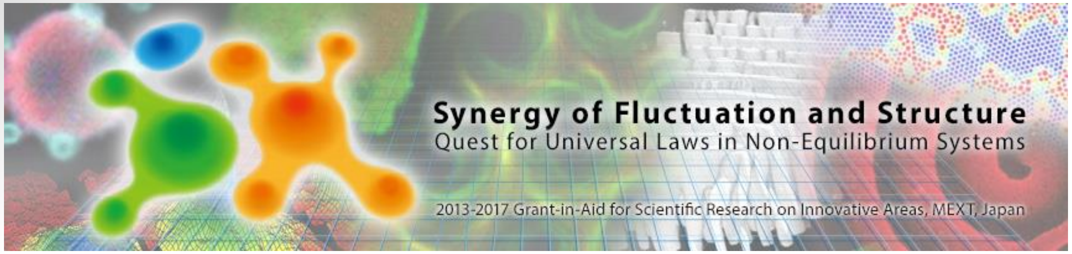

Thermodynamic entropy as a Noether invariant

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See http://arxiv.org/abs/1509.08943

PRL 2016; Editors' suggestion

@ 益川塾、京都産業大学

2016/08/02

2015年７月７日

３輪車上@バンガロール

横倉「熱力学エントロピーを対称性から導出する、
　　という研究はないでしょうか？」

佐々「聞いたことない。いかにも僕が考えそうな
　　問題なのに、考えたこともなかった。
　　でも、待てよ、あり得るわ。うん、あるわ。」

背景：ブラックホールエントロピーをネーター電荷として
　　導出するのは重力業界では有名な話
R. M. Wald, Black hole entropy is the Noether charge, Phys. Rev. D (1993)
（一般相対論１００周年 Phys Rev 記念碑論文のひとつに選出）

基本事項

（ネーターの定理）
対称性があれば、解に沿って保存量がある。

（断熱定理）
相空間の点に対してそれを含むエネルギー面で囲まれた 相空間体積は、準静的操作に対するほとんど全ての解に おいて不変である。

Quick question

What is the symmetry ?

Wait for the slide [36] !

This is the goal of my talk

You will find a surprising result!

Plan of this talk

1. Introduction
2. Adiabatic theorem
3. Noether's theorem
4. Derivation of the symmetry
5. Macroscopic systems
6. Concluding remarks

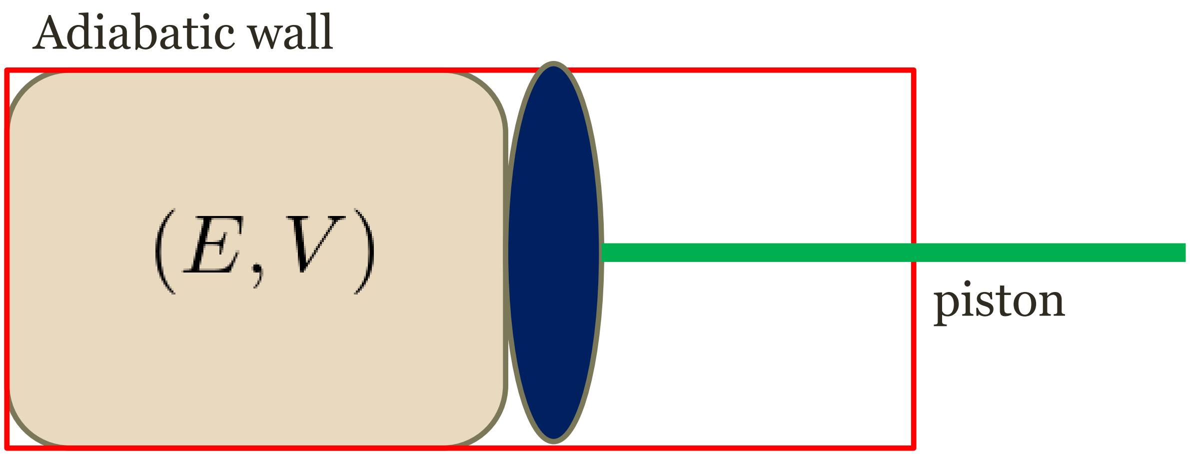
Part II
Adiabatic theorem

Basic concept : entropy

Thermodynamics:

quasi-static adiabatic process:

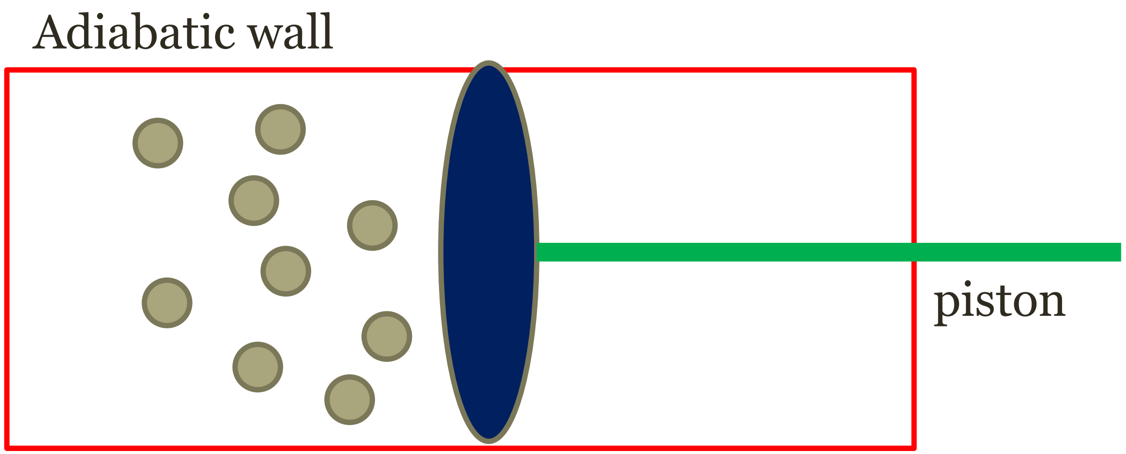
$$(E, V) \to (E', V')$$



$$S(E, V) = S(E', V')$$

Mechanical description

Classical mechanics: $q(t) \in \mathbb{R}^{3N} \quad p(t) \in \mathbb{R}^{3N}$



phase space coordinate $\Gamma = (q, p)$

Hamiltonian

$$H(\Gamma, \alpha) = \sum_i \frac{|\boldsymbol{p}_i|^2}{2m} + \sum_{i<j} V_{\text{int}}(|\boldsymbol{r}_i - \boldsymbol{r}_j|) + \sum_i V_{\text{wall}}(\boldsymbol{r}_i; \alpha)$$

α time-dependent control parameter e.g. $\alpha = V$

Statistical mechanics

Any physical quantity $A(\Gamma)$

$$\langle A\rangle^{\mathrm{mc}}_{E,\alpha} \equiv \frac{1}{\Sigma(E,\alpha)} \int d\Gamma \delta(E - H(\Gamma,\alpha)) A(\Gamma)$$

$$\Sigma(E,\alpha) \equiv \int d\Gamma \delta(E - H(\Gamma,\alpha))$$

Entropy $$S(E,\alpha) \equiv \log \frac{\Omega(E,\alpha)}{N!}$$

$$\Omega(E,\alpha) \equiv \int d\Gamma \theta(E - H(\Gamma,\alpha))$$

Thermodynamic concepts

Inverse temperature $$\beta \equiv \frac{\Sigma(E,\alpha)}{\Omega(E,\alpha)}$$

⇩

Fundamental relation in thermodynamics:

$$dS = \beta dE - \beta \left\langle \frac{\partial H}{\partial \alpha} \right\rangle_{E,\alpha}^{\mathrm{mc}} d\alpha$$

Time-dependent entropy

$\hat{\alpha} = (\alpha(t))_{t=t_\mathrm{i}}^{t_\mathrm{f}}$ protocol : fix

$\Gamma_*(t)$ solution to the Hamiltonian equation

$$E_*(t) \equiv H(\Gamma_*(t), \alpha(t))$$

$$S(E, \alpha) \equiv \log \frac{\Omega(E, \alpha)}{N!}$$

$$S_*(t) \equiv S(E_*(t), \alpha(t))$$

$$\beta \equiv \frac{\Sigma(E, \alpha)}{\Omega(E, \alpha)}$$

$$\beta_*(t) \equiv \beta(E_*(t), \alpha(t))$$

Adiabatic theorem

Quasi-static operation $\alpha(t) = \bar{\alpha}(\epsilon t)$

$$\frac{d\alpha}{dt} = \epsilon \frac{d\bar{\alpha}}{d\tau} = O(\epsilon) \quad \epsilon \to 0$$

$$\tau_{\mathrm{i}} = \epsilon t_{\mathrm{i}}$$

$$\tau_{\mathrm{f}} = \epsilon t_{\mathrm{f}}$$

$$\tau = \epsilon t$$

Ergodic (with respect to the MC measure)

$$\lim_{\epsilon \to 0} [S_*(t_{\mathrm{f}}) - S_*(t_{\mathrm{i}})] = 0$$

Remarks on history

Physical arguments (proofs) : 1905-1915 e.g. Hertz

Kasuga, On the adiabatic theorem for the Hamiltonian System of differential equations in the classical mechanics I-III Proceedings of the Japan Academy 37, 366-382(1961)

Anosov, Averaging in systems of ordinary differential equations with rapidly oscillating solutions, Izv. Akad. Nauk SSSR ser. mat. 24 721-742 (1960)

Lochak and Meunier, Multiphase averaging for classical systems (Springer, 1988)

Ott, Goodness of ergodic adiabatic invariants, Phys. Rev. Lett. 42, 1628-1631 (1979).

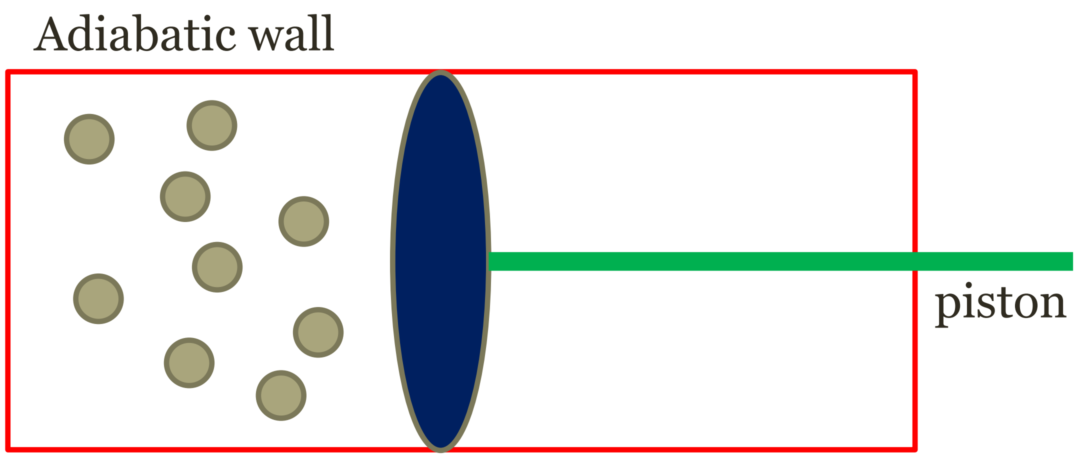
Jarzynski, Multipule-time-scale approach to ergodic adiabatic systems: Another look, Phys. Rev. Lett. 71 839-842 (1993).

Part III
Noether's theorem

Setting up I

$q(t) \in \mathbb{R}^{3N}$ coordinates of particles $q = (\mathbf{r}_1, \cdots, \mathbf{r}_N)$

α control parameter e.g. $\alpha = V$



$\hat{\alpha} = (\alpha(t))_{t=t_\mathrm{i}}^{t_\mathrm{f}}$ protocol : fix

$\hat{q} = (q(t))_{t=t_\mathrm{i}}^{t_\mathrm{f}}$ any trajectory

Setting up II

Action:

$$\mathcal{I}(\hat{q}, \hat{\alpha}) = \int_{t_\mathrm{i}}^{t_\mathrm{f}} dt L(q(t), \dot{q}(t), \alpha(t))$$

Lagrangian:

$$L(q, \dot{q}, \alpha) = T(\dot{q}) - U(q, \alpha)$$

Energy:

$$E(q, \dot{q}, \alpha) = \dot{q}\frac{\partial L}{\partial \dot{q}} - L(q, \dot{q}, \alpha)$$

Transformation I

$G \quad t \to t' = t + \eta\xi(q, \dot{q}, \alpha)$ non-uniform translation of time

η an infinitely small parameter

$$\hat{q} \to \hat{q}' \qquad q'(t') = q(t)$$

Positions are independent of time labeling

$$\hat{\alpha} \to \hat{\alpha}' \qquad \alpha'(t') = \alpha(t')$$

The protocol (as a function of time) is fixed

Transformation II

$$\delta_G \mathcal{I} \equiv \mathcal{I}(\hat{q}', \hat{\alpha}') - \mathcal{I}(\hat{q}, \hat{\alpha})$$

$$= \int_{t_\mathrm{i}}^{t_\mathrm{f}} dt \left[\bar{\delta}_G L + \eta \frac{d(\xi L)}{dt} \right]$$

$$\bar{\delta}_G L \equiv L(q'(t), \dot{q}'(t), \alpha'(t)) - L(q(t), \dot{q}(t), \alpha(t))$$

$$\bar{\delta}_G q(t) \equiv q'(t) - q(t) = -\eta \xi \dot{q}$$

$$= \eta \int_{t_\mathrm{i}}^{t_\mathrm{f}} dt \left\{ -\mathcal{E} \dot{q} \xi + \frac{d}{dt} \left[\xi \left(L - \dot{q} \frac{\partial L}{\partial \dot{q}} \right) \right] \right\}$$

$$\mathcal{E} \equiv \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}}$$

Result

Suppose that there exist $\xi(q, \dot{q}, \alpha)$ and $\psi(q, \dot{q}, \alpha)$

that satisfy $$\delta_G \mathcal{I} = \eta \int_{t_\mathrm{i}}^{t_\mathrm{f}} dt \frac{d\psi}{dt}$$

Noether, 1918; Bessel-Hagen 1921

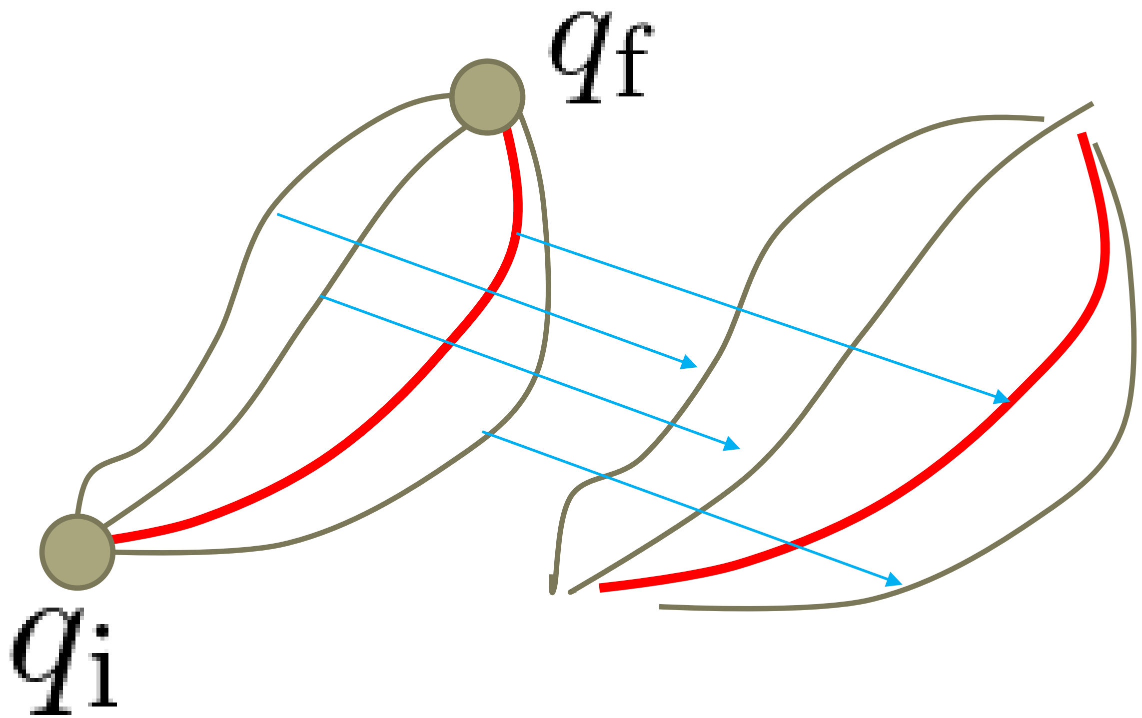
Trautman, CMP, 1967

$$\int_{t_\mathrm{i}}^{t_\mathrm{f}} dt \mathcal{E} \dot{q} \xi = -(\psi + E\xi)\Big|_{t_\mathrm{i}}^{t_\mathrm{f}}$$

$$(\psi_* + E_* \xi_*)\Big|_{t_\mathrm{i}}^{t_\mathrm{f}} = 0$$

$\psi + E\xi$ is an invariant quantity

Remark on the symmetry



Each solution trajectory is transformed to another solution trajectory

Part IV

Derivation of the symmetry

Condition for the Symmetry

$$\delta_G \mathcal{I} = \eta \int_{t_\mathrm{i}}^{t_\mathrm{f}} dt \frac{d\psi}{dt} \qquad t \to t' = t + \eta \xi(q, \dot{q}, \alpha)$$

$$\int_{t_\mathrm{i}}^{t_\mathrm{f}} dt \mathcal{E} \dot{q} \xi = -(\psi + E\xi)|_{t_\mathrm{i}}^{t_\mathrm{f}}$$

$$\frac{dE}{dt} = -\mathcal{E}\dot{q} + \frac{\partial E}{\partial \alpha}\dot{\alpha}$$

↓

$$\int_{t_\mathrm{i}}^{t_\mathrm{f}} dt \left[\frac{d\psi}{dt} + E\dot{\xi} + \dot{\alpha}\frac{\partial E}{\partial \alpha}\xi \right] = 0$$

Do $\xi(q, \dot{q}, \alpha)$ and $\psi(q, \dot{q}, \alpha)$ exist
for any $\hat{q}$ with a fixed quasi-static $\hat{\alpha}$?

Starter

When $\alpha = \mathrm{const.}, \xi = \mathrm{const.}$ and $\psi = 0$ lead to

$$\int_{t_\mathrm{i}}^{t_\mathrm{f}} dt \left[\frac{d\psi}{dt} + E\dot{\xi} + \dot{\alpha} \frac{\partial E}{\partial \alpha} \xi \right] = 0$$

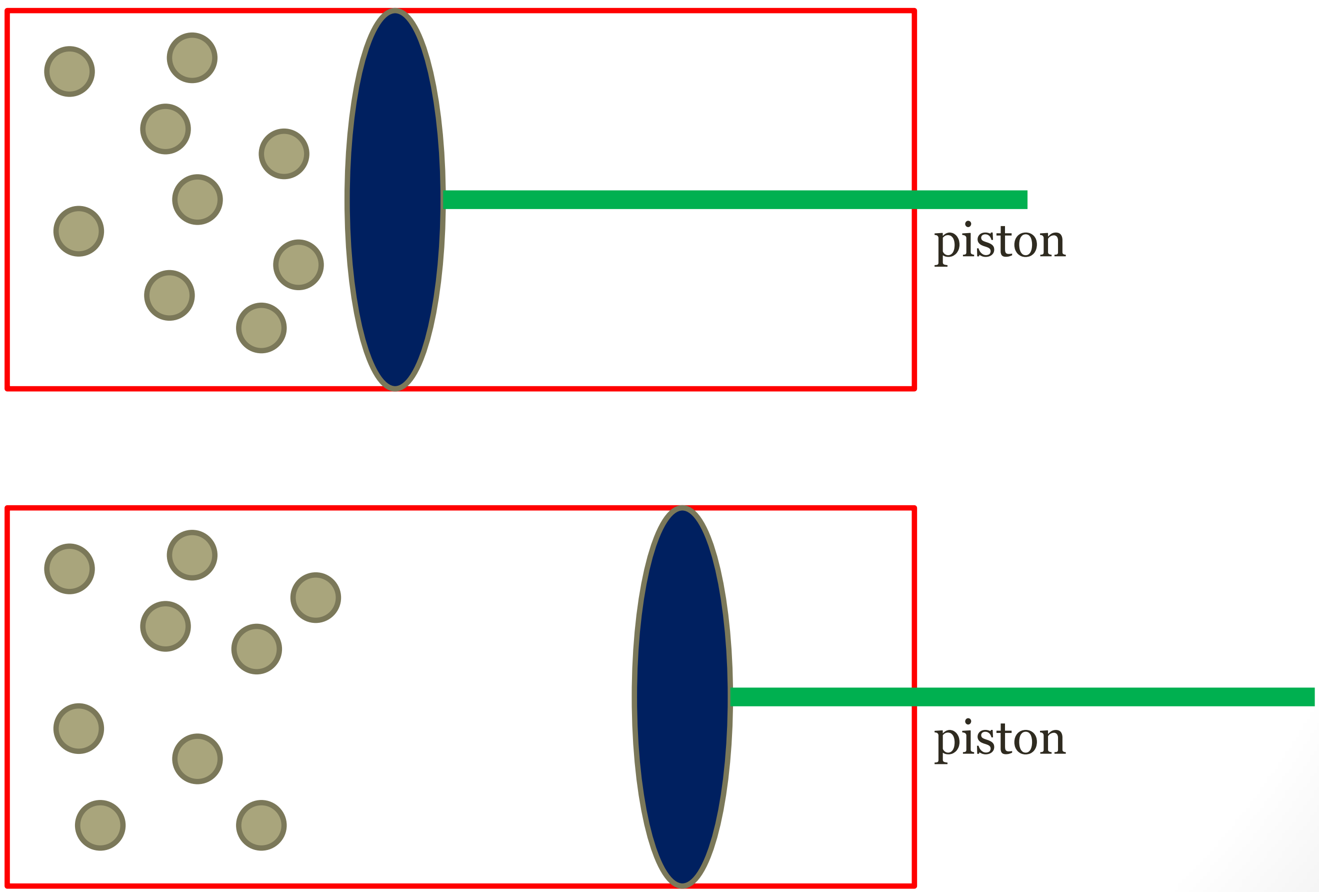
The Noether invariant $\psi + E\xi$ is the energy.

However,

there are no $\xi(q, \dot{q}, \alpha)$ and $\psi(q, \dot{q}, \alpha)$ for a general quasi-static protocol $\hat{\alpha}$.

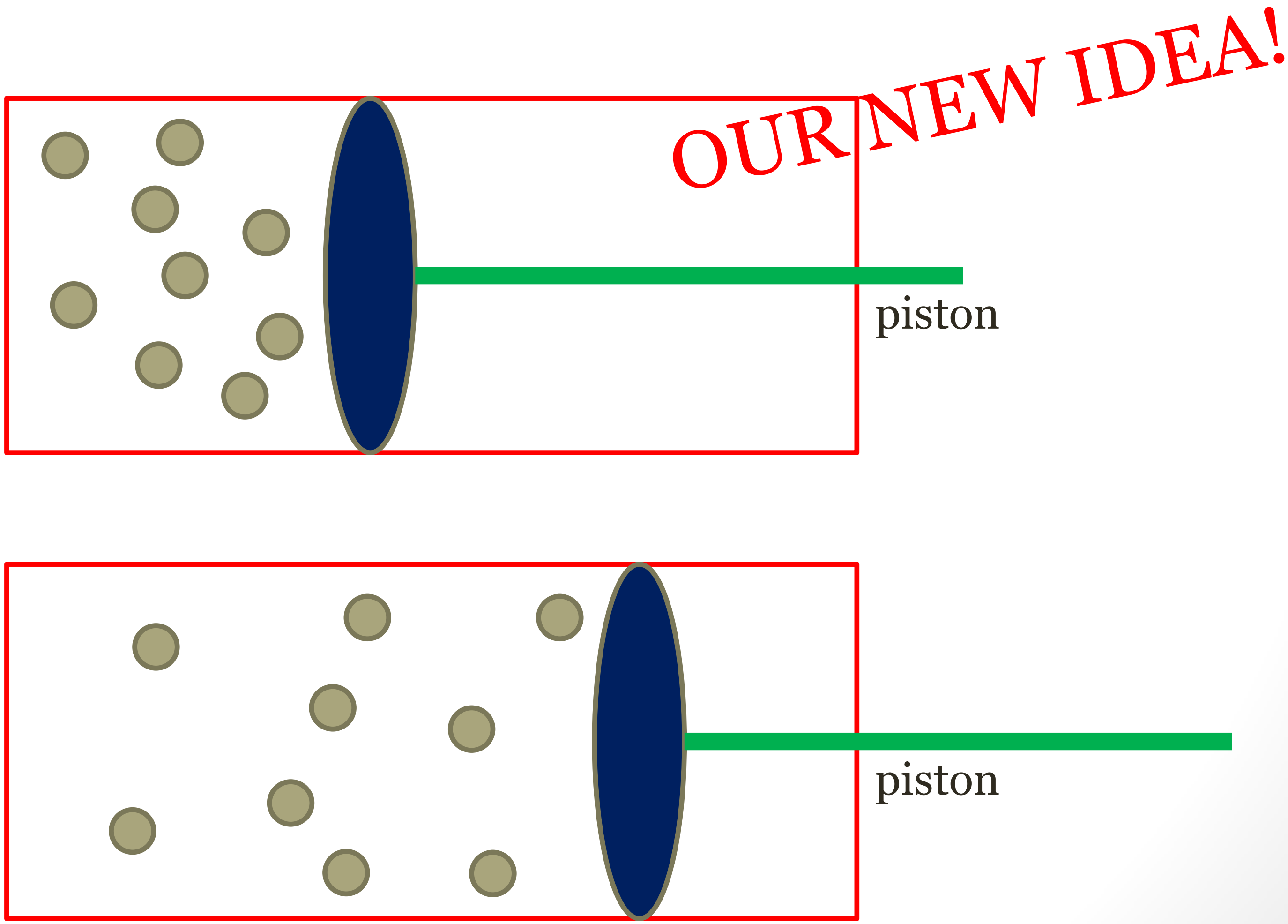
Because

trajectories may be too wild to describe quasi-static processes



We restrict trajectories to

those consistent with quasi-static processes in thermodynamics



Thermodynamically consistent trajectories

$$\hat{q} = (q(t))_{t=t_\mathrm{i}}^{t_\mathrm{f}}$$

→ $$E(t) = H(q(t), p(t); \alpha(t))$$ $$p = \frac{\partial L}{\partial \dot{q}}$$

Two conditions: $\alpha(t) = \bar{\alpha}(\epsilon t)$ (quasi-static operation)

(1) $E(t) = \bar{E}(\epsilon t) + \epsilon g(t)$ which defines a path $(\bar{E}(\tau), \bar{\alpha}(\tau))$

(2) $$\int_{t_\mathrm{i}}^{t_\mathrm{f}} dt \dot{\alpha} \left[\frac{\partial H}{\partial \alpha} - \left\langle \frac{\partial H}{\partial \alpha} \right\rangle_{E(t),\alpha(t)}^{\mathrm{mc}} \right] = 0$$

in the limit $\epsilon \to 0$

Mechanical work = Thermodynamic work

Condition for the Symmetry II

$$\int_{t_\mathrm{i}}^{t_\mathrm{f}} dt \left[\frac{d\psi}{dt} + E\dot{\xi} + \dot{\alpha}\frac{\partial E}{\partial \alpha}\xi \right] = 0$$

$$\xi = \Xi(E(q,\dot{q},\alpha),\alpha) \quad \psi = \Psi(E(q,\dot{q},\alpha),\alpha)$$

$$\int_{\tau_\mathrm{i}}^{\tau_\mathrm{f}} d\tau \left[\frac{d\Psi}{d\tau} + \bar{E}\frac{d\Xi}{d\tau} + \frac{d\bar{\alpha}}{d\tau}\left\langle \frac{\partial E}{\partial \alpha} \right\rangle_{\bar{E}(\tau),\bar{\alpha}(\tau)}^{\mathrm{mc}} \Xi \right] = 0$$

for thermodynamically consistent trajectories

$$\Xi \left(\frac{d\bar{E}}{d\tau} - \frac{d\bar{\alpha}}{d\tau}\left\langle \frac{\partial E}{\partial \alpha} \right\rangle_{\bar{E},\bar{\alpha}}^{\mathrm{mc}} \right) = \frac{d(\Psi + \bar{E}\Xi)}{d\tau}$$

Integrability condition

$$\Xi \left(\frac{d\bar{E}}{d\tau} - \frac{d\bar{\alpha}}{d\tau} \left\langle \frac{\partial E}{\partial \alpha} \right\rangle^{\mathrm{mc}}_{\bar{E},\bar{\alpha}} \right) = \frac{d(\Psi + \bar{E}\Xi)}{d\tau}$$

Total derivative !

$$\left(\frac{\partial \Xi}{\partial \alpha} \right)_E + \frac{\partial}{\partial E} \left(\Xi \left\langle \frac{\partial E}{\partial \alpha} \right\rangle^{\mathrm{mc}}_{E,\alpha} \right)_\alpha = 0$$

(small trick)

$$\frac{\partial}{\partial \alpha} \left(\Xi \beta^{-1} \left(\frac{\partial S}{\partial E} \right)_\alpha \right)_E - \frac{\partial}{\partial E} \left(\Xi \beta^{-1} \left(\frac{\partial S}{\partial \alpha} \right)_E \right)_\alpha = 0$$

Solution

$$\frac{\partial}{\partial \alpha}\left(\Xi\beta^{-1}\left(\frac{\partial S}{\partial E}\right)_{\alpha}\right)_{E} - \frac{\partial}{\partial E}\left(\Xi\beta^{-1}\left(\frac{\partial S}{\partial \alpha}\right)_{E}\right)_{\alpha} = 0$$

$$\left|\frac{\partial(\Xi\beta^{-1}, S)}{\partial(\alpha, E)}\right| = 0$$

$$\Xi = \beta\mathcal{F}(S)$$

$\mathcal{F}$ an arbitrary function

$$\Xi\left(\frac{d\bar{E}}{d\tau} - \frac{d\bar{\alpha}}{d\tau}\left\langle\frac{\partial E}{\partial \alpha}\right\rangle^{\mathrm{mc}}_{\bar{E},\bar{\alpha}}\right) = \frac{d(\Psi + \bar{E}\Xi)}{d\tau}$$

$$\Rightarrow \quad \Psi + E\Xi = \int^{S} dS'\mathcal{F}(S')$$

Summary of Results so far

Transformation:

$$t \to t' = t + \eta\xi(q, \dot{q}, \alpha)$$

$$\xi = \Xi(E(q, \dot{q}, \alpha), \alpha)$$

$$\Xi = \beta\mathcal{F}(S)$$

Symmetry:

$$\delta_G \mathcal{I} = \eta \int_{t_\mathrm{i}}^{t_\mathrm{f}} dt \frac{d\psi}{dt}$$

for thermodynamically consistent trajectories

Noether invariant:

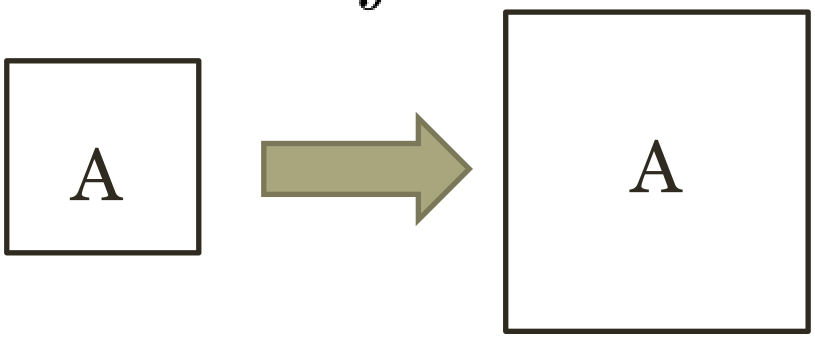
$$\Psi + E\Xi = \int^S dS' \mathcal{F}(S')$$

Part V

Macroscopic systems

Extensive Noether invariant

$$\Psi + E\Xi = \int^{S} dS' \mathcal{F}(S')$$ is an extensive variable



Scaled copy

$$V \to \lambda V$$
$$N \to \lambda N$$

$$\Psi + E\Xi \to \lambda(\Psi + E\Xi)$$
$$E \to \lambda E$$

$$\Psi \to \lambda\Psi$$
$$\Xi \to \Xi$$

$\Xi = \beta\mathcal{F}(S)$ is intensive!

Determination of Ξ

$\Xi = \beta\mathcal{F}(S)$ is intensive!

$\mathcal{F}(S; \mathrm{M}, N)$ is independent of V

$$\Xi_{\mathrm{A}} = \Xi_{\mathrm{B}}$$
$$\beta_{\mathrm{A}} = \beta_{\mathrm{B}}$$

$$\mathcal{F}(S_{\mathrm{A}}; \mathrm{M}_{\mathrm{A}}, N_{\mathrm{A}}) = \mathcal{F}(S_{\mathrm{B}}; \mathrm{M}_{\mathrm{B}}, N_{\mathrm{B}})$$

$$\bar{\mathcal{F}}(s_{\mathrm{A}}; \mathrm{M}_{\mathrm{A}}) = \bar{\mathcal{F}}(s_{\mathrm{B}}; \mathrm{M}_{\mathrm{B}}) \qquad s = S/N$$

$$\mathrm{M}_{\mathrm{A}} = \mathrm{M}_{\mathrm{B}} = \mathrm{M} \quad \Rightarrow \quad \bar{\mathcal{F}}(s; \mathrm{M}) = c(M)$$

Universality

$$\mathcal{F}(S_{\mathrm{A}}; \mathrm{M}_{\mathrm{A}}, N_{\mathrm{A}}) = \mathcal{F}(S_{\mathrm{B}}; \mathrm{M}_{\mathrm{B}}, N_{\mathrm{B}})$$

$$\mathrm{M}_{\mathrm{A}} \neq \mathrm{M}_{\mathrm{B}} \qquad \bar{\mathcal{F}}(s; \mathrm{M}) = c(M)$$

$$c(\mathrm{M}_{\mathrm{A}}) = c(\mathrm{M}_{\mathrm{B}}) = c_*$$

$$\Xi = \beta \mathcal{F}(S; \mathrm{M}, N) = \beta c_*$$

the universal constant of the action-dimension

If there exist no such quantity (like in 19 th century), we may conclude that there is no transformation .....

Universality II

$$\Xi = \beta\mathcal{F}(S;\mathrm{M},N) = \beta c_*$$

the universal constant of the action-dimension

$$c_* = a\hbar$$

Main result

The macroscopic system we study is symmetric
for the transformation $t \rightarrow t + \eta a \beta \hbar$
when the domain of the action is restricted to
"thermodynamically consistent trajectories"

The Noether invariant is then

$$\Psi + E\Xi = a\hbar S + b\hbar N$$

The extensive Noether invariant provides
the unique characterization of thermodynamic entropy

Part VI Concluding remarks

Next problems

What is $t \to t + \eta\hbar\beta$?

Relation with $t + \mathrm{i}\hbar\beta$?

"emergent symmetry"

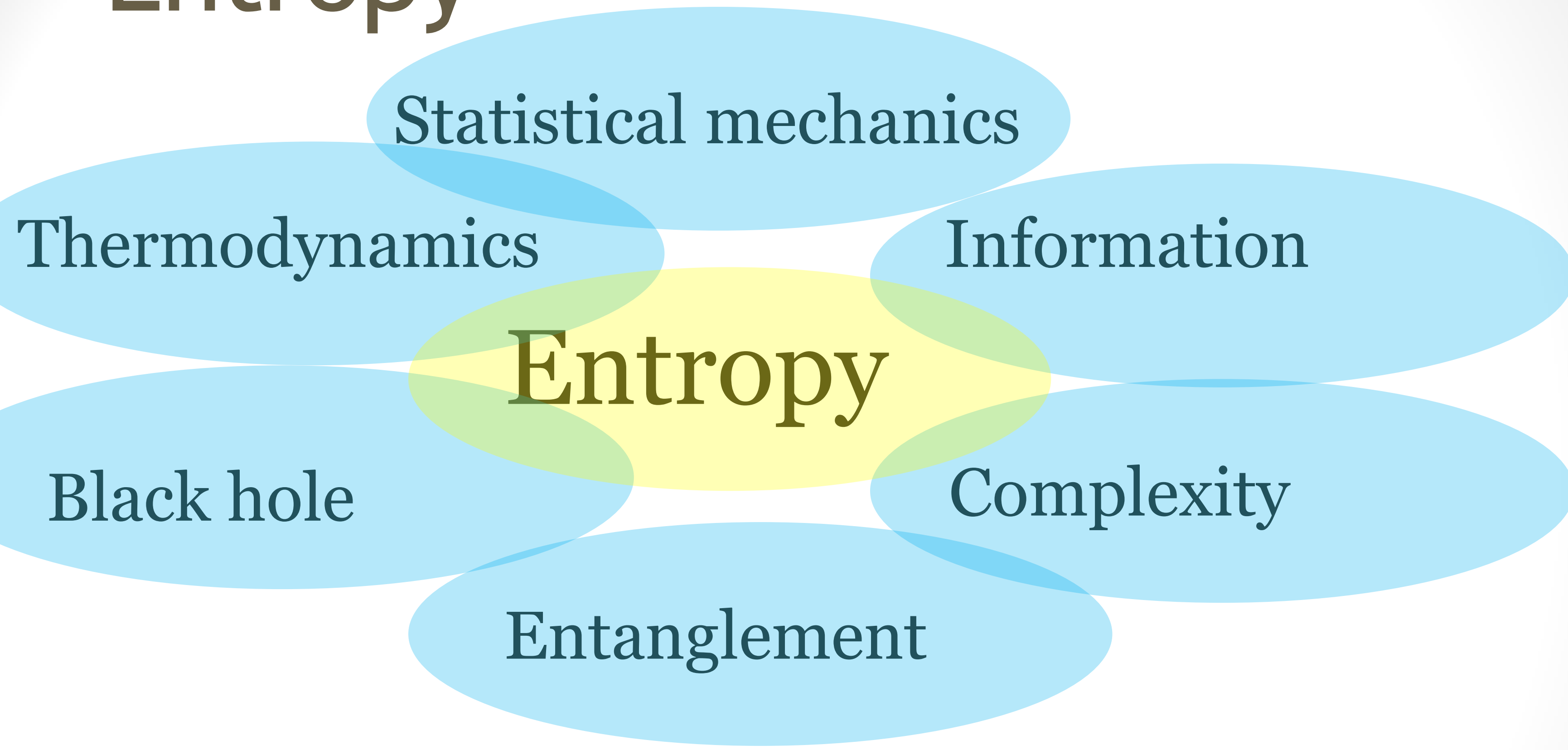
e.g. Symmetry in the action for the perfect fluid

Hael et al, JHEP 2015
Boer et al, JHEP 2015

Connection with black hole entropy (as the Noether charge) ?

Unification with the second law of thermodynamics ?

Entropy


Statistical mechanics
Thermodynamics
Information
Entropy
Black hole
Complexity
Entanglement