beyond geometry? For excited states?

  working in progress [Aoki-Balog-SY]

- How to reconstruct bulk operator?

- Canonical choice of a normalized field?

- Finite temperature? BH?

- Spin 1,2 field? Fermion?

- de Sitter construction? Application to the real world?

⋮

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$\vdots$

Thank you!