Complete gauge invariant action for open superstring field theory

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1. Introduction

To consider the superstring theory as a fundamental theory of the nature, the nonperturbative formulation is needed.

One of the promising candidates is

♠ WZW-like (Open) superstring field theory

♢ A formulation with *no explicit picture changing operator* and working well for the NS (boson) sector: [Berkovits (1995)]

(cf. Hetetrotic [Berkovits, Okawa and Zwiebach (2004)], Type II [Matsunaga (2014)])

However, no one has succeeded to construct complete gauge invariant action including both the NS (boson) and Ramond (fermion) sectors so far.

The purpose of this talk is to construct a complete gauge invariant action for open superstring!

♣ Key points

- Utilizing large Hilbert space
- Restrict the Ramond string field

Plan of the Talk

1. Introduction

2. WZW-like formulation

   2.1 NS sector

   2.2 Ramond sector

3. Complete gauge invariant action

4. Supersymmetry

5. Summary and discussion

2. WZW-like formulation (open superstring)

Superstring SCFT :

$$
\begin{array}{lll}
(X^{\mu}(z), \psi^{\mu}(z)) & \text{Matter} & c = 15 \\
(b(z), c(z)) & \text{Ghost} & c = -26 \\
(\beta(z), \gamma(z)) & \text{Superghost} & c = 11
\end{array} \qquad (1)
$$

Small Hilbertspace :

$$\mathcal{H}_{small} = \{|\text{matter}\rangle \otimes |\text{Fock}(c_{-n}, b_{-m})\rangle \otimes |\text{Fock}(\gamma_{-r}, \beta_{-s})\rangle\}$$

Bosonization :

$$\beta(z) = \ \partial\xi(z)e^{-\phi(z)}, \qquad \gamma(z) = \ e^{\phi(z)}\eta(z).$$

Large Hilbertspace :

$$\mathcal{H}_{large} = \ \{|\text{matter}\rangle \otimes |\text{Fock}(b_{-m}, c_{-n})\rangle \otimes |\text{Fock}(\eta_{-m}, \xi_{-n})\rangle\} \otimes |\text{Fock}(\phi_n)\rangle\}$$

Relation :

$$\mathcal{H}_{large} \ni \Phi = \varphi + \xi_0 a, \tag{2}$$

with $\varphi, a \in \mathcal{H}_{small}$, (or equivalently, $\eta\varphi = \eta a = 0$, $(\eta \equiv \eta_0)$.)

2.1 NS sector [Berkovits]

String field :

$$\mathcal{H}^{(NS)}_{large} \quad \ni \quad |\Phi\rangle = \sum_i |i\rangle \varphi^i(x),$$

where the sum i runs through all the states with $(g,p) = (0,0)$ in $\mathcal{H}^{(NS)}_{large}$, and The string field $|\Phi\rangle$ is Grassmann even ($|\Phi| = 0$). Since the space-time fields $\varphi^i(x)$ are bosons, $|i\rangle$ has to be Grassmann even. We impose the GSO projection:

$$\Phi \;=\; \frac{1}{2}(1 + (-1)^{F_{NS}})\,\Phi\,,$$

with

$$F_{NS} \;=\; \sum_{r>0} \psi^{\mu}_{-r}\psi_{\mu r} + \oint \frac{dz}{2\pi i}\partial\phi(z)\,.$$

Free theory :
Equation of motion (EOM) and gauge tf.

$$Q\eta\Phi = 0, \qquad \delta\Phi = Q\Lambda + \eta\Omega. \tag{3}$$

If we expand as (2) and similarly $\Lambda = \rho - \xi_0\alpha$ with $\eta\rho = \eta\alpha = 0$,

$$\eta\Phi = a, \quad \eta\Lambda = -\alpha, \tag{4}$$

and (3) is equivalent to the conventional formulation:

$$Qa = 0, \qquad \delta a = Q\alpha.$$

The EOM in (3) is derived from the action

$$S_{NS}^{(0)} = \frac{1}{2}\langle Q\Phi, \eta\Phi\rangle, \tag{5}$$

where $\langle A, B\rangle$ is the BPZ inner product in the large Hilbert space, which satisfies

$$\langle B, A\rangle = (-1)^{AB}\langle A, B\rangle,$$
$$\langle QA, B\rangle = -(-1)^{A}\langle A, QB\rangle, \qquad \langle\eta A, B\rangle = -(-1)^{A}\langle A, \eta B\rangle.$$

Counting the ghost number anomaly, $\langle A, B\rangle \neq 0$ iff

$$g(A) + g(B) = 2, \qquad p(A) + p(B) = -1.$$

Ghost and picture numbers :

operator	Φ	Q	b	c	η	ξ	$e^{q\phi}$
(g, p)	$(0, 0)$	$(1, 0)$	$(-1, 0)$	$(1, 0)$	$(1, -1)$	$(-1, 1)$	$(0, q)$

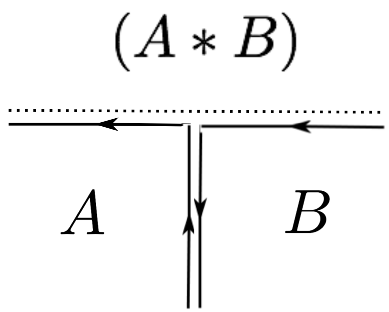
$$S_{NS}^{(0)} = \frac{1}{2}\langle Q\Phi, \eta\Phi\rangle$$

WZW-like action : [Berkovits]
A non-linear extension of (5):

$$S_{NS} = \frac{1}{2}\langle (g^{-1}Qg)(g^{-1}\eta g)\rangle - \frac{1}{2}\int_0^1 dt \langle (\hat{g}^{-1}\partial_t \hat{g})\{(\hat{g}^{-1}Q\hat{g}),(\hat{g}^{-1}\eta\hat{g})\}\rangle, \qquad (6)$$

where $\hat{g} = g(t) = e^{\Phi(t)}$ with $\Phi(1) = \Phi$ and $\Phi(0) = 0$, and $g = \hat{g}(1) = e^{\Phi}$. The product of string fields is defined by using the Witten's $*$-product, which is non-commutative but associative:

$$e^{\Phi} = \mathbb{I} + \Phi + \frac{1}{2}(\Phi * \Phi) + \frac{1}{3!}(\Phi * \Phi * \Phi) + \cdots .$$



For later use, it is convenient to rewrite the action (6) as

$$S_{NS} = -\int_0^1 dt \langle A_t(t), QA_\eta(t)\rangle, \tag{7}$$

where

$$A_\eta(t) = (\eta\hat{g})\hat{g}^{-1}, \qquad A_t(t) = (\partial_t\hat{g})\hat{g}^{-1}.$$

The t-dependence is "topological", and general variation of S_{NS} is given by

$$\delta S_{NS} = -\langle A_\delta, QA_\eta\rangle \qquad \text{with} \qquad A_\delta = (\delta g)g^{-1}\,. \tag{8}$$

From (8) the EOM becomes

$$QA_\eta = 0\,.$$

Gauge symmetry :
We can show that the action (7) is invariant under the gauge transformation

$$\begin{aligned} A_\delta &= Q\Lambda + \eta\Omega - \{A_\eta, \Omega\} \\ &= Q\Lambda + D_\eta\Omega\,, \end{aligned}$$

by using $Q^2 = D_\eta^2 = 0$ and $QA_\eta = D_\eta A_Q$, where $A_Q = (Qg)g^{-1}$.

2.2 Ramond sector

Naive argument

String Field :

$$\mathcal{H}_{large}^{(R)} \quad \ni \quad |\psi\rangle = \sum_i |i\rangle \psi^i(x),$$

where the sum i runs through all the states with $(g,p) = (0,1/2)$ in $\mathcal{H}_{large}$. The string field $|\psi\rangle$ is Grassmann even ($|\psi| = 0$).

Free theory :

$$Q\eta\psi = 0, \qquad \delta\psi = Q\lambda + \eta\omega. \tag{10}$$

Difficulties :

A candidate of the free action vanishes:

$$\langle Q\psi, \eta\psi\rangle \equiv \ 0\,,$$

from picture number counting:

$$\frac{1}{2} + (-1) + \frac{1}{2} \neq -1\,.$$

Similar difficulty? :

Closed bosonic string field: $\Phi \in \mathcal{H}_{small}, \qquad g = 2$

Naive action vanishes,

$$\langle\langle \Phi, Q\Phi \rangle\rangle \equiv 0 \,,$$

from the ghost number counting:

$$2 + 1 + 2 \neq 6 \,.$$

Propagator from the path integral is given by

$$b_0^+ b_0^- \int_0^\infty d\tau \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{-\tau L_0^+ + i\theta L_0^-} \;=\; \frac{b_0^+ b_0^-}{L_0^+} \delta(L_0^-) \,.$$

suggests that we should restrict the string field as

$$L_0^- \Phi \;=\; b_0^- \Phi \;=\; 0 \,,$$

or equivalently

$$\int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{i\theta L_0^-} \Phi \;=\; \Phi \,, \qquad b_0^- c_0^- \Phi \;=\; \Phi \,.$$

The BPZ inner product in the restricted space is given by $\langle\!\langle A, c_0^- B\rangle\!\rangle$, by using which the action can be given as

$$S_0 \;=\; -\frac{1}{2}\langle\!\langle \Phi, c_0^- Q\Phi\rangle\!\rangle \;.$$

Ghost number counting :

$$2+1+1+2 \;=\; 6 \;.$$

Open superstring in the Ramond sector :

Propagator from the path integral in $\mathcal{H}_{small}$

$$b_0 X\int_0^\infty d\tau\; e^{-\tau L_0} \;=\; \frac{b_0 X}{L_0}\;,$$

where

$$X \;=\; -\delta(\beta_0)G_0 + \delta'(\beta_0)b_0\,,$$

is the PCO on the $p=-3/2$ states in the small Hilbertspace.

This suggests that we should use

Restricted string field :

$$\eta\Psi \;=\; 0\,, \qquad XY\Psi \;=\; \Psi\,. \tag{11}$$

with $(g, p) = (1, -1/2)$ and $Y = -c_0\delta'(\gamma_0)$. X and Y satisfy

$$XYX \;=\; X\,, \qquad YXY \;=\; Y\,,$$

and thus XY is a projector. Ψ is related with ψ as $\psi = \chi + \xi_0\Psi$, so it has to be Grassmann odd. We also impose the GSO projection:

$$\Psi \;=\; \frac{1}{2}(1 + \hat{\Gamma}_{11}(-1)^{F_R})\,\Psi\,,$$

with

$$\begin{aligned} \hat{\Gamma}_{11} \;&=2^5\psi_0^0\psi_0^1\cdots\psi_0^9\,, \\ F_R \;&=\; \sum_{n>0}(\psi^{\mu}_{-n}\psi_{n\mu} + \gamma_{-n}\beta_n + \beta_{-n}\gamma_n) + \gamma_0\beta_0\,. \end{aligned}$$

Using the BPZ inner product in the restricted space $\langle\langle A, YB\rangle\rangle$, the action becomes

$$S_0 = -\frac{1}{2}\langle\langle\Psi, YQ\Psi\rangle\rangle\ ,$$

where $\langle\langle\Psi_1, \Psi_2\rangle\rangle$ is non-vanishing iff $p(\Psi_1) + p(\Psi_2) = -2$.

Picture number counting:

$$-\frac{1}{2} - 1 - \frac{1}{2} = -2\ .$$

Restricted space :

Expand Ψ based on the ghost zero-modes

$$\Psi = \sum_{n=0}^{\infty}(\gamma_0)^n(\phi_n + c_0\psi_n)\ .$$

The second condition in (11) restricts Ψ in the form,

$$\Psi = \phi + (\gamma_0 + c_0 G)\psi, \qquad (G = G_0 + 2b_0\gamma_0).$$

Note that X is BRST exact in the large Hilbert space; it is given on the picture number $-3/2$ states as

$$X = \{Q, \Theta(\beta_0)\}, \qquad \Theta(\beta_0) = \xi_0 + \cdots .$$

Or more generally we can introduce

$$\Xi = \xi_0 + (\Theta(\beta_0)\eta\xi_0 - \xi_0)P_{-3/2} + (\xi_0\eta\Theta(\beta_0) - \xi_0)P_{-1/2} ,$$

which is more suitable to use in the large Hilbert space, and $\{Q, \Xi\}$ becomes identical with X on the pictrue number $-3/2$ sates in the small Hilbert space.

Ramond kinetic term :

$$S_R^{(0)} \;=\; -\frac{1}{2}\langle\!\langle \Psi, YQ\Psi \rangle\!\rangle, \qquad \delta^{(0)}\Psi \;=\; Q\lambda,$$

where

$$\eta\Psi \;=0, \qquad XY\Psi \;=\; \Psi,$$
$$\eta\lambda \;=0, \qquad XY\lambda \;=\; \lambda.$$

The relation between BPZ inner products:

$$\langle\!\langle A, B \rangle\!\rangle \;=\; \langle \xi_0 A, B \rangle \;=\; \langle \Xi A, B \rangle.$$

X is BPZ even wrt $\langle\!\langle \cdot, \cdot \rangle\!\rangle$:

$$\langle\!\langle XA, B \rangle\!\rangle \;=\; \langle\!\langle A, XB \rangle\!\rangle.$$

Can we include interactions consistently?

Order by order construction :

Expand the action $S = S_{NS} + S_R$ and gauge transformation as

$$S_{NS} = S_{NS}^{(0)} + g\, S_{NS}^{(1)} + g^2\, S_{NS}^{(2)} + O(g^3)\,,$$
$$S_R = S_R^{(0)} + g\, S_R^{(1)} + g^2\, S_R^{(2)} + O(g^3)\,,$$

and

$$\delta\Phi = \delta^{(0)}\Phi + g\,\delta^{(1)}\Phi + g^2\,\delta^{(2)}\Phi + O(g^3)\,,$$
$$\delta\Psi = \delta^{(0)}\Psi + g\,\delta^{(1)}\Psi + g^2\,\delta^{(2)}\Psi + O(g^3).$$

We attempt to construct them order by order in g, starting from the kinetic terms

$$S_{NS}^{(0)} \;=\; -\frac{1}{2}\,\langle\, \Phi, Q\eta\Phi \,\rangle\,, \qquad S_R^{(0)} \;=\; -\frac{1}{2}\,\langle\!\langle\, \Psi, YQ\Psi \,\rangle\!\rangle\,.$$

which is invariant under tree gauge transformations

$$\delta_\Lambda^{(0)}\Phi = Q\Lambda\,, \qquad \delta_\Lambda^{(0)}\Psi = 0\,,$$

with parameter Λ in the NS sector,

$$\delta^{(0)}_{\Omega}\Phi = \eta\Omega\,, \qquad \delta^{(0)}_{\Omega}\Psi = 0\,,$$

with parameter Ω also in the NS sector and

$$\delta^{(0)}_{\lambda}\Phi = 0\,, \qquad \delta^{(0)}_{\lambda}\Psi = Q\lambda\,,$$

with parameter λ in the Ramond sector.

NS sector:

Expanding the NS action, we obtain

$$S^{(1)}_{NS} = \frac{1}{6}\,\langle\, Q\Phi, [\,\Phi, \eta\Phi\,]\,\rangle\,,$$
$$S^{(2)}_{NS} = \frac{1}{24}\,\langle\, Q\Phi, [\,\Phi, [\,\Phi, \eta\Phi\,]\,]\,\rangle\,.$$

This is invariant at $O(g)$ and $O(g^2)$:

$$\delta^{(0)}S^{(1)}_{NS} + \delta^{(1)}S^{(0)}_{NS} = 0\,, \qquad \delta^{(0)}S^{(2)}_{NS} + \delta^{(1)}S^{(1)}_{NS} + \delta^{(2NS)}S^{(0)}_{NS} = 0\,,$$

under

$$\delta_\Lambda^{(1)}\Phi = -\frac{1}{2}[\,\Phi, Q\Lambda\,]\,, \qquad \delta_\Lambda^{(2NS)}\Phi = \frac{1}{12}[\,\Phi, [\,\Phi, Q\Lambda\,]\,]\,,$$

and

$$\delta_\Omega^{(1)}\Phi = \frac{1}{2}[\,\Phi, \eta\Omega\,]\,, \qquad \delta_\Omega^{(2NS)}\Phi = \frac{1}{12}[\,\Phi, [\,\Phi, \eta\Omega\,]\,]\,.$$

Including Ramond sector :

Cubic order

Counting ghost # and picture #, an allowed candidate of $S_R^{(1)}$ has the form

$$S_R^{(1)} = \alpha_1 \langle\, \Phi, \Psi^2 \,\rangle\,,$$

with a constant α_1 to be determined.

The gauge tf. $\delta^{(1)}$ is determined by requiring

$$\delta^{(0)}S_R^{(1)} + \delta^{(1)}S_{NS}^{(0)} + \delta^{(1)}S_R^{(0)} = 0.$$

The variation under $\delta_\Lambda^{(0)}\Phi$ is given by

$$\delta_\Lambda^{(0)} S_R^{(1)} = \alpha_1 \langle\, Q\Lambda, \Psi^2 \,\rangle = \, - \, \alpha_1 \langle\, \{\, \Psi, \Lambda \,\}, Q\Psi \,\rangle \, . \tag{12}$$

Here, due to the restriction (11), a term of the form

$$\delta S = \langle B, Q\Psi \,\rangle$$

can be rewritten as

$$\begin{aligned} \delta S &= \langle\, \Xi\eta B, XYQ\Psi \,\rangle \\ &= \langle\!\langle\, \eta B, XYQ\Psi \,\rangle\!\rangle = \langle\!\langle\, X\eta B, YQ\Psi \,\rangle\!\rangle \, , \end{aligned}$$

which can be cancelled by the variation of the kinetic term:

$$\delta S_R^{(0)} = \, - \, \langle\!\langle\, \delta\Psi, YQ\Psi \,\rangle\!\rangle.$$

if we take $\delta\Psi = X\eta B$. From (12), we obtain

$$\delta_\Lambda^{(1)} \Psi = \, - \, \alpha_1 \, X\eta \, \{\, \Psi, \Lambda \,\} \, .$$

This $\delta\Psi$ is consistent with the restriction:

$$\eta\,\delta\Psi = 0\,, \qquad XY\delta\Psi = \delta\Psi\,.$$

On the hand, the variation under $\delta_\Omega^{(0)}\Phi$ is given by

$$\delta_\Omega^{(0)} S_R^{(1)} = \alpha_1 \langle\, \eta\Omega, \Psi^2\,\rangle = \alpha_1 \langle\, \Omega, (\eta\Psi)\,\Psi - \Psi\,(\eta\Psi)\,\rangle = 0.$$

Hence $\delta_\Omega^{(1)}\Psi = 0$.

Similarly, the variation under $\delta_\lambda^{(0)}\Psi$ is given by

$$\begin{aligned}
\delta_\lambda^{(0)} S_R^{(1)} &= \alpha_1 \langle\, \Phi, (Q\lambda)\,\Psi\,\rangle + \alpha_1 \langle\, \Phi, \Psi\,(Q\lambda)\,\rangle \\
&= \; - \alpha_1 \langle\, [\,\Psi, \lambda\,], Q\Phi\,\rangle - \alpha_1 \langle\, [\,\Phi, \lambda\,], Q\Psi\,\rangle \\
&= \; - \alpha_1 \langle\, [\,\Psi, \eta\Xi\lambda\,], Q\Phi\,\rangle - \alpha_1 \langle\, [\,\Phi, \eta\Xi\lambda\,], Q\Psi\,\rangle \\
&= \alpha_1 \langle\, \{\Psi, \Xi\lambda\,\}, Q\eta\Phi\,\rangle + \alpha_1 \langle\, \{\,\eta\Phi, \Xi\lambda\,\}, Q\Psi\,\rangle\,,
\end{aligned}$$

which can be canceled by

$$\delta_\lambda^{(1)}\Phi = \alpha_1 \{\Psi, \Xi\lambda\,\}\,, \qquad \delta_\lambda^{(1)}\Psi = \alpha_1\, X\eta\,\{\,\eta\Phi, \Xi\lambda\,\}\,.$$

Quartic order:

Let us construct $S_R^{(2)}$ such that

$$\delta^{(0)}S_R^{(2)} + \delta^{(1)}S_{NS}^{(1)} + \delta^{(1)}S_R^{(1)} + \delta^{(2)}S_{NS}^{(0)} + \delta^{(2)}S_R^{(0)} = 0.$$

The variation of $S_R^{(1)}$ under $\delta_\Lambda^{(1)}$ is given by

$$\begin{aligned}\delta_\Lambda^{(1)}S_R^{(1)} = & -\alpha_1^2\,\langle\,\Phi,(X\eta\,\{\Psi,\Lambda\})\,\Psi\,\rangle - \alpha_1^2\,\langle\,\Phi,\Psi\,(X\eta\,\{\Psi,\Lambda\})\,\rangle\,.\\ & -\frac{\alpha_1}{2}\,\langle\,[\,\Phi,Q\Lambda\,],\Psi^2\,\rangle\end{aligned}$$

Using $X = \{Q,\Xi\,\}$, we can rewrite it as

$$\begin{aligned}\delta_\Lambda^{(1)}S_R^{(1)} = {} & \alpha_1^2\,\langle\,Q\Lambda,\{\,\Psi,\Xi\,\{\eta\Phi,\Psi\}\}\,\rangle + \frac{\alpha_1}{2}\,\langle\,Q\Lambda,[\,\Phi,\Psi^2\,]\,\rangle\\ & + \alpha_1^2\,\langle\,\{\Psi,\Xi\,\{\Psi,\Lambda\}\},Q\eta\Phi\,\rangle + \alpha_1^2\,\langle\,\{\eta\Phi,\Xi\,\{\Psi,\Lambda\}\},Q\Psi\,\rangle\\ & + \alpha_1^2\,\langle\,\{\,\Xi\,\{\eta\Phi,\Psi\},\Lambda\},Q\Psi\,\rangle.\end{aligned}$$

From the first term on the right-hand side we can guess

$$S_R^{(2)} = \alpha_2\,\langle\,\Phi,\{\Psi,\Xi\,\{\eta\Phi,\Psi\}\}\,\rangle\,,$$

with a constant α_2 to be determined. From the variation of $S_R^{(2)}$ under $\delta_\Lambda^{(0)}\Phi$,

$$\delta_\Lambda^{(0)} S_R^{(2)} = \; 2\,\alpha_2 \,\langle\, Q\Lambda, \{\Psi, \Xi\,\{\eta\Phi, \Psi\}\}\,\rangle - \alpha_2 \,\langle\, Q\Lambda, [\,\Phi, \Psi^2\,]\,\rangle\,,$$

we find

$$\begin{aligned}\delta_\Lambda^{(0)} S_R^{(2)} + \delta_\Lambda^{(1)} S_R^{(1)} &= \langle\, \{\Psi, \Xi\,\{\Psi, \Lambda\}\}, Q\eta\Phi\,\rangle \\ &\quad + \langle\, \{\eta\Phi, \Xi\,\{\Psi, \Lambda\}\}, Q\Psi\,\rangle + \langle\, \{\,\Xi\,\{\eta\Phi, \Psi\}, \Lambda\}, Q\Psi\,\rangle\,. \end{aligned} \tag{13}$$

if we take

$$\alpha_1 = \; -\,1\,, \qquad \alpha_2 = \; -\,\frac{1}{2}\,.$$

(13) can be cancelled by

$$\begin{aligned}\delta_\Lambda^{(2R)}\Phi &= \{\Psi, \Xi\,\{\Psi, \Lambda\}\}\,, \\ \delta_\Lambda^{(2)}\Psi &= X\eta\,\{\,\Xi\,\{\eta\Phi, \Psi\}, \Lambda\} + X\eta\,\{\eta\Phi, \Xi\,\{\Psi, \Lambda\}\}\,.\end{aligned}$$

Similarly, the variation of $S_R^{(1)}$ and $S_R^{(2)}$ under $\delta_\Omega^{(1)}\Phi$ are given by

$$\delta_\Omega^{(1)} S_R^{(1)} = \; -\,\frac{1}{2}\,\langle\,[\,\Phi, \eta\Omega\,], \Psi^2\,\rangle = \frac{1}{2}\,\langle\,\eta\Omega, [\,\Phi, \Psi^2\,]\,\rangle\,,$$

and

$$\delta_\Omega^{(0)} S_R^{(2)} = \, - \, \frac{1}{2} \, \langle \, \eta\Omega, \{\Psi, \Xi \, \{\eta\Phi, \Psi\}\} \, \rangle$$
$$= \, - \, \frac{1}{2} \, \langle \, \eta\Omega, \{\Psi, [\, \Phi, \Psi \,] \, \} \, \rangle = \, - \, \frac{1}{2} \, \langle \, \eta\Omega, [\, \Phi, \Psi^2 \,] \, \rangle \, ,$$

respectively. From

$$\delta_\Omega^{(0)} S_R^{(2)} + \delta_\Omega^{(1)} S_R^{(1)} = 0 \, ,$$

we have

$$\delta_\Omega^{(2R)} \Phi = 0 \, , \qquad \delta_\Omega^{(2)} \Psi = 0 \, .$$

Finally, the variation $\delta_\lambda^{(1)} S_{NS}^{(1)}$ is given by

$$\delta_\lambda^{(1)} S_{NS}^{(1)} = \, - \, \frac{1}{2} \, \langle \, \delta_\lambda^{(1)} \Phi, \{Q\Phi, \eta\Phi\} \, \rangle = \frac{1}{2} \, \langle \, \{\Psi, \Xi\lambda \, \}, \{Q\Phi, \eta\Phi\} \, \rangle$$
$$= \, - \, \frac{1}{2} \, \langle \, \Xi\lambda, [\, \{Q\Phi, \eta\Phi\}, \Psi \,] \, \rangle \, .$$

While the variation $\delta_\lambda^{(1)} S_R^{(1)}$ is given by

$$\delta_\lambda^{(1)} S_R^{(1)} = \langle\, \{\Psi, \Xi\lambda\,\}, \Psi^2\,\rangle \\ + \langle\, \Phi, (X\eta\,\{\,\eta\Phi, \Xi\lambda\,\}\,)\,\Psi\,\rangle + \langle\, \Phi, \Psi\,(X\eta\,\{\,\eta\Phi, \Xi\lambda\,\}\,)\,\rangle\,.$$

The first term on the right-hand side vanishes:

$$\langle\, \{\Psi, \Xi\lambda\,\}, \Psi^2\,\rangle = \,-\,\langle\, \Xi\lambda, \Psi^3\,\rangle + \langle\, \Xi\lambda, \Psi^3\,\rangle = 0\,.$$

The remaining terms are summarized as

$$\delta_\lambda^{(1)} S_R^{(1)} = \,-\,\langle\, \eta\Phi, (\{Q,\Xi\}\,\{\,\eta\Phi, \Xi\lambda\,\}\,)\,\Psi\,\rangle + \langle\, \eta\Phi, \Psi\,(\{Q,\Xi\}\,\{\,\eta\Phi, \Xi\lambda\,\}\,)\,\rangle \\ = \langle\, \Xi\lambda, [\,\eta\Phi, \{Q,\Xi\}\,\{\,\eta\Phi, \Psi\}\,]\,\rangle\,.$$

Similarly, we find

$$\delta_\lambda^{(0)} S_R^{(2)} = \,-\,\frac{1}{2}\,\langle\, \Phi, \{Q\eta\Xi\lambda, \Xi\,\{\eta\Phi, \Psi\}\}\,\rangle - \frac{1}{2}\,\langle\, \Phi, \{\Psi, \Xi\,\{\eta\Phi, Q\eta\Xi\lambda\}\}\,\rangle \\ = \langle\, \Xi\lambda, Q\,\{\eta\Phi, \Xi\,\{\eta\Phi, \Psi\}\}\,\rangle + \frac{1}{2}\,\langle\, \Xi\lambda, Q\,\{\,[\,\Phi, \eta\Phi\,], \Psi\}\,\rangle\,,$$

by rewriting $Q\lambda$ as $Q\eta\Xi\lambda$. Then

$$
\begin{aligned}
&\delta_\lambda^{(1)} S_{NS}^{(1)} + \delta_\lambda^{(1)} S_R^{(1)} + \delta_\lambda^{(0)} S_R^{(2)} \\
&= -\frac{1}{2}\langle\, \Xi\lambda, [\, \{Q\Phi, \eta\Phi\}, \Psi\,]\,\rangle + \langle\, \Xi\lambda, [\, \eta\Phi, \{Q, \Xi\}\,\{\, \eta\Phi, \Psi\}\,]\,\rangle \\
&\quad + \langle\, \Xi\lambda, Q\,\{\eta\Phi, \Xi\,\{\eta\Phi, \Psi\}\}\,\rangle + \frac{1}{2}\langle\, \Xi\lambda, Q\,\{\,[\,\Phi, \eta\Phi\,], \Psi\}\,\rangle \\
&= -\langle\, \{\Psi, \Xi\,\{\eta\Phi, \Xi\lambda\}\} + \{\,\Xi\,\{\eta\Phi, \Psi\}, \Xi\lambda\}, Q\eta\Phi\,\rangle + \frac{1}{2}\langle\, [\,\Phi, \{\Psi, \Xi\lambda\}\,], Q\eta\Phi\,\rangle \\
&\quad - \langle\, \{\,\eta\Phi, \Xi\,\{\eta\Phi, \Xi\lambda\}\}, Q\Psi\,\rangle - \frac{1}{2}\langle\, \{[\,\Phi, \eta\Phi\,], \Xi\lambda\}, Q\Psi\,\rangle\,.
\end{aligned}
$$

These terms can be canceled by

$$
\begin{aligned}
\delta_\lambda^{(2)}\Phi &= -\{\Psi, \Xi\,\{\eta\Phi, \Xi\lambda\}\} - \{\,\Xi\,\{\eta\Phi, \Psi\}, \Xi\lambda\} + \frac{1}{2}[\,\Phi, \{\Psi, \Xi\lambda\}\,]\,, \\
\delta_\lambda^{(2)}\Psi &= -X\eta\,\{\,\eta\Phi, \Xi\,\{\eta\Phi, \Xi\lambda\}\} - \frac{1}{2}X\eta\,\{[\,\Phi, \eta\Phi\,], \Xi\lambda\}\,.
\end{aligned}
$$

A summary so far:

NS action:

$$S_{NS} = \frac{1}{2}\langle\, Q\Phi, \eta\Phi\,\rangle \;+\; \frac{1}{6}\langle\, Q\Phi, [\,\Phi, \eta\Phi\,]\,\rangle \;+\; \frac{1}{24}\langle\, Q\Phi, [\,\Phi, [\,\Phi, \eta\Phi\,]\,]\,\rangle \;+\; \cdots\,.$$

Ramond action:

$$S_R = \; -\frac{1}{2}\langle\!\langle\, \Psi, YQ\Psi\,\rangle\!\rangle - \langle\, \Phi, \Psi^2\,\rangle - \frac{1}{2}\langle\, \Phi, \{\Psi, \Xi\,\{\eta\Phi, \Psi\}\}\,\rangle \;+\; \cdots\,.$$

The gauge tf. with Λ:

$$\delta_\Lambda\Phi = Q\Lambda - \frac{1}{2}[\,\Phi, Q\Lambda\,] \;+\; \frac{1}{12}[\,\Phi, [\,\Phi, Q\Lambda\,]\,] + \{\Psi, \Xi\,\{\Psi, \Lambda\}\} + \;\cdots\,,$$

$$\delta_\Lambda\Psi = X\eta\,\{\,\Psi, \Lambda\,\} \;+ X\eta\,\{\,\Xi\,\{\eta\Phi, \Psi\}, \Lambda\} + X\eta\,\{\eta\Phi, \Xi\,\{\Psi, \Lambda\}\} \;+\; \cdots\,.$$

The gauge tf. with Ω:

$$\delta_\Omega\Phi = \eta\Omega + \;\frac{1}{2}[\,\Phi, \eta\Omega\,] \;+\; \frac{1}{12}[\,\Phi, [\,\Phi, \eta\Omega\,]\,] \;+\; \cdots\,,$$

$$\delta_\Omega\Psi = 0 + \;\cdots\,,$$

The gauge tf. with λ:

$$\begin{aligned}\delta_\lambda \Phi = & -\{\Psi, \Xi\lambda\} - \{\Psi, \Xi\{\eta\Phi, \Xi\lambda\}\} \\ & - \{\Xi\{\eta\Phi, \Psi\}, \Xi\lambda\} + \frac{1}{2}[\Phi, \{\Psi, \Xi\lambda\}] + \cdots, \\ \delta_\lambda \Psi = & Q\lambda - X\eta\{\eta\Phi, \Xi\lambda\} \\ & - X\eta\{\eta\Phi, \Xi\{\eta\Phi, \Xi\lambda\}\} - \frac{1}{2} X\eta\{[\Phi, \eta\Phi], \Xi\lambda\} + \cdots .\end{aligned}$$

3. Complete gauge invariant action

We finally obtain a complete action S as

$$S = -\frac{1}{2}\langle\!\langle\, \Psi, YQ\Psi \,\rangle\!\rangle - \int_0^1 dt\, \langle\, A_t(t), QA_\eta(t) + (\, F(t)\Psi\,)^2 \,\rangle\,,$$

where linear map F is given by

$$\begin{aligned} F(t)\Psi &= \Psi + \Xi\,\{A_\eta(t), \Psi\} + \Xi\,\{A_\eta(t), \Xi\,\{A_\eta(t), \Psi\}\} + \ldots \\ &= \sum_{n=0}^{\infty} \underbrace{\Xi\,\{A_\eta(t), \Xi\,\{A_\eta(t), \ldots, \Xi\,\{A_\eta(t)}_{n}, \Psi\}\ldots\}\}\,, \end{aligned}$$

which satisfies $D_\eta F = F\eta$.

We can show that the action S is invariant under the gauge tf.

$$\begin{aligned} (\delta e^{\Phi})e^{-\Phi} &= Q\Lambda + D_\eta\Omega + \{F\Psi, F\Xi\,(\,\{F\Psi, \Lambda\} - \lambda\,)\}\,, \\ \delta\Psi &= Q\lambda + X\eta\, F\Xi\, D_\eta\,(\,\{F\Psi, \Lambda\} - \lambda\,)\,. \end{aligned}$$

4. Supersymmetry [Kunitomo, in priparation]

Perturbative construction

At the linearized level

$$\delta_{\mathcal{Q}}^{(0)}\Phi \;=\; \mathcal{Q}\Xi\Psi, \qquad \delta_{\mathcal{Q}}^{(0)}\Psi \;=\; X\mathcal{Q}\eta\Phi\,,$$

where

$$\mathcal{Q} \;=\; \epsilon_\alpha q^\alpha\,, \qquad q^\alpha \;=\; \oint \frac{dz}{2\pi i}\,\sigma^\alpha(z)e^{-\frac{\phi}{2}}(z)\,.$$

Requiring the invariance, we find

$$\delta_{\mathcal{Q}}^{(1)}\Phi \;=\frac{1}{2}[\Phi, \mathcal{Q}\Xi\Psi] - \mathcal{Q}\Xi[\Phi,\Psi] + \{\Psi, \Xi\mathcal{Q}\Phi\}\,,$$

$$\delta_{\mathcal{Q}}^{(1)}\Psi \;=\; -\frac{1}{2}X\eta[\Phi, \mathcal{Q}\Phi] + X\eta[\Phi, \Xi\mathcal{Q}\eta\Phi]\,.$$

$$
\begin{aligned}
\delta_{\mathcal{Q}}^{(2)}\Phi &= \frac{1}{12}[\Phi,[\Phi,\mathcal{Q}\Xi\Psi]] - [\Xi[\Phi,\Psi],\Xi\mathcal{Q}\eta\Phi] + \frac{1}{2}[\Xi[\Phi,\Psi],\mathcal{Q}\Phi] \\
&+ \frac{1}{2}\{[\Phi,\Psi],\Xi\mathcal{Q}\Phi\} + \frac{1}{2}\{\Psi,\Xi\{\eta\Phi,\Xi\mathcal{Q}\Phi\}\} + \frac{1}{2}\{\Psi,\Xi[\Phi,\Xi\mathcal{Q}\eta\Phi]\} \\
&- \frac{1}{2}\mathcal{Q}\Xi[\Phi,\Xi\{\eta\Phi,\Psi\}] - \frac{1}{2}\mathcal{Q}\Xi[\eta\Phi,\Xi[\Phi,\Psi]]\,, \\
\delta_{\mathcal{Q}}^{(2)}\Psi &= \frac{1}{6}X\eta[\Phi,[\Phi,\mathcal{Q}\Phi]] + \frac{1}{2}X\eta[\Phi,\Xi[\mathcal{Q}\Phi,\eta\Phi]] + \frac{1}{2}X\eta\{\eta\Phi,\Xi[\Phi,\Xi\mathcal{Q}\eta\Phi]\} \\
&+ \frac{1}{2}X\eta[\Phi,\Xi[\eta\Phi,\Xi\mathcal{Q}\eta\Phi]]\,,
\end{aligned}
$$

and so on. We can finally find the action is invariant under

$$
\begin{aligned}
A_{\delta_{\mathcal{Q}}} &= g(\mathcal{Q}\Xi(g^{-1}F\Psi g))g^{-1} + \{F\Psi, F\Xi A_{\mathcal{Q}}\}\,, \\
\delta_{\mathcal{Q}}\Psi &= X\eta F\Xi D_\eta A_{\mathcal{Q}}\,,
\end{aligned}
$$

where

$$
A_{\delta_{\mathcal{Q}}} \;=\; (\delta_{\mathcal{Q}}g)g^{-1}, \qquad A_{\mathcal{Q}} \;=\; (\mathcal{Q}g)g^{-1}\,.
$$

We can show that it satisfies the algebra

$$
\begin{aligned}
A_{[\delta_{\mathcal{Q}_1},\delta_{\mathcal{Q}_2}]} &= f\xi\left(QA_{p_{12}} + [F\Psi, F\Xi\left(p_{12}F\Psi + [F\Psi, A_{p_{12}}]\right)]\right) \\
&\quad + Q\Lambda_{12} + \{F\Psi, F\Xi\{F\Psi, \Lambda_{12}\}\} + D_\eta\Omega_{12}\,, \\
[\delta_{\mathcal{Q}_1}, \delta_{\mathcal{Q}_2}]\Psi &= X\eta F\Xi\left(p_{12}F\Psi + [F\Psi, A_{p_{12}}]\right) \\
&\quad + X\eta F\Xi D_\eta\{F\Psi, \Lambda_{12}\}\,,
\end{aligned}
$$

up to EOM, where

$$
p_{12} \;=\; -[\mathcal{Q}_1, \mathcal{Q}_2] \;=\; (\epsilon_1 C\bar{\gamma}_\mu\epsilon_2)\oint\frac{dz}{2\pi i}\psi^\mu(z)e^{-\phi(z)}\,,
$$

and $A_{p_{12}} = (p_{12}g)g^{-1}$. The linear map f acts on the NS string field A as

$$
fA \;=\; \frac{1}{1-\xi(\eta - D_\eta)}A\,.
$$

The gauge parameters Λ_{12} and Ω_{12} are given by

$$\Lambda_{12} = f\xi(\mathcal{Q}_1 F\Xi\mathcal{Q}_2 A_\eta - \mathcal{Q}_2 F\Xi\mathcal{Q}_1 A_\eta - [F\Xi\mathcal{Q}_1 A_\eta, F\Xi\mathcal{Q}_2 A_\eta]) + f\xi A_{p_{12}} \,,$$

$$\begin{aligned}\Omega_{12} &= \delta_{\mathcal{Q}_1}\Omega_{\mathcal{Q}_2} - [A_{\delta_{\mathcal{Q}_1}}, \Omega_{\mathcal{Q}_2}] - \delta_{\mathcal{Q}_2}\Omega_{\mathcal{Q}_1} + [A_{\delta_{\mathcal{Q}_2}}, \Omega_{\mathcal{Q}_1}] + [\Omega_{\mathcal{Q}_1}, D_\eta \Omega_{\mathcal{Q}_2}] \\ &\quad - f\xi\left(Q\Lambda_{12} + \{F\Psi, F\Xi\{F\Psi, \Lambda_{12}\}\}\right) \,,\end{aligned}$$

with

$$\Omega_{\mathcal{Q}} = f\xi\left(g(\mathcal{Q}\Xi(g^{-1}F\Psi g))g^{-1} + \{F\Psi, F\Xi A_{\mathcal{Q}}\}\right) \,,$$

5. Summary and discussion

Summary :

♠ We have constructed complete gauge invariant action, including both the NS and Ramond sectors, for open superstring field theory.

♠ The key points are to utilize the large Hilbert space and to restrict the Ramond string field. The gauge symmetry is compatible with the restriction.

Remaining tasks:

- ★ Confirm the perturbative amplitudes are correctly reproduced.
  For tree level 4- and 5-point amplitudes,
  Kunitomo, Okawa, Sukeno, and Takezaki, in preparation.

- ★ Quantize à la Batalin-Vilkovisky. Compute loop amplitudes.
  A_∞ structure is usefuel. [Erler-Okawa-Takezaki, arXiv:1602.02582]

- ★ Extend the formulation to heterotic string field theory.
  Partially given by [Goto, and Kunitomo, arXiv:1606.07194]

- ★ Clarify the relation to the Sen's formulation.

- ★ Extend the formulation to type II string field theory.
  Study Ramond-Ramond sector, (and also AdS/CFT?).

The susy tf. can be rewritten as

$$\begin{aligned} A_{\delta_{\mathcal{Q}}} &= f\xi(\mathcal{Q}F\Psi + [F\Psi, F\Xi\mathcal{Q}A_\eta]) + D_\eta\Omega_{\mathcal{Q}}\,, \\ \delta_{\mathcal{Q}}\Psi &= X\eta F\Xi\mathcal{Q}A_\eta\,, \end{aligned}$$

with

$$\Omega_{\mathcal{Q}} \;=\; f\xi\left(g(\mathcal{Q}\Xi(g^{-1}F\Psi g))g^{-1} + \{F\Psi, F\Xi A_{\mathcal{Q}}\}\right)\,,$$

up to EOM. [Erler, private communication]