新しいラージN極限とインスタントン

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畔柳さん、花田さん、本多さん、松尾さんとの共同研究に基づく。arXiv:1307.0809 [hep-th] (accepted by JHEP)

From Strings to M !!

D-branes M-branes

- BLG theory & ABJM theory for M2-branes (2007, 2008)
- 5d/6d equivalence for M5-branes (2010)

Large-N limit?

4d SU(N) super Yang-Mills theory

- In usual, it means ‘t Hooft limit:

$$N \to \infty\,, \quad \lambda = g_{YM}^2 N\ :\ \text{fixed}$$

- 1/N expansion = genus expansion

(related to string theory!)

$$F = \sum_{g=0}^{\infty} F_g(\lambda) N^{2-2g}$$

- The perturbative series (of λ) may have a finite radius of convergence at large-N.

→ Analytic continuation to strong ‘t Hooft coupling?

(cf. AdS/CFT correspondence in string theory. M-theory?)

- Various nice properties (factorization, integrability, etc…)

Super Yang-Mills theory

- Action

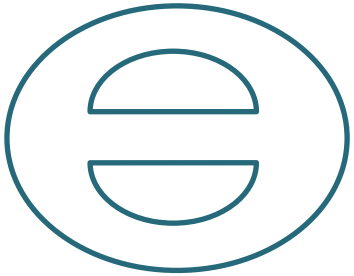
SU(N)

$$S_{YM} = \frac{N}{\lambda} \int d^4x \, \overset{\downarrow}{\mathrm{Tr}} \left[\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} (D_\mu \phi)^2 + \ldots \right]$$

- N-dependence of amplitudes

- Planar diagram (genus 0)

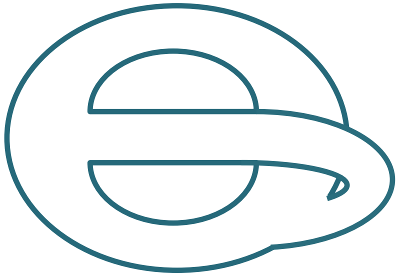
dominant!



N^{-3} × N^2 × N^3 = N^2

(propagators) (vertices) (index loops)

- Non-planar diagram (genus 1)

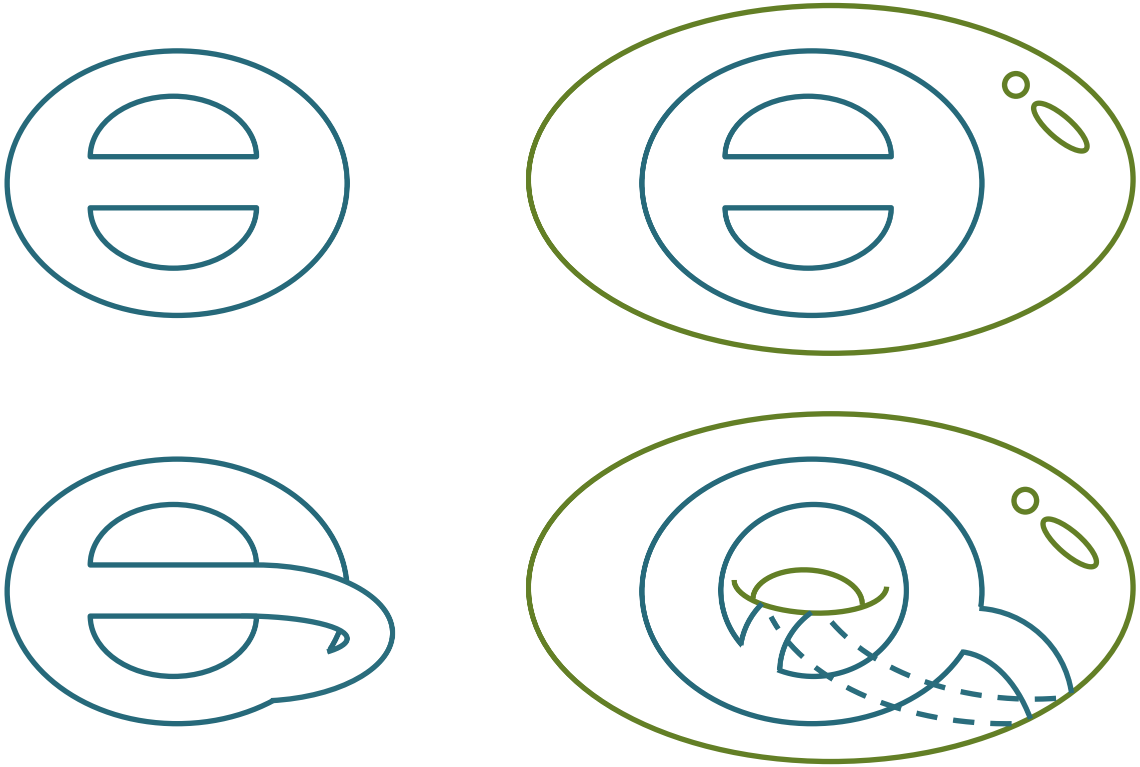


N^{-3} × N^2 × N^1 = N^0

(propagators) (vertices) (index loops)

✓ This is why the large-N limit is called "the planar limit".

Topology



sphere
(genus 0)

torus
(genus 1)

- More generally,
  – (# propagators) + (# vertices) + (# index loops)
  = 2-2(# genus) : It is known as Euler number.

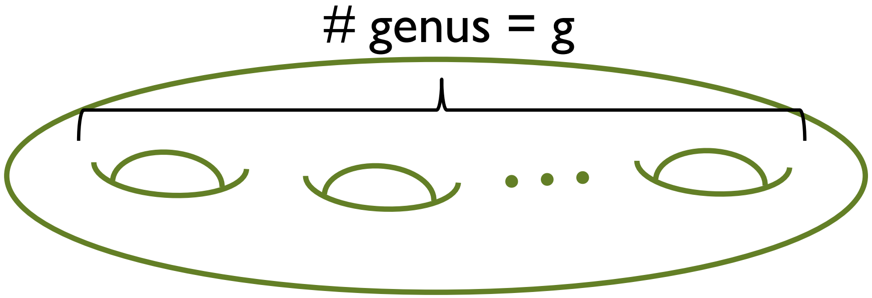
N^{2-2g}

Relation to string theory

- We consider the diagrams which can be drawn on genus-g surface, called "genus-g diagrams".
- They correspond to diagrams with g closed string loops. The amplitudes: $N^{2-2g} \sim g_{YM}^{2(2g-2)} \sim g_s^{2g-2}$ (with λ fixed)
- For example, in AdS/CFT correspondence, 4d N=4 SYM is equivalent to IIB string on $AdS_5 \times S^5$ (by considering the bunch of D3-branes and $\alpha'/R_{AdS}^2 \sim \lambda^{-1/2}$).

genus-g surface

= g closed string loops



String to M?

- In string theory (in AdS/CFT correspondence),
  - classical gravity = planar limit

    ($g_s \to 0$, $\alpha' \to 0$) (Only genus-0 diagrams dominate.)
  - g_s correction = 1/N correction

    (string coupling)
  - α' correction = 1/λ correction

    (~ string length^2)
- But what about M-theory?
  - We need to consider the region of $g_{YM}^2 \sim g_s \gtrsim \mathcal{O}(1)$
  - Is there any large-N limit??

Another large-N limit

$$N \to \infty\,, \quad \lambda = g_{YM}^2 N \sim \boxed{N^p}$$

- p=0 case

- It is nothing but the 't Hooft limit.

- p>0 case

- We call it "very strongly coupled large-N limit".
- 1/N expansion is different from genus expansion.
- Application for M-theory? (p=1): $g_{YM}^2 \sim g_s \gtrsim \mathcal{O}(1)$

  (In 't Hooft limit, g_{YM} and g_s must become zero.)
- Instanton effect becomes finite: $\exp\left[-8\pi^2/g_{YM}^2\right] = \mathcal{O}(1)$

  (In 't Hooft limit, it is exponentially suppressed.)

Conjecture

- The very strongly coupled large-N limit is well-defined and essentially the same as the 't Hooft limit.
- More precisely: large-N limit and strong 't Hooft coupling limit commute.
- When there is no "phase transition" (or as long as one considers the same point in the moduli space), the analytic continuation from the planar limit gives the right answer.

[Azeyanagi-Fujita-Hanada '12]
[Azeyanagi-Hanada-Honda-Matsuo-SS '13]

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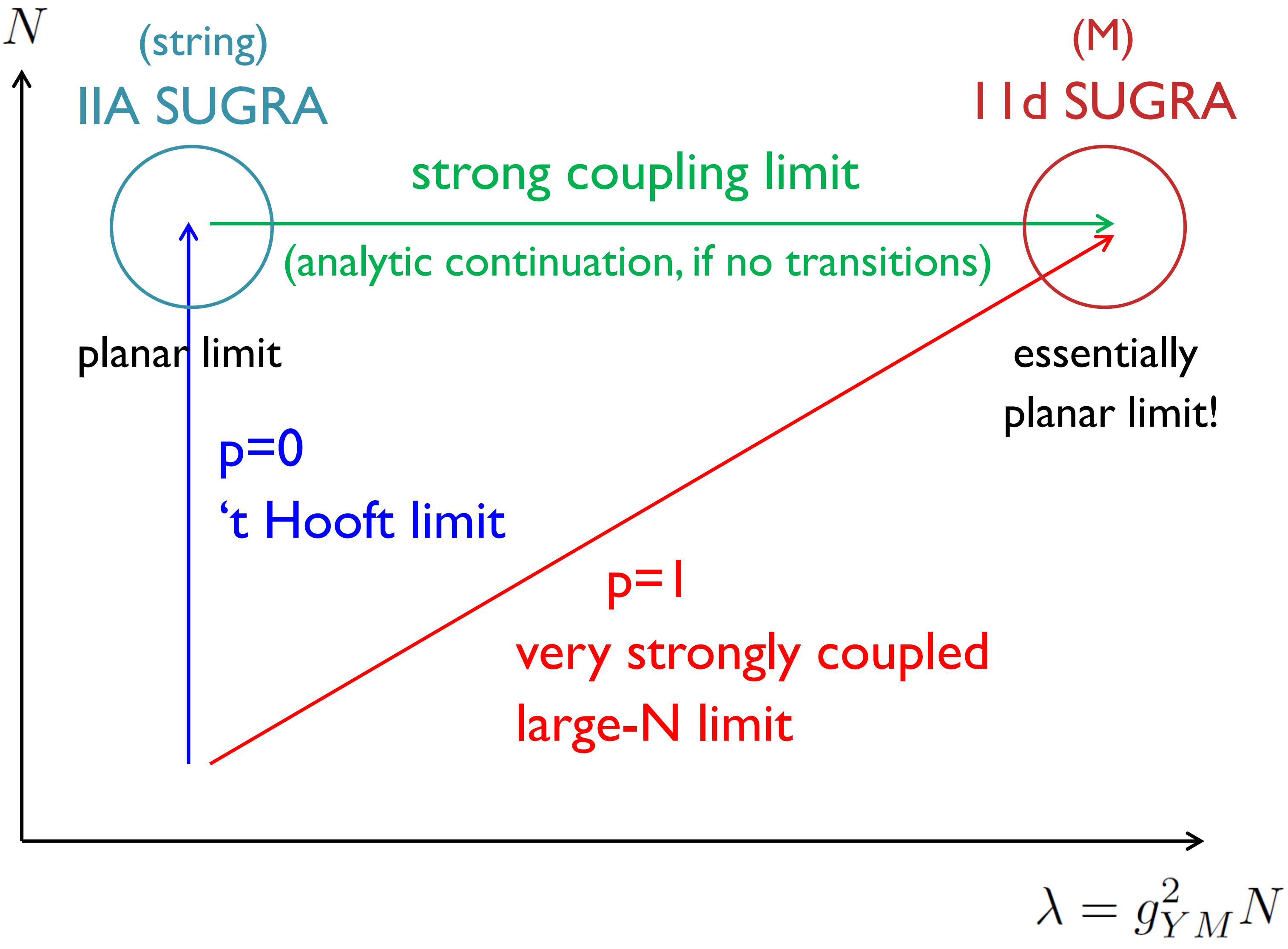


N

(string)

IIA SUGRA

(M)

11d SUGRA

strong coupling limit

(analytic continuation, if no transitions)

planar limit

essentially
planar limit!

p=0
't Hooft limit

p=1
very strongly coupled
large-N limit

$\lambda = g_{YM}^2 N$

Observation:

Analytic continuation to M-theory

Effective theory on M2-branes

- ABJM theory (dual to M-theory on $AdS_4 \times S^7/Z_k$)

$$S_{\rm ABJM} = \frac{k}{2\pi}\int d\tau d^2x \left(\mathrm{Tr}\Big[(D_\mu\Phi^\dagger_A)(D^\mu\Phi^A) + i\Psi^{\dagger A}\gamma^\mu D_\mu\Psi_A\Big] + \mathcal{L}^{(1)}_{\rm CS} - \mathcal{L}^{(2)}_{\rm CS} - V_B - V_F\right),$$

Chern-Simons level

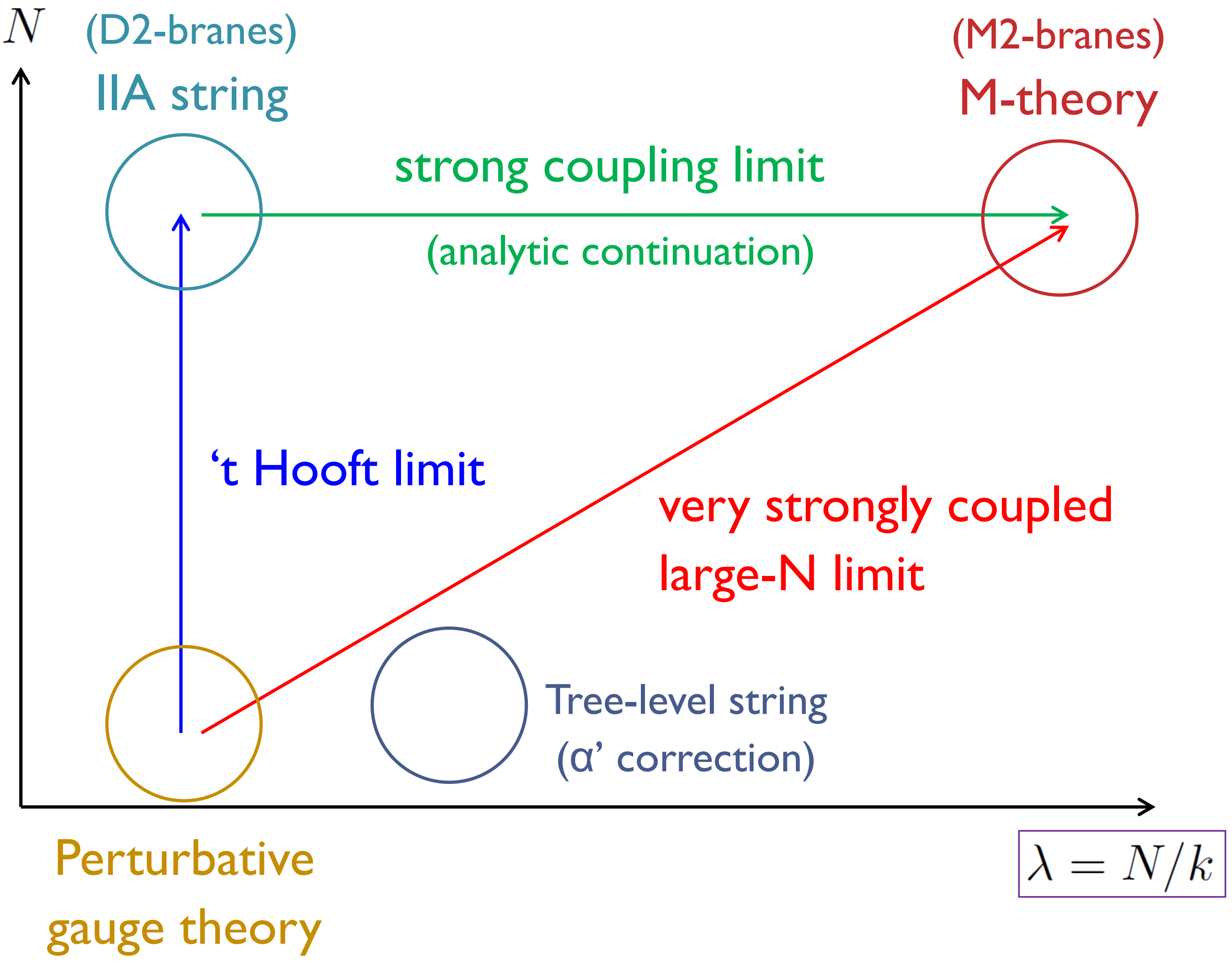
U(N) × U(N) gauge group

where $A, B, C, D = 1, \ldots, 4$

$$\mathcal{L}^{(i)}_{\rm CS} = \frac{1}{2}\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left[A^{(i)}_\mu\partial_\nu A^{(i)}_\rho + \frac{2}{3}A^{(i)}_\mu A^{(i)}_\nu A^{(i)}_\rho\right],$$

$$V_B = \frac{1}{3}\mathrm{Tr}\Big[\Phi^\dagger_A\Phi^A\Phi^\dagger_B\Phi^B\Phi^\dagger_C\Phi^C + \Phi^A\Phi^\dagger_A\Phi^B\Phi^\dagger_B\Phi^C\Phi^\dagger_C + 4\Phi^A\Phi^\dagger_B\Phi^C\Phi^\dagger_A\Phi^B\Phi^\dagger_C - 6\Phi^A\Phi^\dagger_B\Phi^B\Phi^\dagger_A\Phi^C\Phi^\dagger_C\Big],$$

$$V_F = i\mathrm{Tr}\Big[\Phi^\dagger_A\Phi^A\Psi^{\dagger B}\Psi_B - \Phi^A\Phi^\dagger_A\Psi_B\Psi^{\dagger B} - 2\Phi^\dagger_A\Phi^B\Psi^{\dagger A}\Psi_B + 2\Phi^A\Phi^\dagger_B\Psi_A\Psi^{\dagger B} - \epsilon^{ABCD}\Phi^\dagger_A\Psi_B\Phi^\dagger_C\Psi_D + \epsilon_{ABCD}\Phi^A\Psi^{\dagger B}\Phi^C\Psi^{\dagger D}\Big],$$



N

(D2-branes)
IIA string

(M2-branes)
M-theory

strong coupling limit

(analytic continuation)

't Hooft limit

very strongly coupled
large-N limit

Tree-level string
(α' correction)

Perturbative
gauge theory

$\boxed{\lambda = N/k}$

Free energy

- We can calculate it by using ABJM matrix model

  (which is derived by using localization technique).

  [Kapustin-Willett-Yaakov '09]

- It agrees with SUGRA predictions.

  (We checked it smoothly connects them and perturbative ones.)

  [Drukker-Marino-Putrov '10]

  [Hanada-Honda-Honma-Nishimura-SS-Yoshida '12]

✓ In IIA string region: $(N^{\frac{1}{5}} \ll k \ll N)$

$$F = \frac{\pi\sqrt{2}}{3}\frac{N^2}{\sqrt{\lambda}}$$

✓ In M-theory region: $(k \ll N^{\frac{1}{5}})$

$$F = \frac{\pi\sqrt{2}}{3}\sqrt{k}N^{3/2}$$

the same expression!

- Analytic continuation from $\lambda = \mathcal{O}(1)$ to $\lambda = \mathcal{O}(N)$

Planar part is dominant!

even in M-theory region!

$$F = \sum_{g=0}^{\infty} \boxed{F_g(\lambda)} N^{2-2g}$$

N-dependent in M-theory region

- AdS/CFT tells us that, at strong coupling, α'-expansion (1/λ-expansion) is good, at least in IIA string region.
- Then, only the leading term in each F_g is important.

For F_0 (g=0) : $\frac{\pi\sqrt{2}}{3}\frac{N^2}{\sqrt{\lambda}}$, For F_g (g>0) : $c_g \frac{N^{2-2g}}{\lambda^{\frac{1}{2}-g}}$

- The planar part (g=0) dominates even outside the planar limit : $\lambda \sim N^p \quad (0 < p \leq 1)$

What about M5-branes?

- Effective field theory is not known yet.
- Recently, however, the special class (called "class S") of 4d N=2 theories has been widely studied as the theory of M5-brane on 2d punctured Riemann surface.

[Gaiotto '09]

special (simple) examples

- 4d N=2* SYM : main topic of this seminar

mass deformation

- 4d N=4 SYM (maximal SUSY) :

Effective theory on D3-branes (IIB string on $AdS_5 \times S^5$)

Example 1:
4d N=4 SYM

AdS_5/CFT_4 correspondence

[Maldacena '97]

- 4d N=4 SYM is equivalent to IIB string on $AdS_5 \times S^5$:

$$g_{YM}^2 \sim g_s \,, \quad \alpha'/R_{AdS}^2 \sim \lambda^{-1/2} \qquad \boxed{\lambda = g_{YM}^2 N}$$

- Perturbative string picture is valid when

$$g_s \ll 1 \,, \quad \lambda \gg 1$$

- In usual, one takes the 't Hooft limit first and then consider strong 't Hooft coupling. (tree-level string)
- Or one consider large-but-finite-N with λ=O(1), so that 1/N expansion and string loop expansion coincide.
- However, such limit is not required for the validity of the weakly coupled gravity description.

Very strongly coupled large-N

[Azeyanagi-Fujita-Hanada ’12]

When there is gravity dual:

$$
\begin{aligned}
\tilde{F}(\lambda, N) &= F(g_s, \alpha') \\
&= g_s^{-2} F_0(\alpha') + F_1(\alpha') + g_s^2 F_2(\alpha') + \cdots \\
&= g_s^{-2} (f_{0,0} + f_{0,1}\alpha' + f_{0,2}\alpha'^2 + \cdots) \\
&\quad + (f_{1,0} + f_{1,1}\alpha' + f_{1,2}\alpha'^2 + \cdots) \\
&\quad + g_s^2 (f_{2,0} + f_{2,1}\alpha' + f_{2,2}\alpha'^2 + \cdots) + \cdots
\end{aligned}
$$

- $f_{0,0}$ dominates as long as $g_s \ll 1 \,, \quad \lambda \gg 1$
- At $1 \ll \lambda \ll N$, it is simply the same expression as the planar limit. So the very strongly coupled limit exists!!

SUGRA =

- In addition, the analytic continuation to $N < \lambda$ can be confirmed by using S-duality.

Example 2 (main topic):
4d N=2* SYM

[Azeyanagi-Hanada-Honda-Matsuo-SS '13]

Free energy

- 4d N=2* SYM can be obtained by mass deformation of N=4 SYM. Here we consider N=2* U(kN) SYM.

- Free energy is calculated by using Nekrasov's formula.

$$\mathcal{Z}_{\mathcal{N}=2^*} = \int d^{kN}a \left(\prod_{\substack{i,j=1 \\ i<j}}^{kN} (a_i - a_j)^2 \right) Z^{(\mathrm{pert})}_{\mathcal{N}=2^*}(a_i, m) \left| Z^{(\mathrm{inst})}_{\mathcal{N}=2^*}(a_i, \tilde{m}) \right|^2 \exp\left(-\frac{8\pi^2}{g_p^2} \sum_{i=1}^{kN} a_i^2 \right)$$

perturbative instanton classical

- ✓ g_p is YM coupling and a_i are Coulomb parameters (moduli).
- ✓ Classical part : almost the same as N=4 case
- ✓ Perturbative part : 1-loop quantum effect
- ✓ Instanton part : non-perturbative contribution

Perturbative part

$$F^{(\mathrm{pert})} = -(kN)^2(1+m^2)\left(\frac{1}{2}\log\frac{\lambda_p(1+m^2)}{16\pi^2}+\frac{1}{4}+\gamma\right)$$

- This sector takes the same expression in the 't Hooft limit and the very strongly coupled large-N limit. That is, they are related by "analytic continuation".

- The conditions in this expression are as follows:

✓ We need to use the saddle point method for the integral of *a*, so we assume $g_p^2 \ll 1$, or equivalently, $\boxed{\lambda_p = g_p^2(kN) \ll N}$

✓ We use the spectral density of *a* for $\boxed{\lambda_p \gg 4\pi^2 m^2/(m^2+1)}$ which obeys the semi-circle law.

[Douglas-Shenker '95]
[Russo-Zarembo '12]

One-instanton part

$\boxed{F_Y^{(\mathrm{inst})} = -\log(\mathcal{Z}_Y/\mathcal{Z}_\emptyset)}$ for the Young tableau $Y = (\underset{b}{\Box}, \emptyset, \cdots, \underset{a}{\emptyset})$

$$F_Y^{(\mathrm{inst})} \simeq \frac{8\pi^2}{g_p^2} - \log\left[\frac{(1-\tilde{m})^2}{(2-\tilde{m})\tilde{m}}\right] \underbrace{-kN\int da\,\rho(a)\log\left[\frac{(2-i(b-a)-\tilde{m})(i(b-a)-\tilde{m})}{(2-i(b-a))i(b-a)}\right]}_{O(\log g_p^2)\ \text{: interactions between instantons}}$$

- Compared to the perturbative part, it is subdominant in the both limits. But a different point is:

✓ In the ‘t Hooft limit, it is exponentially suppressed as ~ e^{-N}.

✓ In the very strongly coupled large-N limit (g_p~1), it may give comparable contribution $O(N^0)$ as the genus-one diagrams in the perturbative sector.

(Whole of) free energy

- By summing up all the parts and by taking into account the multi-instanton configurations, one obtains

$$F = \boxed{-\log \mathcal{Z}_{\emptyset}} + kN \log \left| 1 + \sum_{\tilde{Y}} e^{-\boxed{F^{(\mathrm{inst})}_{\tilde{Y}}}} \right|^2$$

perturbative part

instanton with only $Y_1 \neq \emptyset$

✓ Note that the interaction between instantons is negligible.

✓ Then the free energy for generic tableaux decomposes to a sum of contribution from each eigenvalue *a* as

$$F^{(\mathrm{inst})}_{Y} \simeq \textstyle\sum_{Y_i \neq \emptyset} F^{(\mathrm{inst})}_{Y_i} \qquad Y = (Y_1, \ldots, Y_{kN})$$

- In both limits the planar part (~ N^2) is dominant and they are related by "analytic continuation".

Further evidence: Orbifold equivalence

[Azeyanagi-Hanada-Honda-Matsuo-SS '13]

4d N=4 SYM

[Kachru-Silverstein ’98]

$$\begin{tikzcd}
\text{4d N=4 U(kN) SYM} \arrow[r, leftrightarrow, "\text{AdS/CFT}"] \arrow[d, leftrightarrow, "\text{orbifold equivalence}"'] & \text{IIB string on } AdS_5 \times S^5 \arrow[d, leftrightarrow, "\text{orbifold equivalence}"] \\
\text{4d N=4 } [U(N)]^k \text{ SYM} \arrow[r, leftrightarrow, "\text{AdS/CFT}"'] & \text{IIB string on } AdS_5 \times S^5/Z_k
\end{tikzcd}$$

In the gauge side, correlation func. of Z_k-invariant operators coincide with that in the orbifolded theory.

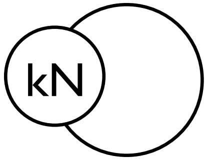
In the gravity side, Z_k-invariant modes do not distinguish these two theories.

- In this discussion, the planar limit is not really necessary: classical gravity discussion is the key.
- From the gauge theory viewpoint, the equivalence is gone as soon as the nonplanar diagrams are taken into account.

4d N=2* SYM

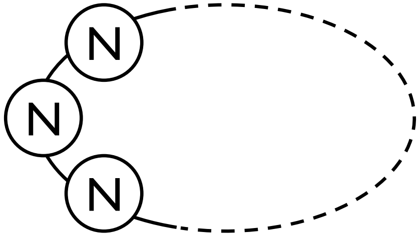
[Azeyanagi-Hanada-Honda-Matsuo-SS '13]

"parent": N=2* U(kN) gauge theory



$\downarrow$ Z_k orbifolding

"daughter": N=2 $[U(N)]^k$ necklace quiver



- When the daughter theory keeps N=2 SUSY, one can easily confirm the orbifold equivalence by using the Nekrasov's formula. In both large-N limits,

$$F_p(\lambda, N) = kF_d(\lambda, N)$$

- The equivalence holds at each instanton sector.
(The sector with the same total number of instantons.)

Towards SYM with less SUSY

“parent”
4d N=2 SYM

Z_k orbifolding

“daughter”
YM with less SUSY

You can use Nekrasov’s formula.

You can take it to be non-SUSY!

$$F_p(\lambda, N) = kF_d(\lambda, N)$$

- The orbifold equivalence requires that the vacuum structures of the parent and daughter theories be properly related.
- When the number of instantons and anti-instantons is O(1), the vacuum structures don’t change, so the equivalence holds.
- However, when it becomes O(N), the vacuum structures in the very strongly coupled large-N limit are modified and hence careful identification of the right vacua is required.

Conclusion and Discussion

- Both in the very strongly coupled large-N limit and in the ‘t Hooft limit, the planar sector is dominant.
- In addition, the two large-N limits are smoothly related by analytic continuation. (No transitions in our cases.)

➢ Application for 4d N=2 theories in “class S”

- Gauge/gravity correspondence in M-theory?

On-shell action of 11d SUGRA
(Gaiotto-Maldacena geometry)

$\overset{??}{=}$

Free energy of
4d N=2 gauge theory

- 11d SUGRA, corresponding to “planar” in gauge theory side, *may* know the instantons in the large-N limit with fixed g_{YM}!

[Azeyanagi-Hanada-Honda-Matsuo-SS, in progress]