Random volumes from matrices

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2016.2.15 @京都産業大学 益川塾

Based on the work with

Masafumi Fukuma and **Sotaro Sugishita** (Kyoto Univ.)

[arXiv:1503.08812][JHEP 1507 (2015) 088] “Random volumes from matrices”
[arXiv:1504.03532] “Putting matters on the triangle-hinge models”
M. Fukuma, S. Sugishita and N.U. (in preparation / work in progress)

Introduction

String theory : strong candidate for unified theory including QG

M theory : one of the aspects of the string theory

⟺ supermembrane theory in 11D spacetime

⟺ the random walk of a membrane (random volumes)

Random volume theory : 3D QG theory coupled to scalar fields

$$Z = \sum_{\text{topologies}} \int [\mathcal{D}g_{\alpha\beta}][\mathcal{D}X] e^{-S}$$

$$S = \int d^3\sigma \sqrt{g} \left(\Lambda + \kappa R + g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu \right)$$

In this talk, we discuss the discretized approach to random volume theory.

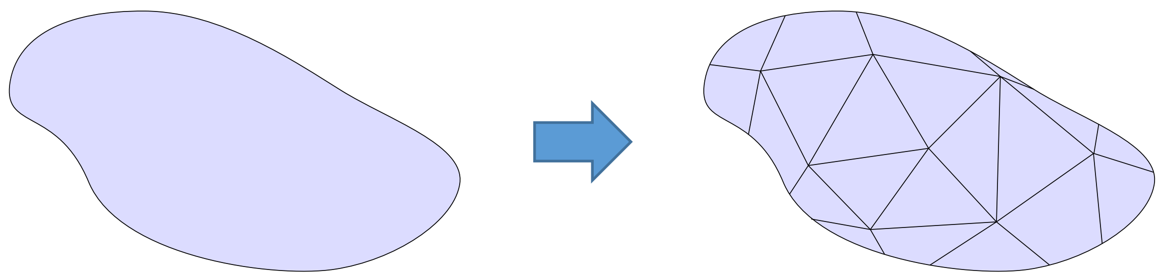
Discretized approach (random surfaces)

Random surface theory : 2D QG theory coupled to scalar fields

$$Z = \sum_{\text{topologies}} \int [\mathcal{D}g_{\alpha\beta}][\mathcal{D}X] e^{-S}$$

$$S = \int d^2\sigma \sqrt{g} \left(\Lambda + \kappa R + g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu \right)$$

We approximate 2D surfaces as triangular decompositions:



2D closed surface triangular decomposition

Then $\log Z$ is defined as the sum of all distinct connected triangular decompositions.

$$\log Z = \sum_{\substack{\text{triangular} \\ \text{decompositions}}} e^{-S}$$

We now consider the model such that the free energy can be realized as the sum of triangular decompositions.

Discretized approach (random surfaces)

This form of free energy can be obtained by the matrix models.

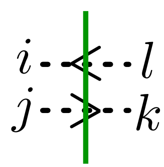
$$S(M) = \frac{1}{2}\mathrm{tr}M^2 - \frac{\lambda}{3}\mathrm{tr}M^3$$
$$= \frac{1}{2}M_{ij}M_{ji} - \frac{\lambda}{3}M_{ij}M_{jk}M_{ki}$$

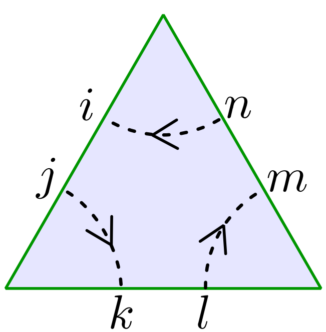
$M = (M_{ij})$: Hermitian matrix

$$\log Z = \sum_{\substack{\text{triangular}\\ \text{decompositions}}} e^{-S}$$

Feynman rules

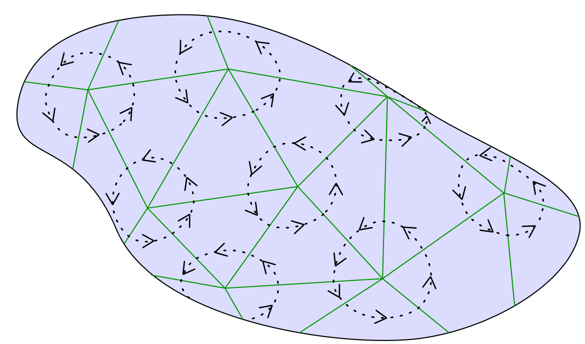
$$\langle M_{ij}M_{kl}\rangle_0 = \delta_{il}\delta_{jk}$$





$$= \lambda\delta_{jk}\delta_{lm}\delta_{ni}$$

Feynman diagram



This model dynamically generates triangular decompositions.

Discretized approach (random surfaces)

$$S(M) = \frac{1}{2}\mathrm{tr}M^2 - \frac{\lambda}{3}\mathrm{tr}M^3$$
$$= \frac{1}{2}M_{ij}M_{ji} - \frac{\lambda}{3}M_{ij}M_{jk}M_{ki}$$

$M = (M_{ij})$: Hermitian matrix

$$S = \frac{1}{2}\mathrm{tr}X^2 - \frac{\lambda}{3}\mathrm{tr}X^3$$
$$= \frac{1}{2}x_i^2 - \frac{\lambda}{3}x_i^3$$

$$X = \mathrm{diag}(x_1, \ldots, x_N)$$

<u>This model can be solved analytically:</u>

diagonalization :

$$M = UXU^{-1} \quad \begin{cases} X = \mathrm{diag}(x_1, \ldots, x_N) \\ U \in U(N) \end{cases}$$

$$(dM) = \left(\prod_i dx_i\right)(dU)\prod_{i<j}(x_i - x_j)^2$$

effective action :

$$Z = \int \left(\prod_i dx_i\right) e^{-S_{\mathrm{eff}}(X)}$$

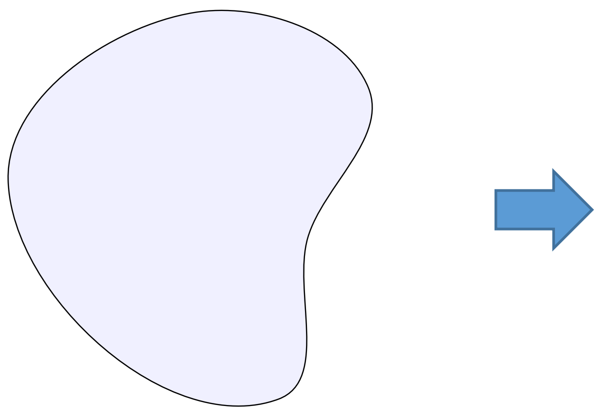
$$S_{\mathrm{eff}}(X) = S(X) - 2\sum_{i<j}\ln|x_i - x_j|$$

Large N analysis can be performed by the saddle point method.

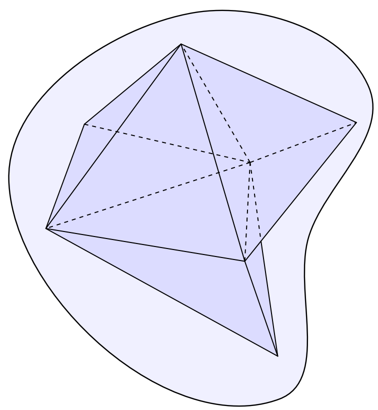
Discretized approach of 3D random volumes

The discretized approach to 3D random volume theory
can be obtained in the similar way as that of 2D random surface theory.

We approximate 3D volumes as tetrahedral decompositions:



closed 3D volume



tetrahedral decomposition

Then $\log Z$ is defined as the sum of
all distinct connected tetrahedral decompositions
of 3D manifolds.

$$\log Z = \sum_{\substack{\text{tetrahedral} \\ \text{decompositions}}} e^{-S}$$

One approach (tensor models)

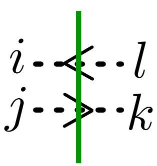
[Ambjorn et al.-Sasakura 1991]

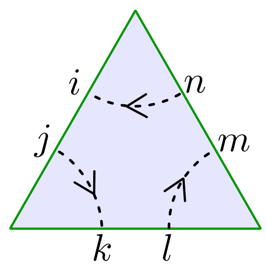
Matrix models

$$S(M) = \frac{1}{2}\mathrm{tr}M^2 - \frac{\lambda}{3}\mathrm{tr}M^3$$
$$= \frac{1}{2}M_{ij}M_{ji} - \frac{\lambda}{3}M_{ij}M_{jk}M_{ki}$$

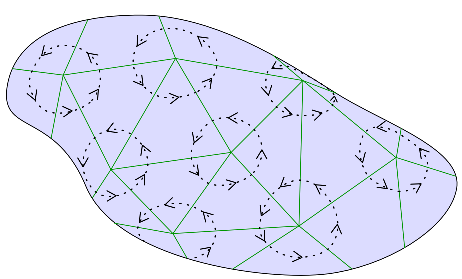
$M = (M_{ij})$: Hermitian matrix

$$\langle M_{ij}M_{kl}\rangle_0 = \delta_{il}\delta_{jk}$$





Feynman diagrams



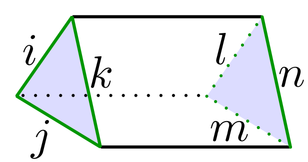
triangular decompositions

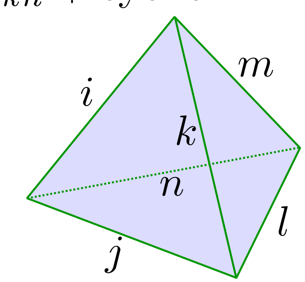
Tensor models

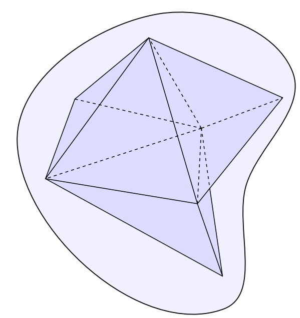
$$S(T) = \frac{1}{2}T_{ijk}T_{ijk}$$
$$- \frac{\lambda}{4}T_{ijk}T_{kml}T_{mjn}T_{lni}$$

$T = (T_{ijk})$: tensor

$$\langle T_{ijk}T_{lmn}\rangle_0 = \delta_{il}\delta_{jm}\delta_{kn} + \text{cyclic}$$







tetrahedral decompositions

One approach (tensor models)

Matrix models

Analytic property

diagonalization

$$M = UXU^{-1} \quad \begin{cases} X = \mathrm{diag}(x_1, \ldots, x_N) \\ U \in U(N) \end{cases}$$

$$(dM) = \left(\prod_i dx_i\right)(dU)\prod_{i<j}(x_i - x_j)^2$$

effective action

$$Z = \int \left(\prod_i dx_i\right) e^{-S_{\mathrm{eff}}(X)}$$

$$S_{\mathrm{eff}}(X) = S(X) - 2\sum_{i<j}\ln|x_i - x_j|$$

→ large N analysis can be performed by the saddle point method

Tensor models

Analytic property

no such analogues

→ **We propose a new class of "matrix models" which generate 3D random volumes.**

Plan of talk

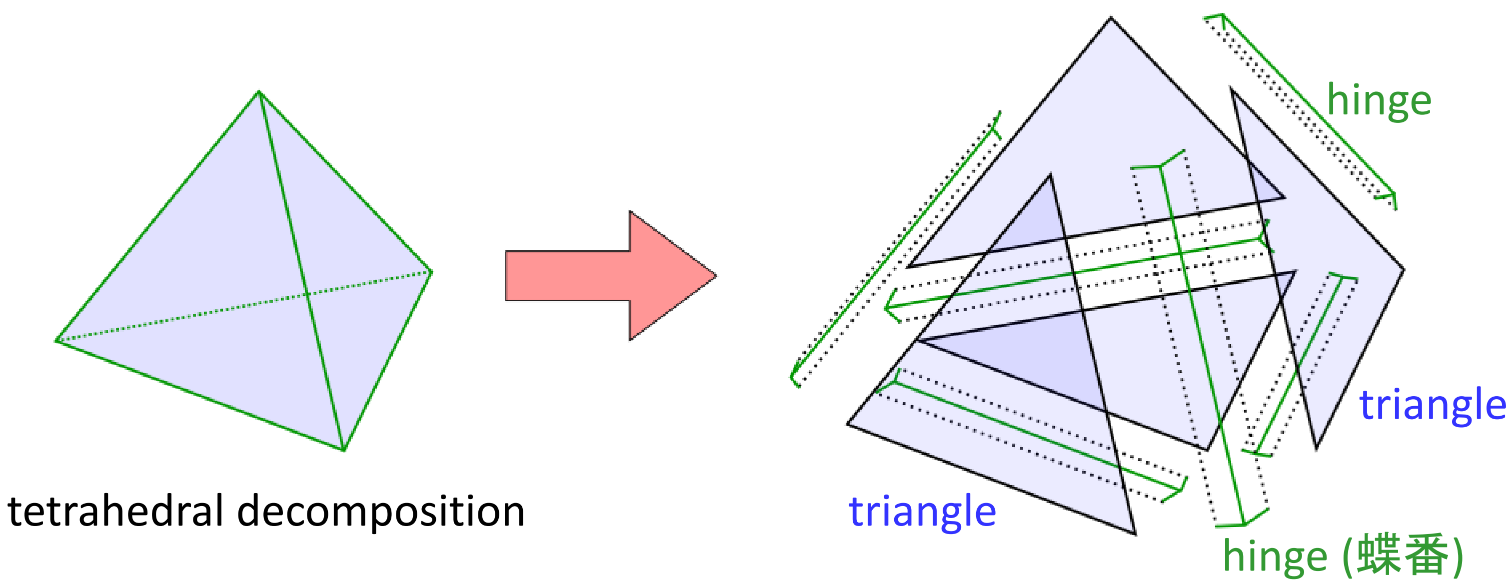
1. Introduction
2. The models (triangle-hinge models)
   - Model definition
   - Algebraic construction
3. General forms of the free energy
4. Matrix ring
5. Restricting to manifolds
6. Assigning matter degrees of freedom

Main idea

Using <u>triangles</u> (instead of tetrahedra) as building blocks.

That is, we decompose a tetrahedral decomposition to
a collection of triangles glued together along multiple hinges.

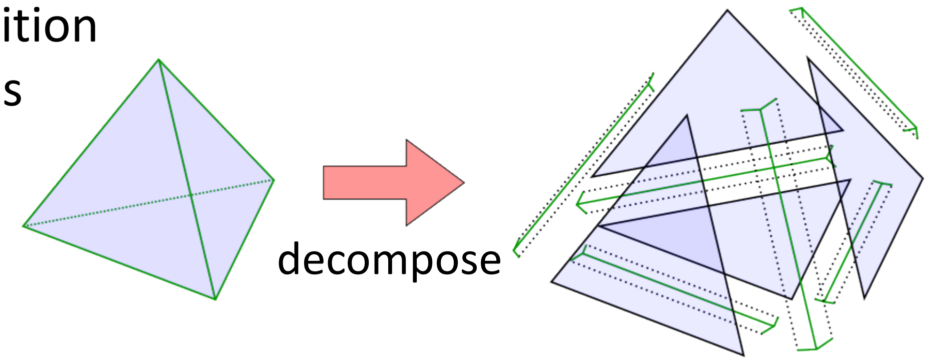
[cf: Chung-Fukuma-Shapere 1993]



Strategy

We construct tetrahedral decompositions in the following steps:

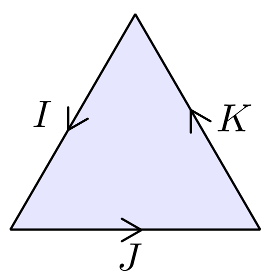
(1) Decompose a tetrahedral decomposition to a collection of triangles and hinges



(2) Assign real numbers C^{IJK} and $Y_{I_1 \ldots I_k}$:

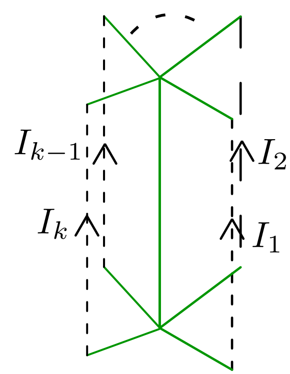
<u>triangles</u>

$$C^{IJK} \sim$$



<u>k - hinges</u>

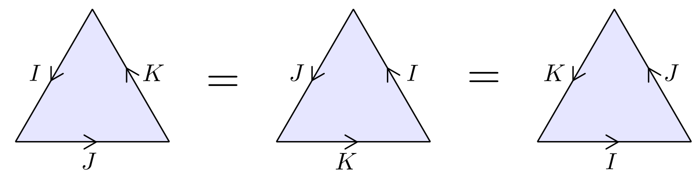
$$Y_{I_1 \ldots I_k} \sim$$



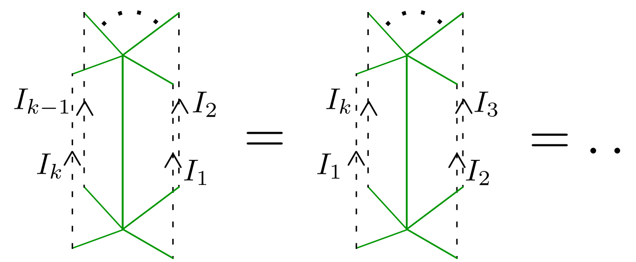
Here, C^{IJK} and $Y_{I_1 \ldots I_k}$ have two symmetries:

The properties of C^{IJK} and $Y_{I_1 \ldots I_k}$

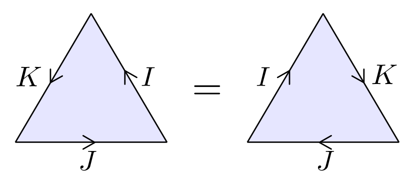
(1) rotation: these are cyclically symmetric.



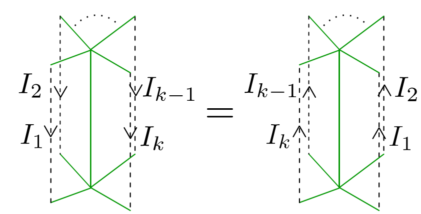
$$\begin{cases} C^{IJK} = C^{JKI} = C^{KIJ} \\ Y_{I_1 \ldots I_k} = Y_{I_2 \ldots I_k I_1} = \ldots \end{cases}$$

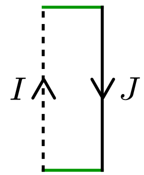


(2) flip: these are invariant under flipping.



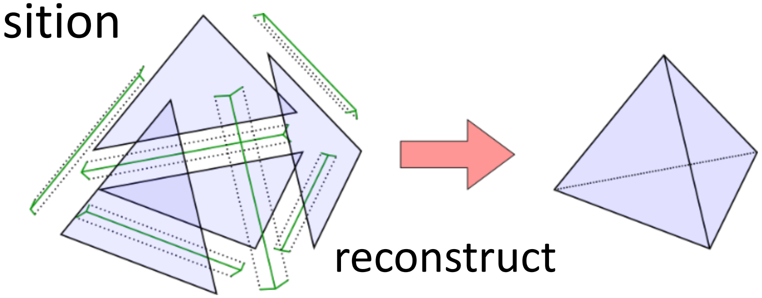
$$\begin{cases} C^{LMN} T_L{}^I T_M{}^J T_N{}^K = C^{KJI} \\ T_{I_1}{}^{J_1} \ldots T_{I_k}{}^{J_k} Y_{J_1 \ldots J_k} = Y_{I_k \ldots I_1} \end{cases}$$

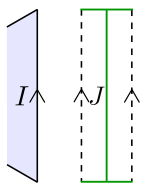


Here, $T_I{}^J \sim$  is the "direction revering tensor", which satisfies $T^2 = 1$.

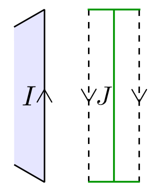
Strategy

(3) Reconstruct the original tetrahedral decomposition by gluing triangles and hinges along edges with the tensors ${\delta_I}^J$ and ${T_I}^J$



${\delta_I}^J \sim$ 

same direction

${T_I}^J \sim$ 

opposite direction

(4) Assign Boltzmann weight $w(\gamma)$ to diagram γ

$$w(\gamma) \equiv \frac{1}{S(\gamma)} \sum_{\{\text{indices}\}} \left[\prod_{f:\ \text{triangle}} C^{IJK}(f) \prod_{h:\ \text{hinge}} Y_{I_1 \ldots I_k}(h) \right]$$

The indices are contracted when two edges are identified

(${T_I}^J$ to be inserted when necessary)

Model definition (strategy)

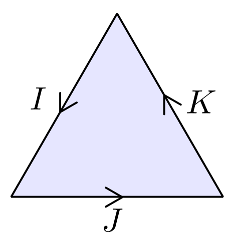
<u>dynamical variables</u>

$$A=(A_I),\ B=(B^I)\quad \text{with}\quad A_I=T_I{}^J A_J,\ B^I=B^J T_J^I$$

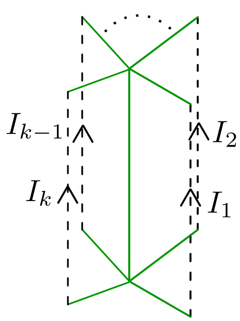
<u>action</u>

$$S(A,B)=\frac{1}{2}A_I B^I-\frac{\lambda}{6}C^{IJK}A_I A_J A_K-\sum_{k\geq 2}\frac{\mu_k}{2k}B^{I_1}\ldots B^{I_k}Y_{I_1\ldots I_k}$$

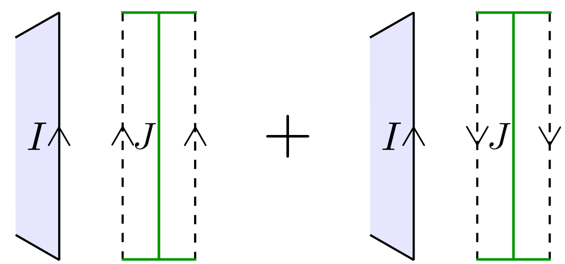
<u>Feynman rules</u>



$$=\lambda C^{IJK}$$



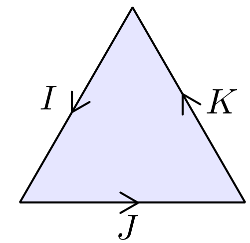
$$=\mu_k Y_{I_1\ldots I_k}$$

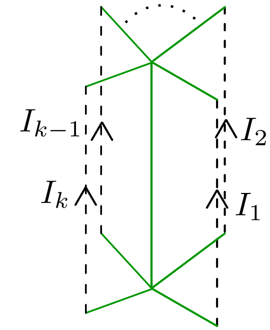


glued: $\langle A_I B^J\rangle_0=\delta_I{}^J+T_I{}^J$

$\frac{1}{2}(\delta_I{}^J+T_I{}^J)$: projection matrix

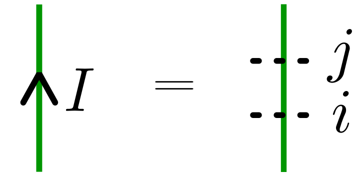
$A,\ B$: invariant under T





Triangles and hinges have orientations.

➡ In order to represent the orientations,
we set the index I to be double index ; $I = (i, j)$.



$$\begin{cases} A_I,\ B^I & \text{: single index} \\ C^{IJK} \\ Y_{I_1 \ldots I_k} \\ T_I{}^J \end{cases}$$

$$S(A,B) = \frac{1}{2} A_I B^I - \frac{\lambda}{6} C^{IJK} A_I A_J A_K - \sum_{k \geq 2} \frac{\mu_k}{2k} B^{I_1} \ldots B^{I_k} Y_{I_1 \ldots I_k}$$

➡

$$\begin{cases} A_{ij},\ B^{ij} & \text{: double index} \\ C^{(ij)(kl)(mn)} \\ Y_{(i_1 j_1)\ldots(i_k j_k)} \\ T_{ij}{}^{kl} = \delta_i{}^l \delta_j{}^k \end{cases}$$

$$S(A,B) = \frac{1}{2} A_{ij} B^{kl} - \frac{\lambda}{6} C^{(ij)(kl)(mn)} A_{ij} A_{kl} A_{mn} - \sum_{k \geq 2} \frac{\mu_k}{2k} B^{i_1 j_1} \ldots B^{i_k j_k} Y_{(i_1 j_1)\ldots(i_k j_k)}$$

Algebraic construction

The building blocks $C^{(ij)(kl)(mn)}$ and $Y_{(i_1 j_1)\ldots(i_k j_k)}$ can be obtained from a semisimple associative algebra $\mathcal{A}$.

linear space with product $" \times "$
satisfying associativity : $(a \times b) \times c = a \times (b \times c)$

In the following, we take a basis $\{e_i\}$ $\left(\mathcal{A} = \bigoplus_i \mathbb{R} e_i\right)$.

The product is expressed as $e_i \times e_j = y_{ij}{}^k e_k$ ($y_{ij}{}^k$: structure constants)

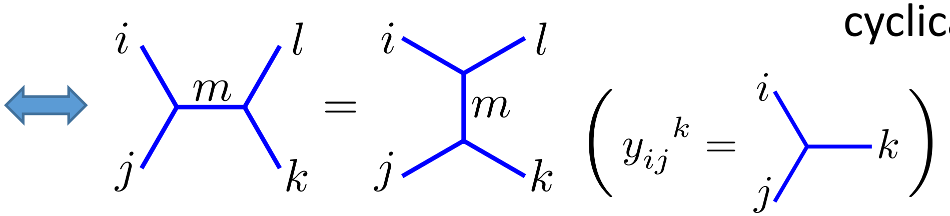
$y_{ij}{}^k$ have two important properties.

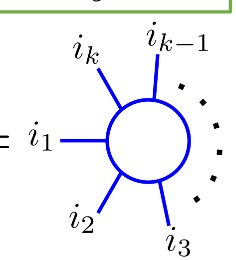
(1) associativity

$$(e_i \times e_j) \times e_k = e_i \times (e_j \times e_k)$$

$$y_{ij}{}^k y_{mk}{}^l = y_{im}{}^l y_{jk}{}^m \quad \Rightarrow \quad y_{i_1 \ldots i_k} \equiv y_{i_1 j_1}{}^{j_k} y_{i_2 j_2}{}^{j_1} \ldots y_{i_k j_k}{}^{j_{k-1}}$$

cyclically symmetric

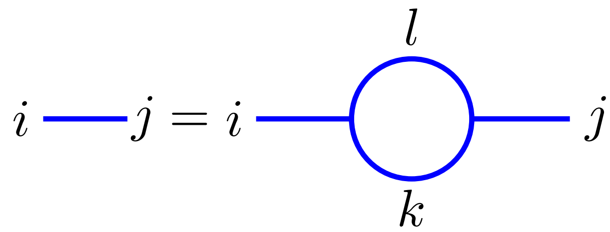


$y_{i_1 i_2 \ldots i_k} =$ 

The models (triangle-hinge models)

(2) metric

$g_{ij} \equiv y_{ik}{}^{l} y_{jl}{}^{k}$ ⟺

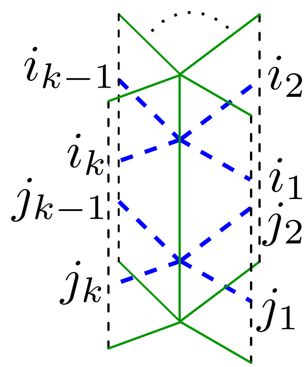


$\mathcal{A}$ is semisimple ⟺ $g = (g_{ij})$ has $g^{-1} = (g^{ij})$ [Fukuma-Hosono-Kawai 1994]

⇒

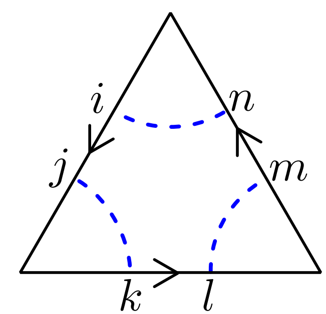
$Y_{(i_1 j_1)\dots(i_k j_k)}$ can be defined as

$$Y_{(i_1 j_1)\dots(i_k j_k)} = y_{i_1 \dots i_k} y_{j_k \dots j_1}$$



And we can choose $C^{(ij)(kl)(mn)}$ as

$$C^{(ij)(kl)(mn)} = g^{jk} g^{lm} g^{ni}$$



These have the symmetries

rotation : $\begin{cases} C^{(ij)(kl)(mn)} = C^{(kl)(mn)(ij)} = C^{(mn)(ij)(kl)} \\ Y_{(i_1 j_1)\dots(i_k j_k)} = Y_{(i_2 j_2)\dots(i_k j_k)(i_1 j_1)} = \dots \end{cases}$

flip : $\begin{cases} C^{(ij)(kl)(mn)} = C^{(nm)(lk)(ji)} \\ Y_{(i_1 j_1)\dots(i_k j_k)} = Y_{(j_k i_k)\dots(j_1 i_1)} \end{cases}$

Model definition (summary)

This model is characterized by an associative algebra $\mathcal{A}$.

<u>dynamical variables</u>

$$A = (A_{ij}),\ \ B = (B^{ij}) \quad \text{: real symmetric matrices}$$

<u>action</u>

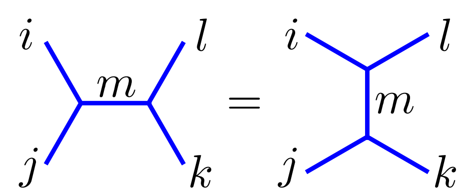
$$S(A,B) = \frac{1}{2} A_{ij} B^{ij} - \frac{\lambda}{6} g^{jk} g^{lm} g^{ni} A_{ij} A_{kl} A_{mn} - \sum_{k\geq 2} \frac{\mu_k}{2k} y_{i_1 \ldots i_k} y_{j_k \ldots j_1} B^{i_1 j_1} \ldots B^{i_k j_k}$$

Here, $y_{i_1 \ldots i_k} \equiv y_{i_1 j_1}{}^{j_k} y_{i_2 j_2}{}^{j_1} \cdots y_{i_k j_k}{}^{j_{k-1}}$ and $g^{-1} = (g^{ij})$ is the inverse of $g = (g_{ij})$.

<u>algebraic properties</u>

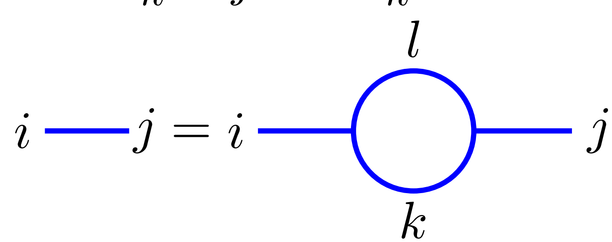
(1) associativity

$$y_{ij}{}^{k} y_{mk}{}^{l} = y_{im}{}^{l} y_{jk}{}^{m} \quad \Leftrightarrow$$



(2) metric

$$g_{ij} \equiv y_{ik}{}^{l} y_{jl}{}^{k} \quad \Leftrightarrow$$



Plan of talk

1. Introduction
2. The models (triangle-hinge models)
3. General forms of the free energy
4. Matrix ring
5. Restricting to manifolds
6. Assigning matter degrees of freedom

Index function and index networks

action

$$S(A,B)=\frac{1}{2}A_{ij}B^{ij}-\frac{\lambda}{6}g^{jk}g^{lm}g^{ni}A_{ij}A_{kl}A_{mn}-\sum_{k\geq 2}\frac{\mu_k}{2k}y_{i_1\ldots i_k}y_{j_k\ldots j_1}B^{i_1j_1}\ldots B^{i_kj_k}$$

The free energy of this model has the following form:

$$\log Z=\sum_{\gamma}\frac{1}{S(\gamma)}\lambda^{s_2(\gamma)}\left(\prod_{k\geq 2}\mu_k^{s_1^k(\gamma)}\right)\mathcal{F}(\gamma)$$

- γ : connected diagram
- $S(\gamma)$: symmetry factor
- $s_2(\gamma)$: # of triangles
- $s_1^k(\gamma)$: # of k -hinges

$\mathcal{F}(\gamma)$: a function of g^{ij} and $y_{i_1\ldots i_k}$ (thus a function of ${y_{ij}}^k$)

"index function of diagram γ "

We claim :

$\mathcal{F}(\gamma)$ is given as the product of 2D topological invariants, which are defined around the vertices of the diagram γ .

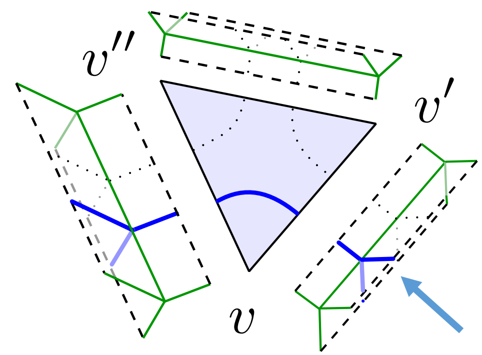
General forms of the free energy

This fact can be shown in two steps:

(1) $\mathcal{F}(\gamma)$ is given as the product of the contributions from vertices :

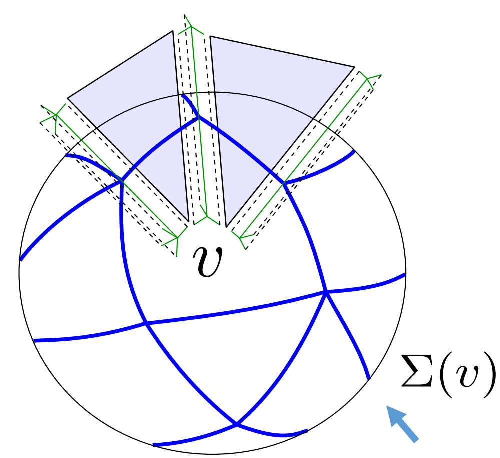
$$\mathcal{F}(\gamma) = \prod_{v:\ \text{vertex of}\ \gamma} \zeta(v)$$

The index lines on two different hinges
are connected (through an intermediate triangle)
if and only if the hinges share the same vertex of γ .



The connected components of the index lines (index network)
have a one-to-one correspondence to the vertices of γ .

$$\mathcal{F}(\gamma) = \prod_{v:\ \text{vertex of}\ \gamma} \zeta(v)$$



Each connected component of the index networks
can be regarded as a <u>closed 2D surface</u> enclosing a vertex.
$\Sigma(v)$ (not necessarily a sphere)

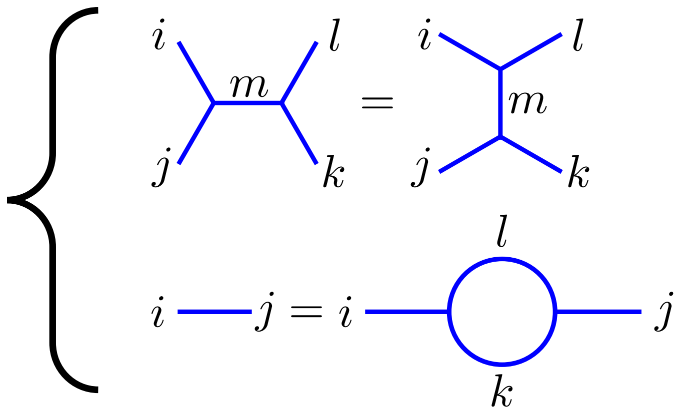
General forms of the free energy

This fact can be shown in two steps:

(2) Each contribution $\zeta(v)$ is a 2D topological invariant of the 2D closed surface enclosing the vertex v :

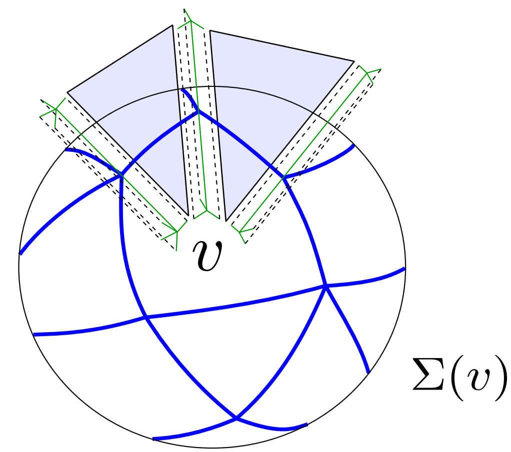
$$\zeta(v) = \mathcal{I}_{g(v)} \quad (g(v)\text{: genus of the 2D surface})$$

The contribution $\zeta(v)$ for the index network around v is invariant under the following deformations:



(associativity)

(definition of metric)



These deformations generate 2D topology-preserving local moves. [Fukuma-Hosono-Kawai 1994]

⇒ $\zeta(v)$ is the 2D topological invariant of $\Sigma(v)$ associated with $\mathcal{A}$.

$$\zeta(v) = \mathcal{I}_{g(v)}[\mathcal{A}] \quad (g(v)\text{: genus of } \Sigma(v) \text{ })$$

Plan of talk

1. Introduction
2. The models (triangle-hinge models)
3. General forms of the free energy
4. Matrix ring
5. Restricting to manifolds
6. Assigning matter degrees of freedom

Matrix ring

In this section, we discuss a simple (and important) example of associative algebra $\mathcal{A}$.

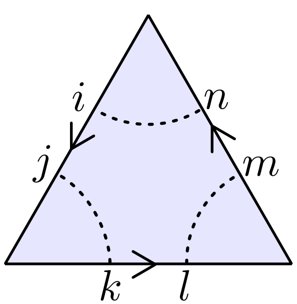
Set $\mathcal{A}$ to be a matrix ring: $\mathcal{A} = M_n(\mathbb{R}) = \bigoplus_{a,b} \mathbb{R} e_{ab} \quad (e_{ab} \times e_{cd} = \delta_{bc} e_{ad})$

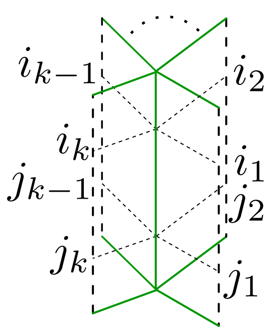
the set of $n \times n$ real matrices — the usual matrix product

Then the variables in the model can be illustrated as the thickened diagrams.

$$\begin{cases} A_{ij}, \ B^{ij} \\ y_{i_1 \ldots i_k} \\ g^{ij} \end{cases}$$

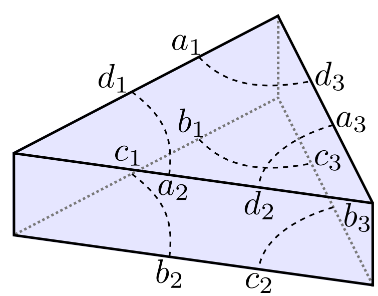
interaction terms

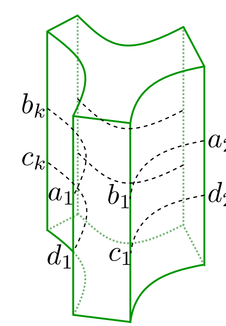




$$\begin{cases} A_{abcd}, \ B^{abcd} \ (A_{abcd} = A_{cdab}, \ B^{abcd} = B^{cdab}) \\ y_{a_1 b_1 \ldots a_k b_k} = n \delta_{b_1 a_2} \delta_{b_2 a_3} \ldots \delta_{b_k a_1} \\ g^{abcd} = \frac{1}{n} \delta^{ad} \delta^{bc} \end{cases}$$

interaction terms





Model definition (matrix ring)

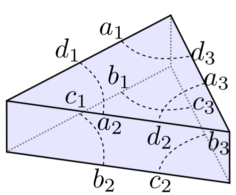
dynamical variables

$$A=(A_{abcd}),\ \ B=(B^{abcd})\ \ \ (A_{abcd}=A_{cdab},\ \ B^{abcd}=B^{cdab})$$

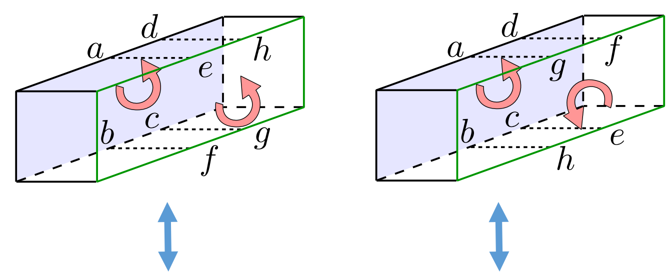
action

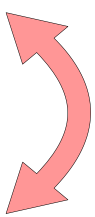
$$S(A,B)=\frac{1}{2}A_{abcd}B^{abcd}-\frac{\lambda}{6n^3}A_{bacd}A_{dcef}A_{feab}-\sum_{k\geq 2}\frac{n^2\mu_k}{2k}B^{a_1a_2b_2b_1}B^{a_2a_3b_3b_2}\dots B^{a_ka_1b_1b_k}$$

Feynman rules

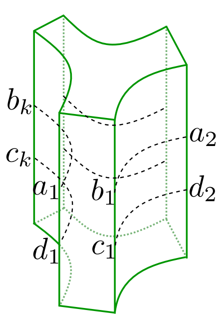


$$=\frac{\lambda}{n^3}\delta^{d_1a_2}\delta^{d_2a_3}\delta^{d_3a_1}\delta^{b_3c_2}\delta^{b_2c_1}\delta^{b_1c_3}$$

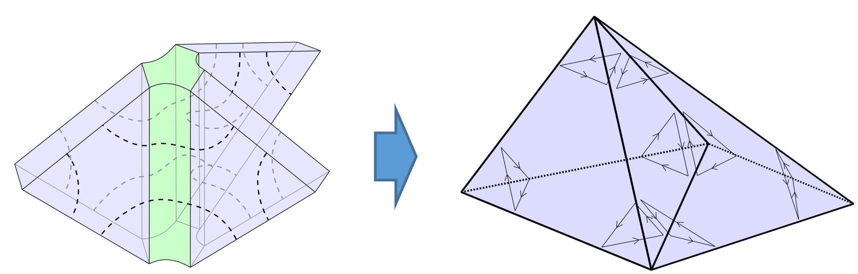




glued: $\langle A_{abcd}B^{efgh}\rangle_0=\delta_a{}^e\delta_b{}^f\delta_c{}^g\delta_d{}^h+\delta_a{}^g\delta_b{}^h\delta_c{}^e\delta_d{}^f$



$$=n^2\mu_k\delta_{b_1a_2}\dots\delta_{b_ka_1}\delta_{d_kc_{k-1}}\dots\delta_{d_1c_k}$$



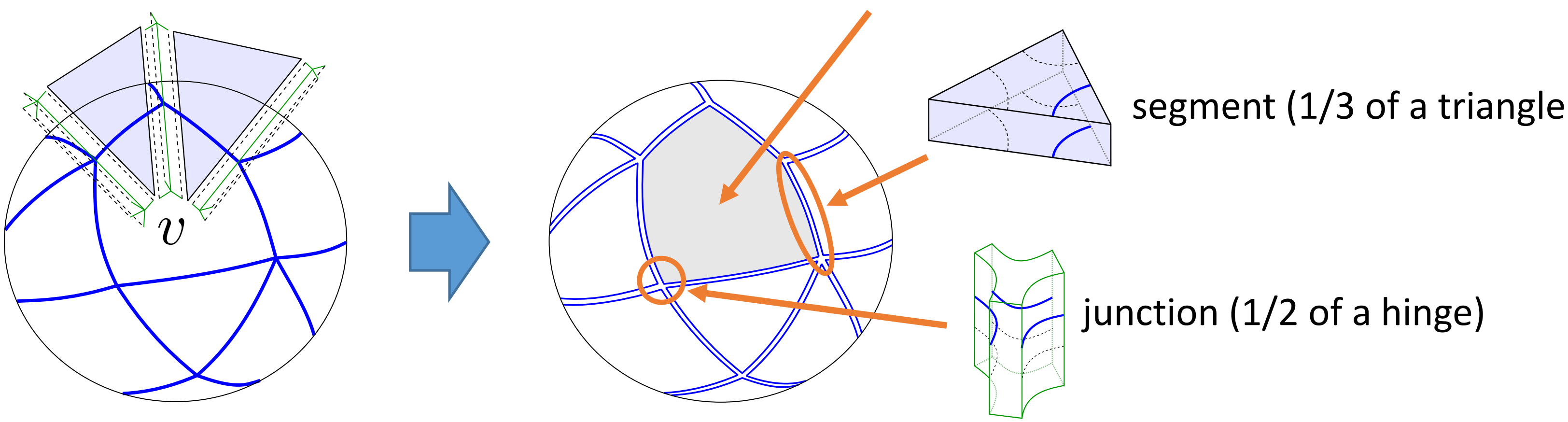
In this case, we can calculate $\zeta(v) = \mathcal{I}_{g(v)}$ exactly.

action

$$S(A,B) = \frac{1}{2} A_{abcd} B^{abcd} - \frac{\lambda}{6n^3} A_{bacd} A_{dcef} A_{feab} - \sum_{k \geq 2} \frac{n^2 \mu_k}{2k} B^{a_1 a_2 b_2 b_1} B^{a_2 a_3 b_3 b_2} \ldots B^{a_k a_1 b_1 b_k}$$

The n dependence appears in three ways (from index loops, triangles and hinges).

- Each index network becomes a collection of index loops, and each index loop gives the factor n ⇒ $n^{\#\{\text{polygon}\}}$



- Each triangle gives the factor n^{-3} and each triangle gives three segments ⇒ $n^{-\#\{\text{segment}\}}$
- Each hinge has the factor n^2 and each hinge gives two junctions ⇒ $n^{\#\{\text{junction}\}}$

⇒ $$\zeta(v) = n^{\#\{\text{polygon}\} - \#\{\text{segment}\} + \#\{\text{junction}\}} = n^{2-2g(v)}$$

Plan of talk

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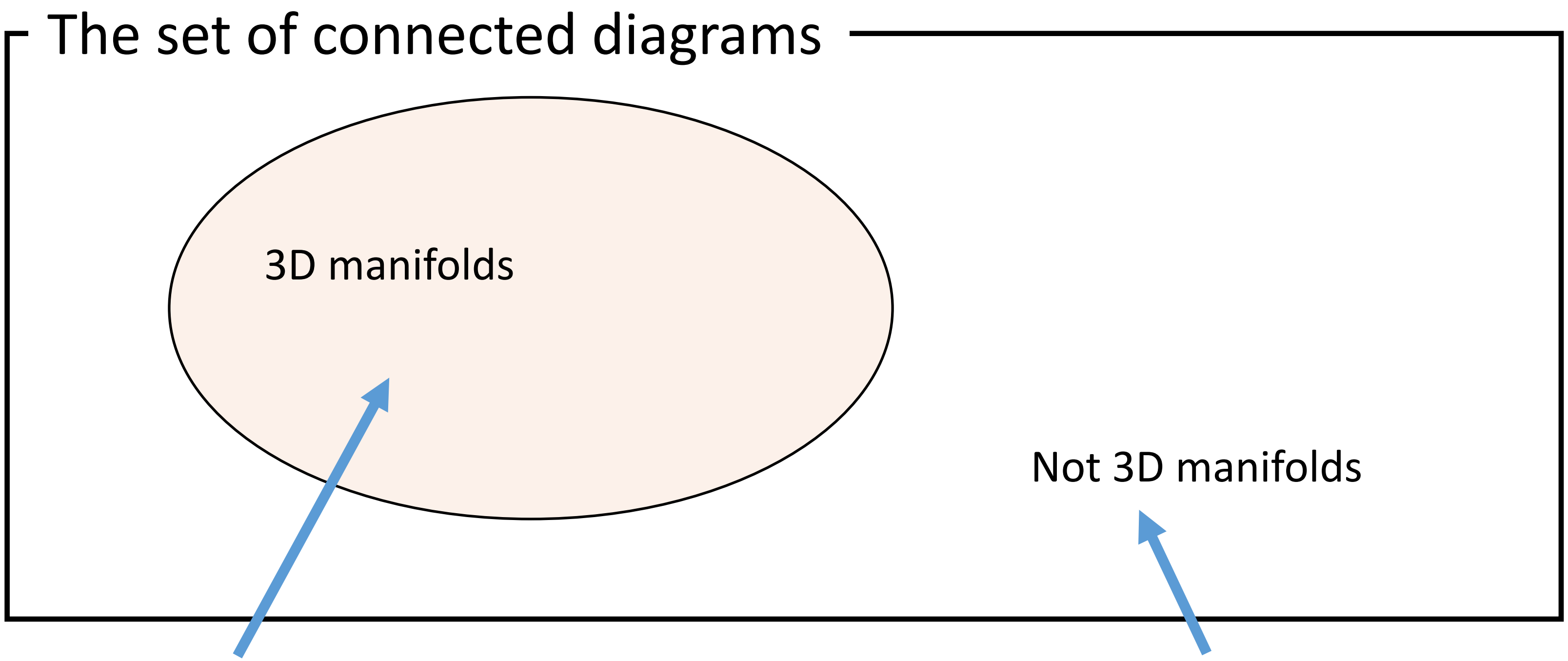
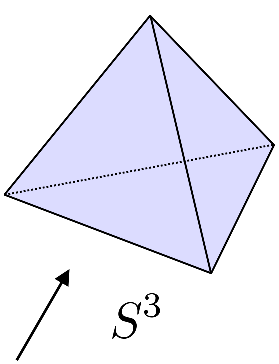
Feynman diagrams



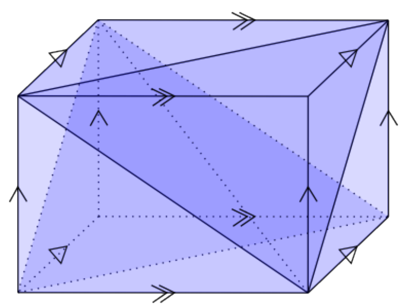
Some diagrams represent 3D manifolds:

But some do ***not***.

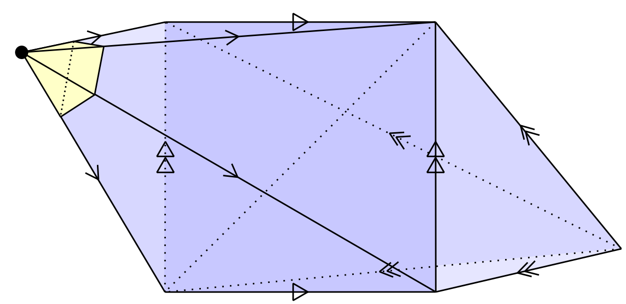


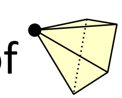
S^3

(Two tetrahedra glued at four faces)

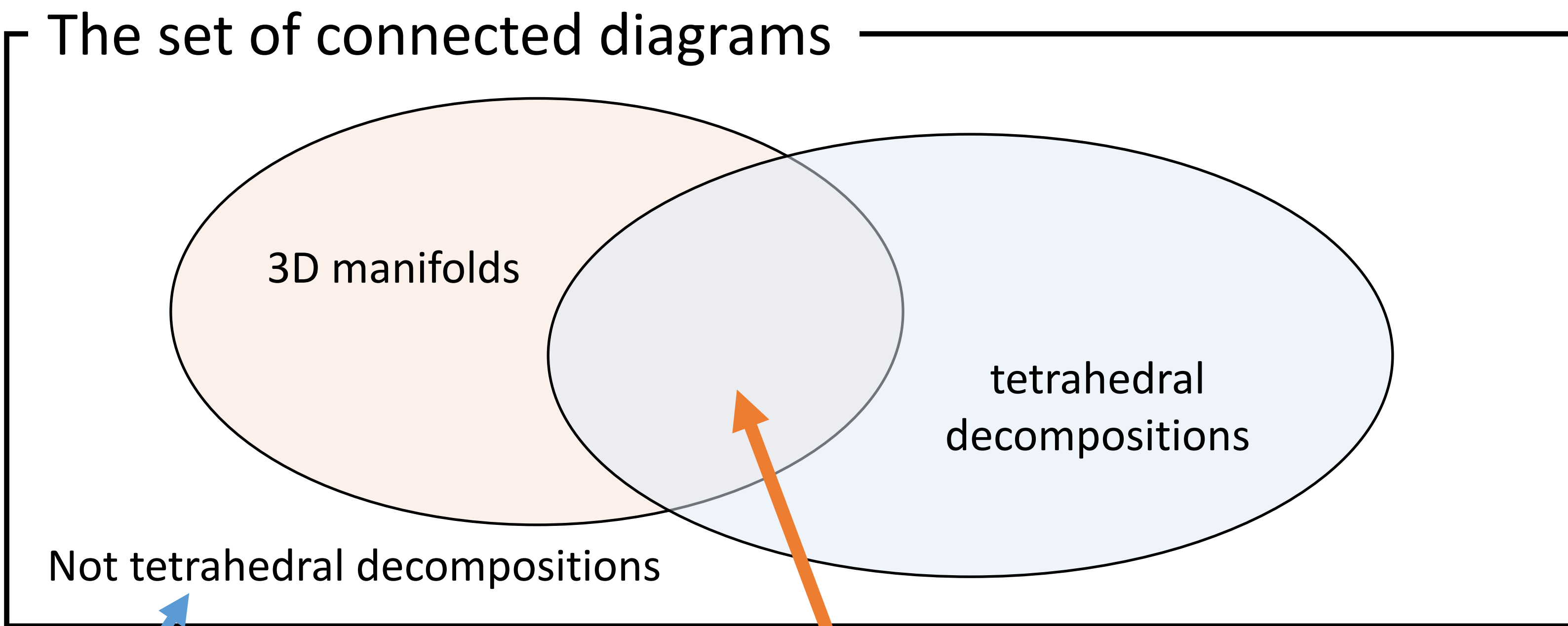


T^3

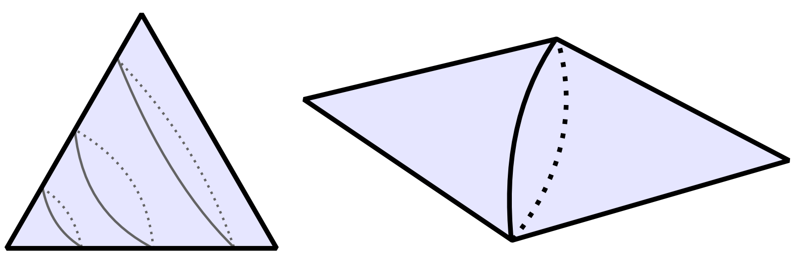


boundary of  : T^2

Feynman diagrams



There are diagrams which do ***not*** represent tetrahedral decompositions.



The free energy of 3D gravity:

$$\sum_{\substack{\text{tetrahedral}\\\text{decompositions}}} e^{-S}$$

of <u>3D manifold</u>

We restrict diagrams to "<u>tetrahedral decompositions of 3D manifold</u>".

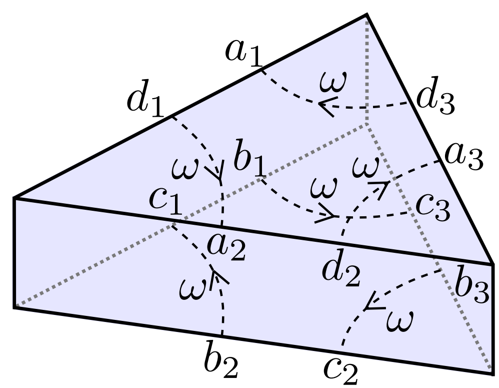
How to restrict diagrams

Again we set $\mathcal{A}$ to be a matrix ring: $\mathcal{A} = M_{3m}(\mathbb{R}) \quad (n = 3m)$

and change the form of C :

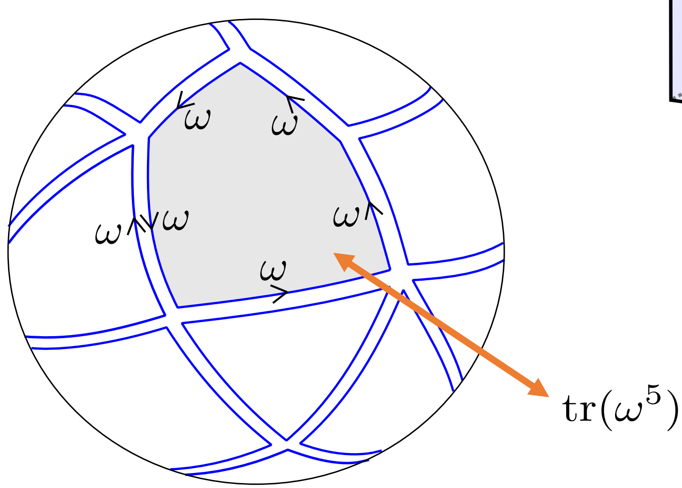
$$C^{(a_1 b_1 c_1 d_1)(a_2 b_2 c_2 d_2)(a_3 b_3 c_3 d_3)} = \frac{1}{n^3} \delta^{d_1 a_2} \delta^{d_2 a_3} \delta^{d_3 a_1} \delta^{b_3 c_2} \delta^{b_2 c_1} \delta^{b_1 c_3}$$

$$C^{(a_1 b_1 c_1 d_1)(a_2 b_2 c_2 d_2)(a_3 b_3 c_3 d_3)} = \frac{1}{n^3} \omega^{d_1 a_2} \omega^{d_2 a_3} \omega^{d_3 a_1} \omega^{b_3 c_2} \omega^{b_2 c_1} \omega^{b_1 c_3}$$



$$\omega = \begin{pmatrix} 0 & 1_m & 0 \\ 0 & 0 & 1_m \\ 1_m & 0 & 0 \end{pmatrix}$$

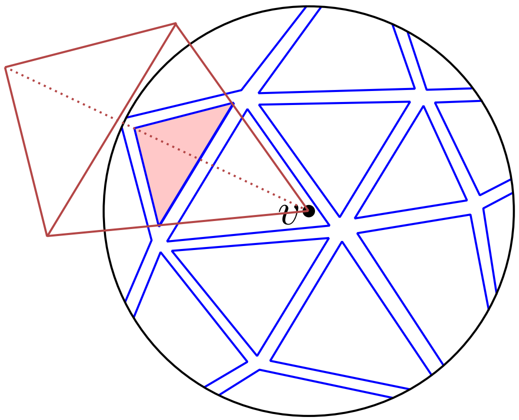
This changes the triangle term:



Every index loop becomes the trace of ω .

Restricting to manifolds

$$\omega = \begin{pmatrix} 0 & 1_m & 0 \\ 0 & 0 & 1_m \\ 1_m & 0 & 0 \end{pmatrix}$$ is the shift matrix.



⇒ $$\mathrm{tr}(\omega^l) = \begin{cases} 3m \ (l = 0 \mod 3) \\ 0 \quad (l \neq 0 \mod 3) \end{cases}$$

If we take the limit $\begin{cases} n \to \infty \ (n = 3m) \\ \frac{n}{\lambda}, \ n^2 \mu_k : \ \text{fixed} \end{cases}$

⇒ The dominant contributions : the diagrams with $l = 3$ for every index loop $(*)$

(to be proved in the next slide)

⇓

Each index loop represents a corner of a tetrahedron.

⇓

These diagrams represent tetrahedral decompositions.

Restricting to manifolds

Proof of $(*)$

We recall $l = 3\bar{l}$ and define the following numbers.

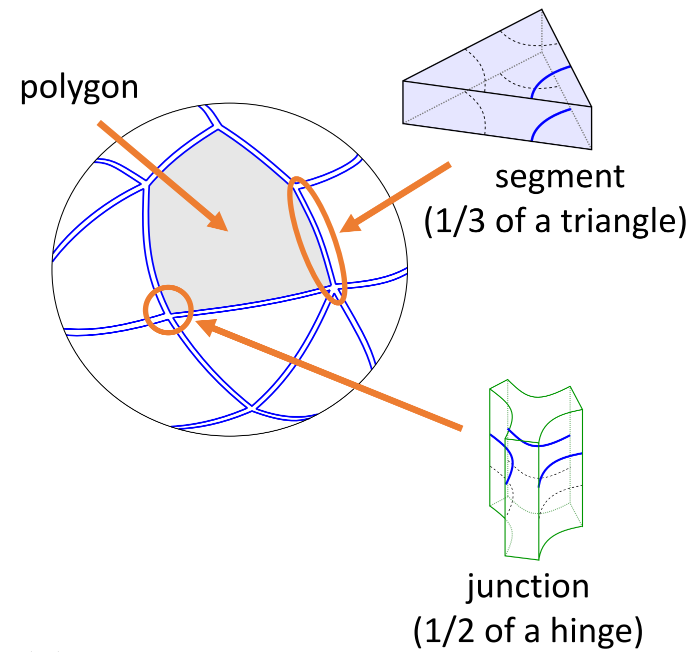
$$\begin{cases} s_2(\gamma) & : \text{\# of triangles} \\ s_1^k(\gamma) & : \text{\# of } k \text{ - hinges} \\ s_0(\gamma) & : \text{\# of vertices} \end{cases} \qquad \begin{cases} t_2^l(v) & : \text{\# of } l \text{ - gons} \\ t_1(v) & : \text{\# of segments} \\ t_0^k(v) & : \text{\# of } k \text{ - junctions} \end{cases}$$

⇒ These satisfy $\sum_v t_1(v) = 3s_2, \ \sum_v t_0^k(v) = 2s_1^k, \ \sum_{l\geq 3} l t_2^l(v) = 2t_1(v)$

The Boltzmann weight is expressed as

$$w(\gamma) = \frac{1}{S(\gamma)} \lambda^{s_2} \prod_{k\geq 2} \mu_k^{s_1^k} \prod_v n^{2-2g(v)}$$

$$= \frac{1}{S(\gamma)} \prod_v \left[\left(\prod_{k\geq 2} (\lambda^2 \mu_k)^{\frac{1}{2} t_0^k(v)} \right) \left(\frac{n}{\lambda}\right)^{2-2g(v)} \left(\frac{1}{\lambda}\right)^{\frac{1}{3} d(v)} \right]$$



Here, $d(v) = 2t_1(v) - 3\sum_{l\geq 3} t_2^l(v) = \sum_{l\geq 3} (l-3) t_2^l(v) \geq 0$

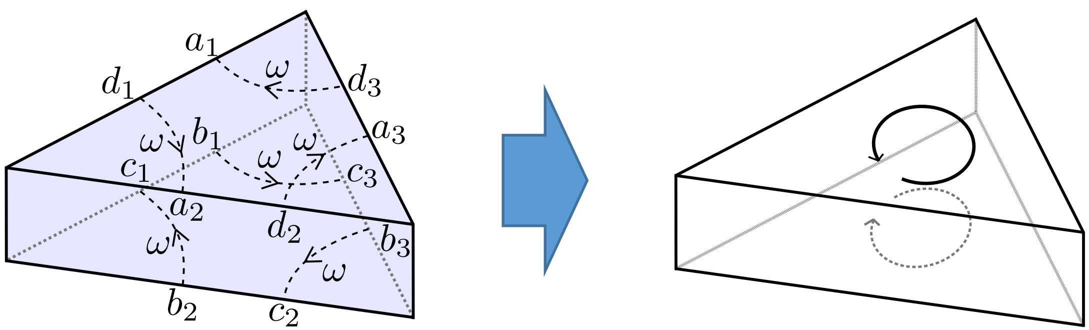
If we take the limit $\begin{cases} n \to \infty \ (n = 3m) \\ \frac{n}{\lambda}, \ n^2 \mu_k : \ \text{fixed} \end{cases}$

⇒ The dominant contributions : the diagrams with $d(v) = 0$

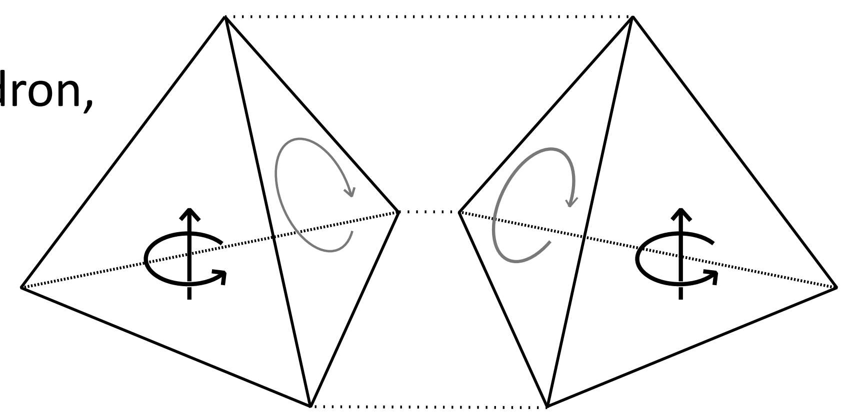
⇔ the diagrams with $l = 3$ for every index loop

Triangle-hinge models generates 3D oriented tetrahedral decompositions.

Two triangles (upper and lower side of a thickened triangle) always have opposite orientations.



If we define local orientation to each tetrahedron, two tetrahedra glued at their faces always have the same orientation.



The Boltzmann weight of the diagram γ is expressed as

$$w(\gamma) = \frac{1}{S(\gamma)} \prod_v \left[\left(\prod_{k\geq 2} (\lambda^2 \mu_k)^{\frac{1}{2} t_0^k(v)} \right) \left(\frac{n}{\lambda}\right)^{2-2g(v)} \left(\frac{1}{\lambda}\right)^{\frac{1}{3}d(v)} \right]$$

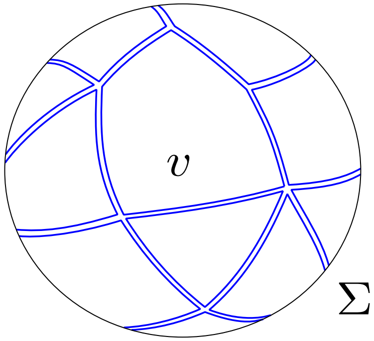
If we further take the limit $\frac{n}{\lambda} \to \infty$,
the dominant contributions: the diagrams with $g(v) = 0 \text{ for } {}^\forall v$

The genus of $\Sigma(v)$

Each vertex of these diagrams has a neighborhood of S^2 .

This neighborhood is homeomorphic to B^3.

These diagrams represent 3D manifolds.



We can single out tetrahedral decompositions of orientable 3D manifolds by taking the large n limit of the parameters.

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Assigning matter degrees of freedom

We have discussed 3D “pure gravity”.

In order to assign matter degres of freedom, we set $\mathcal{A}$ to be a tensor product of algebras :

$$\begin{cases} \mathcal{A} = \mathcal{A}_{\text{grav}} \otimes \mathcal{A}_{\text{matt}} \\ C = C_{\text{grav}} C_{\text{matt}} \ \text{(factorized)} \end{cases}$$

$$\mathcal{A}_{\text{grav}} = M_{3m}(\mathbb{R})$$

$$C_{\text{grav}}^{(a_1 b_1 c_1 d_1)(a_2 b_2 c_2 d_2)(a_3 b_3 c_3 d_3)} = \frac{1}{n^3} \omega^{d_1 a_2} \omega^{d_2 a_3} \omega^{d_3 a_1} \omega^{b_3 c_2} \omega^{b_2 c_1} \omega^{b_1 c_3}$$

The index function $\mathcal{F}(\gamma) = \mathcal{F}(\gamma; \mathcal{A})$ also factorizes :

$$\mathcal{F}(\gamma; \mathcal{A}) = \mathcal{F}(\gamma; \mathcal{A}_{\text{grav}}) \mathcal{F}(\gamma; \mathcal{A}_{\text{matt}})$$

- $\mathcal{A}_{\text{grav}}$ restricts the diagrams to tetrahedral decompositions of 3D manifold
- $\mathcal{A}_{\text{matt}}$ can represent matter degrees of freedom

In the following, we only discuss $\mathcal{A}_{\text{matt}}$ (and omit the index of $\mathcal{A}_{\text{grav}}$)

Assigning matter degrees of freedom

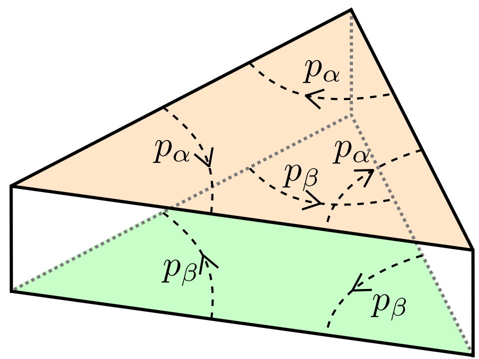
Set $\mathcal{A}_{\rm matt}$ also to be a matrix ring : $\mathcal{A}_{\rm matt} = M_q(\mathbb{R}) = \bigoplus_{1\le\alpha,\beta\le q} \mathbb{R}e_{\alpha\beta}$

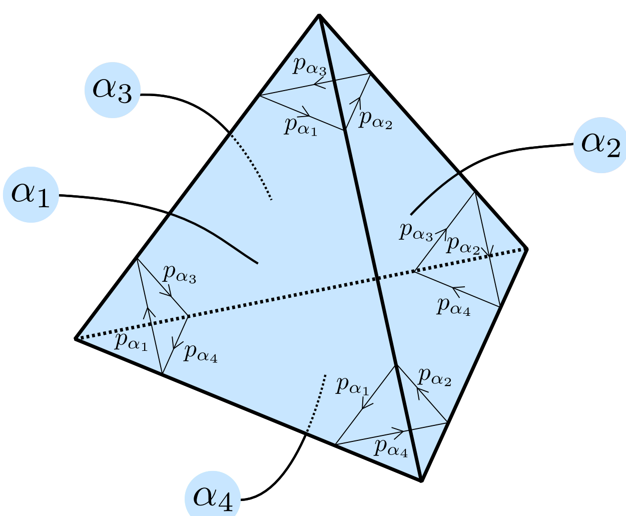
and choose C to have the form :

$$\lambda C^{(\alpha_1\beta_1\gamma_1\delta_1)(\alpha_2\beta_2\gamma_2\delta_2)(\alpha_3\beta_3\gamma_3\delta_3)} = \sum_{\alpha,\beta=1}^{q} \lambda_{\alpha\beta}\, p_\alpha^{\delta_1\alpha_2} p_\alpha^{\delta_2\alpha_3} p_\alpha^{\delta_3\alpha_1} p_\beta^{\beta_3\gamma_2} p_\beta^{\beta_2\gamma_1} p_\beta^{\beta_1\gamma_3}$$

$(p_\alpha)^{\beta\gamma} = \delta_\alpha{}^\beta \delta_\alpha{}^\gamma$: projection matrix

This changes the triangle term:





The four index loops in each tetrahedron gives the factor

$$\mathrm{tr}(p_{\alpha_1}p_{\alpha_2}p_{\alpha_3})\mathrm{tr}(p_{\alpha_2}p_{\alpha_1}p_{\alpha_4})\mathrm{tr}(p_{\alpha_1}p_{\alpha_3}p_{\alpha_4})\mathrm{tr}(p_{\alpha_3}p_{\alpha_2}p_{\alpha_4})$$

$$= \begin{cases} 1 \ (\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4) \\ 0 \ (\text{otherwise}) \end{cases}$$

➡ Each tetrahedron has a single color.

Assigning matter degrees of freedom

Interaction term ($\mathcal{A}_{\mathrm{matt}}$ part) : $\lambda C^{(\alpha_1\beta_1\gamma_1\delta_1)(\alpha_2\beta_2\gamma_2\delta_2)(\alpha_3\beta_3\gamma_3\delta_3)} = \sum_{\alpha,\beta=1}^{q} \lambda_{\alpha\beta} p_\alpha^{\delta_1\alpha_2} p_\alpha^{\delta_2\alpha_3} p_\alpha^{\delta_3\alpha_1} p_\beta^{\beta_3\gamma_2} p_\beta^{\beta_2\gamma_1} p_\beta^{\beta_1\gamma_3}$

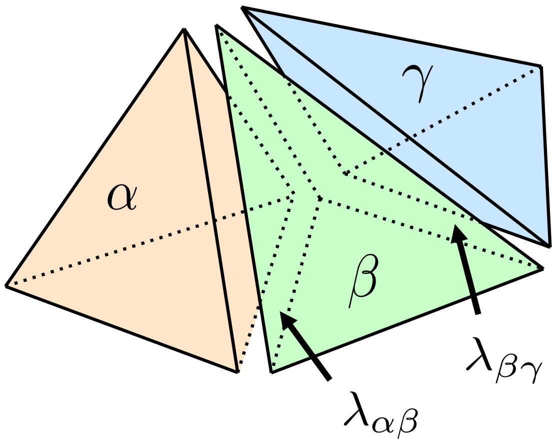
Feynman rules

(1) Assign a color $\alpha = 1, \ldots, q$ to each tetrahedron

(2) If two adjacent tetrahedra have colors α and β, multiply $\lambda_{\alpha\beta}$ ($\mathcal{A}_{\mathrm{matt}}$ part)

(3) Sum over all distinct tetrahedral decompositions of 3D manifold ($\mathcal{A}_{\mathrm{grav}}$ part)

This is the q -state spin system on 3D random volumes.



Example : $q = 2$ ⟺ The Ising model coupled to 3D QG

Comment

In the similar way, we can put local spin systems on simplices of arbitrary dimensions.

(tetrahedra, triangles, edges and vertices)

Summary

- We proposed a new class of matrix models which generate 3D random volumes.
- The models are characterized by semisimple associative algebras $\mathcal{A}$.
- Although most of the Feynman diagrams do not represent manifolds,
  we can reduce the possible diagrams
  to those representing <u>tetrahedral decompositions of 3D manifolds</u>.
- We can assign matter degrees of freedom on simplices of any dimensions.

Future directions

- Further developing the analytic treatment
- Inventing a machinery to restrict the diagrams to a particular topology
- Assigning matter degrees of freedom corresponding to
  the target space coordinates X^{μ} and investigating the critical behaviors