Ghost Problems in Massive Gravity in terms of the Hidden Local Symmetry

Taichiro KUGO

October 7, 2013

益川塾セミナー

in collaboration with

Nobuyoshi OHTA

1 Introduction

Cosmological Constant Problem:

$$\begin{aligned} \text{Higgs Condensation} \sim & \quad (\ 100\,\text{GeV}\ )^4 \\ \text{QCD Chiral Condensation} \sim & \quad (\ 100\,\text{MeV}\ )^4 \end{aligned} \qquad (1)$$

These seem not contributing to the Cosmological Constant!

$\Longrightarrow$ Massive Gravity: an idea toward resolving it

However, Massive Gravity has its own problems:

- van Dam-Veltman-Zakharov (vDVZ) discontinuity

  Its $m \to 0$ limit does not coincides with the Einstein gravity.

- Boulware-Deser ghost

$$\underbrace{10}_{h_{\mu\nu}} - \underbrace{(1+3)}_{N,\ N^i} \quad = \quad 6 \quad = \quad \underbrace{5}_{\text{massive spin2}} \quad + \quad \underbrace{1}_{\text{BD ghost}} \qquad (2)$$

Let us focus on the BD ghost problem here.

2 vDVZ discontinuity and Vainshtein mechanism

$$S = \frac{1}{2} T^{\mu\nu} \frac{d_{\mu\nu,\rho\sigma}}{p^2 + m^2} T^{\rho\sigma} \tag{3}$$

massive case

$$d^m_{\mu\nu,\rho\sigma} = \frac{1}{2} \left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \frac{2}{3}\eta_{\mu\nu}\eta_{\rho\sigma} \right) \tag{4}$$

where, in fact, $\eta_{\mu\nu} \to \eta_{\mu\nu} + p_\mu p_\nu / m^2$.
massless case

$$d^0_{\mu\nu,\rho\sigma} = \frac{1}{2} \left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma} \right) \tag{5}$$

light bending becomes $3/4$ compared with the massless case!
Vainshtein pointed out that the linear approximation is not valid inside the radius

$$R_V = (R_S m^{-4})^{1/5} \tag{6}$$

the potential around the mass is:

$$\begin{aligned} R_V \leq r \leq m^{-1} & \qquad \text{the above is true} \\ R_S \leq r \leq R_V & \qquad \text{almost the same as } m = 0 \text{ case} \end{aligned} \tag{7}$$

3 Fierz-Pauli massive gravity (linearized)

Einstein-Hilbert action

$$\mathcal{L}_{\text{EH}} = \sqrt{-g}R \tag{8}$$

$$\mathcal{L} = \left[\mathcal{L}_{\text{EH}}\right]_{\text{quadratic part in } h_{\mu\nu}} + \underbrace{\left[-\frac{m^2}{4}(h_{\mu\nu}^2 - ah^2)\right]}_{= \ \mathcal{L}_{\text{FP}}^{\text{mass}}(a=1)} \tag{9}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{10}$$

In Fierz-Pauli theory with $a = 1$, there are only 5 modes describing properly massive spin 2 particle.

$\because$) No time derivative apperas for h_{00}, h_{0i} in $\mathcal{L}_{\rm EH} \rightarrow \mathcal{L}_{\rm EH}$ is linear in $N,\ N_i$.
If $a = 1$, the mass term $\mathcal{L}_{\rm FP}^{\rm mass}$ is also clearly linear in $N \sim h_{00}$!
$\Longrightarrow$

- N_i can be solved algebraically and be eliminated.
- N e.o.m. $\frac{\delta S}{\delta N} = 0$ gives 1 constraint on other fields since S is linear in N

so that

$$\underbrace{10}_{h_{\mu\nu}} - \underbrace{3}_{N_i} - (\underbrace{1}_{N} + \underbrace{1}_{\text{constraint}}) = 5 \tag{11}$$

Nonlinear completion of this theory was proposed by
dGRT: de Rham-Gabadadze-Tolley, Phys. Rev. Lett. 106 (2011)
which is claimed to be free of BD ghost on arbitrary background and to connect smoothly to Einstein gravity as $m \rightarrow 0$ by Vaishtein mechanism.

4 Arkani-Hamed-Georgi-Schwartz : Stückelberg formalism

Ann. Phys. 305 (2003) 96; the work preceding to dRGT.
AHGS have rewritten the Fierz-Pauli theory into GC invariant form: GC

invariance is realized as a Fake Symmetry, or Hidden Local Symmetry.

The simplest case is the "two site model", in which case easiest way to understand is to regard it as "space-time filling d-brane" in $D = d + 1$ dimensional target space-time.

$$\begin{aligned} \text{Target Space :} &\quad X^M \quad \text{with metric} \quad G_{MN}(X) \\ \text{brane (world sheet) :} &\quad x^\mu \quad \text{with metric} \quad g_{\mu\nu}(x) \end{aligned} \tag{12}$$

Embedding function

$$X^M = Y^M(x) \tag{13}$$

Induced metric on the brane

$$f_{\mu\nu}(x) = \partial_\mu Y^M(x) \cdot G_{MN}\big(Y(x)\big) \cdot \partial_\nu Y^N(x) \tag{14}$$

From world volume viewpoint,

$$\begin{aligned} Y^M(x) \quad &: \quad D \text{ scalar functions} \\ G_{MN}(Y(x)) \quad &: \quad \tfrac{D(D+1)}{2} \text{ scalar functions} \\ &\text{then,} \Rightarrow f_{\mu\nu}(x) : \text{GC tensor} \end{aligned} \tag{15}$$

From now on, we take

$$G_{MN}(X) = \eta_{MN} \quad \text{Flat Minkowski target space} \tag{16}$$

$$\begin{aligned} \mathcal{L}_{\rm AHGS} &= \mathcal{L}_{\rm EH} + \mathcal{L}^{\rm mass}_{\rm AHGS} \\ \mathcal{L}^{\rm mass}_{\rm AHGS} &= -\frac{m^2}{4}\sqrt{-g}g^{\mu\nu}g^{\alpha\beta}\left(H_{\mu\alpha}H_{\nu\beta} - aH_{\mu\nu}H_{\alpha\beta}\right) \end{aligned} \tag{17}$$

where

$$\begin{aligned} H_{\mu\nu} &= g_{\mu\nu} - f_{\mu\nu} \\ &= g_{\mu\nu} - \partial_\mu Y^M \cdot \eta_{MN} \cdot \partial_\nu Y^N \end{aligned} \tag{18}$$

is a GC tensor and the AHGS lagrangian $\mathcal{L}_{\rm AHGS}$ is GC invariant. This is achieved by the introduction of the mapping function $Y^M(x)$ which is analogous to $g^{5M}(x)$ from deconstruction point of view.

$$Y^M(x) = x^\mu\, \delta^M_\mu + \phi^M(x) \tag{19}$$

$\phi^M = 0$: "Unitary Gauge" (or, "static gauge" from brane viewpoint)

$$\Longrightarrow \quad \partial_\mu Y^M(x) = \delta^M_\mu \quad \Longrightarrow \quad f_{\mu\nu}(x) = \eta_{\mu\nu} \tag{20}$$

This mass term reduces to $\mathcal{L}^{\rm mass}_{\rm FP}$ at linearized level.

We can see more explicitly the absence of BD-ghost in this AHGS formulation of massive gravity.

Since there is

$$\text{Fake Symmetry} = \text{Hidden Local Symmetry} = \text{GC invariance} \tag{21}$$

Any gauge can be adopted, they are all gauge-equivalent, so we will take "R_ξ-gauge".

Generally, before fixing gauge,

$$f_{\mu\nu} = \partial_\mu Y^M \eta_{MN} \partial_\nu Y^N = \eta_{\mu\nu} + \partial_\mu \phi_\nu(x) + \partial_\nu \phi_\mu(x) + \partial_\mu \phi^M \cdot \partial_\nu \phi_M(x)$$

so that

$$\begin{aligned} H_{\mu\nu} &\equiv g_{\mu\nu} - f_{\mu\nu} \\ &= h_{\mu\nu} - \partial_\mu \phi_\nu - \partial_\nu \phi_\mu - \partial_\mu \phi^M \cdot \partial_\nu \phi_M(x) \end{aligned} \tag{22}$$

with $\phi_\mu \equiv \eta_{\mu M}\phi^M$. Then the AHGS mass term for $a=1$ takes the following form up to quadratic terms:

$$\mathcal{L}^{\rm mass}_{\rm AHGS}\Big|_{\rm qudratic} = \mathcal{L}^{\rm mass}_{\rm FP}(h_{\mu\nu}) - m^2\phi^\mu(\partial^\nu h_{\mu\nu} - \partial_\mu h) - \frac{m^2}{4}(\partial_\mu\phi_\nu - \partial_\nu\phi_\mu)^2 + (1-a)m^2\left[-(\partial_\mu\phi^\mu)^2 + h\,\partial_\mu\phi^\mu\right]$$

Let us introduce a scalar field π writing

$$\phi_\mu(x) \equiv \frac{1}{m}A_\mu(x) - \frac{1}{m^2}\partial_\mu\pi(x) \tag{23}$$

Then the AHGS mass term now takes the form

$$\mathcal{L}^{\rm mass}_{\rm AHGS}\Big|_{\rm qudratic} = \mathcal{L}^{\rm mass}_{\rm FP}(h_{\mu\nu}) - (mA^\mu - \partial^\mu\pi)(\partial^\nu h_{\mu\nu} - \partial_\mu h) - \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + (1-a)\left[-(\partial_\mu A^\mu)^2 + 2\frac{1}{m}\partial A\cdot\Box\pi - \frac{(\Box\pi)^2}{m^2} + h\,(m\partial_\mu A^\mu - \Box\pi)\right]$$

Note that the dipole ghsot term $(\Box\pi)^2$ appears unless $a=1$!

Clearly this system is invariant under the GC and additional U(1) gauge transgformation independently of a value:

$$\delta h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu, \quad \delta A_\mu = m\xi_\mu + \partial_\mu\Lambda, \quad \delta\pi = m\Lambda \tag{24}$$

Hereafter we consider only the case of Fierz-Pauli value $a = 1$. We make the shift $h_{\mu\nu} \to h_{\mu\nu} + \frac{2}{D-2}\eta_{\mu\nu}\pi$ in the Einstein-Hilbert term

$$\mathcal{L}_{\rm EH}\Big|_{\rm quadr} = \frac{1}{4}h^{\mu\nu}\Big[\partial_\mu\partial_\nu h - \partial_\mu h_\nu - \partial_\nu h_\mu + \Box h_{\mu\nu} + \eta_{\mu\nu}(\partial_\lambda h^\lambda - \Box h)\Big], \quad (25)$$

to cancel the mixing of π and $h_{\mu\nu}$ and to produce normal kinetic term for π. We used the notation

$$h_\mu = \partial^\nu h_{\mu\nu}, \quad h = h^\mu_\mu. \quad (26)$$

To get rid of the mixing terms of $h_{\mu\nu}$, A_μ, π further, it is convenient to take the "R_ξ-gauge":

$$\begin{aligned}\mathcal{L}_{\rm GF} + \mathcal{L}_{\rm FP} &= -i\delta_{\rm B}\left[\bar{c}^\mu\Big(h_\mu - x\partial_\mu h - \alpha m A_\mu + \frac{\alpha}{2}B_\mu\Big)\right] \\ &\quad - i\delta_{\rm B}\left[\bar{c}\Big(\partial A - m\beta(yh + z\pi) + \frac{\beta}{2}B\Big)\right] \end{aligned} \quad (27)$$

where the parameters α, β as well as x, y, z are gauge parameters. Suitable choice of x, y, z can resolve the mixings.

The GC and U(1) gauge trf are lifted to the BRS trf:

$$
\begin{aligned}
&\delta_{\mathrm{B}} h_{\mu\nu} = \delta\lambda\Big(\partial_\mu c_\nu + \partial_\nu c_\mu - \frac{2}{D-2}\eta_{\mu\nu} m c\Big), \\
&\delta_{\mathrm{B}} A_\mu = \delta\lambda(m c_\mu + \partial_\mu c), \qquad \delta_{\mathrm{B}}\pi = \delta\lambda\, m c, \\
&\delta_{\mathrm{B}} c_\mu = \delta\lambda\, c^\rho \partial_\rho c_\mu, \qquad \delta_{\mathrm{B}} \bar{c}_\mu = i\delta\lambda\, B_\mu, \qquad \delta_{\mathrm{B}} B_\mu = 0, \\
&\delta_{\mathrm{B}} c = \delta\lambda\, c^\rho \partial_\rho c, \qquad \delta_{\mathrm{B}} \bar{c} = i\delta\lambda B, \qquad \delta_{\mathrm{B}} B = 0,
\end{aligned} \tag{28}
$$

Propagators:

$w \equiv 2(D-1)/(D-2)$.

$h_{\mu\nu}$-sector:

$$
\begin{aligned}
&h_{\mathrm{TT}} : \ \text{transverse-traceless } \tfrac{(D+1)(D-2)}{2}\text{-modes} && -\frac{1}{p^2+m^2}, \\
&\partial h_{\mathrm{T}} : \ \text{S-transverse } (D-1)\text{-modes} && -\frac{1}{p^2+\alpha m^2}, \\
&\partial\partial h + h : \ \text{SS and trace } (1+1)\text{-modes} && -\frac{1}{p^2+\alpha\beta m^2} \\
& && -\frac{1}{p^2+2\beta w m^2},
\end{aligned} \tag{29}
$$

A_μ-π-sector:

$$\begin{aligned} A_{\rm T}: &\ \text{massive vector } (D-1)\text{-modes} && -\frac{1}{p^2+\alpha m^2} \\ \partial A: &\ \text{S 1-mode} && -\frac{1}{p^2+\alpha\beta m^2} \\ \pi: &\ \text{scalar 1-mode} && -\frac{1}{p^2+2\beta w m^2} \end{aligned} \tag{30}$$

Faddeev-Popov ghost sector:

$$\begin{aligned} \bar{c}_{\rm T},\quad c_{\rm T}: &\ \text{massive } 2\times(D-1)\text{-modes} && -\frac{1}{p^2+\alpha m^2} \\ \partial\bar{c},\quad \partial c: &\ \text{S } (1+1)\text{-modes} && -\frac{1}{p^2+\alpha\beta m^2} \\ \bar{c},\quad c: &\ \text{scalar } (1+1)\text{-modes} && -\frac{1}{p^2+2\beta w m^2} \end{aligned} \tag{31}$$

Counting of physical degrees of freedom:

$$\underbrace{10\ +\ 4\ +\ 1}_{g_{\mu\nu}+A_\mu+\pi} - (\underbrace{\ 4\ +\ 4\ }_{\text{GCghosts:}c_\mu+\bar{c}_\mu}) - (\underbrace{\ 1\ +\ 1\ }_{\text{U(1)ghosts:}c+\bar{c}}) = 5 \qquad ! \tag{32}$$

Or, in D-dimensional space-time,

$$\underbrace{\frac{D(D+1)}{2} \;+\; D \;+\; 1}_{g_{\mu\nu}+A_\mu+\pi} - (\underbrace{D \;\;+\;\; D}_{\text{GCghosts}:c_\mu+\bar{c}_\mu}) - (\underbrace{\;1\;\;+\;\;1\;}_{\text{U(1)ghosts}:c+\bar{c}}) = \frac{(D+1)(D-2)}{2}$$

Note that U(1) gauge invariance was a fake gauge symmetry which was brought into the system by introducing the Stückelberg scalar π.

But it gave subtracting 2 modes $c+\bar{c}$.

Isn't this STRANGE ?

The point is that usually

$$H_{\mu\nu} \supset \partial_\mu \phi_\nu \supset \partial_\mu \partial_\nu \pi \tag{33}$$

so that

$$\begin{aligned} H_{\mu\nu}^2 \supset \partial_\mu \partial_\nu \pi \cdot \partial^\mu \partial^\nu \pi, \quad H^2 \supset \Box\pi \cdot \Box\pi \\ \Rightarrow \quad H_{\mu\nu}^2 - a\, H^2 \supset (1-a)\Box\pi \cdot \Box\pi \end{aligned} \tag{34}$$

That is, When $a \neq 1$ there appears Higher Derivative Term so that the single field π actually contains (1+1)- modes! (one of them is of negative metric.)

So the problem is boiled down to confirm that the absence of higher derivative term for π.

5 "Ghost-free" massive gravity of de Rham-Gabadadze-Tolley

PRL 106 (2011)

$$\mathcal{L} = \mathcal{L}_{\mathrm{EH}} - \frac{m^2}{4}\sqrt{-g}\,U(g_{\mu\nu}, H_{\mu\nu}) \tag{35}$$

dRGT have determined their mass term U as follows:
Focussing on the derivative term of π, set $A_\mu = 0$ in

$$\phi_\mu(x) = \frac{1}{m}A_\mu(x) - \frac{1}{m^2}\partial_\mu\pi(x) \quad \Rightarrow \quad \phi^M(x) = -\frac{1}{m^2}\partial_\mu\pi(x) \tag{36}$$

and so

$$H_{\mu\nu} = h_{\mu\nu} + 2\Pi_{\mu\nu} - \Pi_\mu{}^\rho\Pi_{\rho\nu}, \qquad \Pi_{\mu\nu} \equiv \partial_\mu\partial_\nu\pi \tag{37}$$

They require that

$$\sqrt{-g}\,U(g, H)\Big|_{h_{\mu\nu}=0} \quad \text{be a total derivative} \tag{38}$$

Define a symmetric tensor $K_{\mu\nu}$ such that $K^{\mu}{}_{\nu} = g^{\mu\rho}K_{\rho\nu}$ satisfies

$$H^{\mu}{}_{\nu} = 2K^{\mu}{}_{\nu} - K^{\mu}{}_{\beta}K^{\beta}{}_{\nu}$$
$$\Rightarrow K^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - \sqrt{\delta^{\mu}{}_{\nu} - H^{\mu}{}_{\nu}} \quad (39)$$

Then clearly $K_{\mu\nu}$ is $\Pi_{\mu\nu} = \partial_\mu\partial_\nu\pi$ on the flat background.

$$K_{\mu\nu}\Big|_{h_{\mu\nu}=0} = \Pi_{\mu\nu} \quad (40)$$

dRGT demands that U be a polynomial in $K_{\mu\nu}$ tensor such that it becomes a total derivative on flat background; i.e., when $K_{\mu\nu} \rightarrow \Pi_{\mu\nu}$: Clearly

$$\det(\delta^{\mu}{}_{\nu} + \lambda\Pi^{\mu}{}_{\nu}) = 1 + \lambda U^{(1)}(\Pi) + \lambda^2 U^{(2)}(\Pi) + \lambda^3 U^{(3)}(\Pi) + \lambda^4 U^{(4)}(\Pi) \quad (41)$$

give $U^{(n)}(\Pi)$ $(n = 1, 2, 3, 4)$ which are total derivatives:

$$U^{(1)}(\Pi) = \varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\nu\rho\sigma}\Pi^{\mu}{}_{\alpha} = 3!\,[\Pi]$$
$$U^{(2)}(\Pi) = \varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\rho\sigma}\Pi^{\mu}{}_{\alpha}\Pi^{\nu}{}_{\beta}$$
$$= 2\left([\Pi^2] - [\Pi]^2\right) \quad \rightarrow \text{ Fierz-Pauli}$$
$$U^{(3)}(\Pi) = \varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\sigma}\Pi^{\mu}{}_{\alpha}\Pi^{\nu}{}_{\beta}\Pi^{\rho}{}_{\gamma}$$
$$U^{(4)}(\Pi) = \varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\Pi^{\mu}{}_{\alpha}\Pi^{\nu}{}_{\beta}\Pi^{\rho}{}_{\gamma}\Pi^{\sigma}{}_{\delta} \quad (42)$$

Now the dRGT mass term is given:

$$\sqrt{-g}U(g,\,H) = \sqrt{-g}\left(2U^{(2)}(K) + \alpha_3 U^{(3)}(K) + \alpha_4 U^{(4)}(K)\right) \tag{43}$$

minimal model

$$\begin{aligned} &\alpha_3 = \alpha_4 = 0. \\ &2U^{(2)}(K) = \underbrace{\langle H^2\rangle - \langle H\rangle^2}_{\text{AHGS mass term}} + \frac{1}{2}\big(\langle H^3\rangle - \langle H^2\rangle\langle H\rangle\big) + \cdots \end{aligned} \tag{44}$$

6 Vierbein formalism

Hinterbichler-Rosen, arXiv:1203.5783[hep-th]

dRGT action is equivalent to

$$\mathcal{L} = \mathcal{L}_{\rm EH}(g_{\mu\nu} = e^a_\mu \eta_{ab}\, e^b_\nu) + U(e) \tag{45}$$

where the mass term is given by

$$U(e) = \det(e^a_\mu + \lambda\, b^a_\mu)\Big|_{\lambda^n\ \to\ \text{arbitrary parameters}\ \alpha_n} \tag{46}$$

in terms of the induced vierbein

$$b^a_\mu = u^a{}_A\, \partial_\mu Y^M(x) \cdot E^A_M\big(Y(x)\big) \equiv u^a{}_A\, \hat{b}^A_\mu \tag{47}$$

where $E^A_M(X)$ is the vierbein in the Taget Space, which we henceforth take flat one $E^A_M(X) = \delta^A_M$. $u^a{}_A$ is the Stückelberg field for LL; $u^a{}_A \in SO(3,1)$.
$u^a{}_A$ can be solved algebraically, since it appears only in the mass term because $\mathcal{L}_{\rm EH}$ is LL invariant:

$$\Rightarrow\ u\hat{b}e^{-1} = \sqrt{\eta e^{T\,-1}\hat{b}^T\eta\hat{b}e^{-1}} \tag{48}$$

Plugging this back into the mass term

$$\begin{aligned}
\det(e^a_\mu + \lambda\, u^a_A\, \hat{b}^A_\mu) &= \det e \cdot \det(1^a_b + \lambda\, u^a_A\, \hat{b}^A_\mu e^\mu_b) \\
&= \det e \cdot \det(1 + \lambda\, \sqrt{\eta e^{T\,-1}\hat{b}^T\eta\hat{b}e^{-1}}) \\
&= \det e \cdot \det(1 + \lambda e\, \sqrt{e^{-1}\eta e^{T\,-1}\hat{b}^T\eta\hat{b}\; e^{-1}}) \\
&= \sqrt{-g} \cdot \det(1 + \lambda\sqrt{g^{-1}f})
\end{aligned} \tag{49}$$

7 'Proof' of Absence of BD ghost by Hinterbichler-Rosen

in Unitary Gauge in vierbein formalism

$$\begin{cases} u^a{}_A = \delta^a{}_A & \text{for LL} \\ Y^M(x) = x^\mu\, \delta^M_\mu & \text{for GC} \end{cases} \quad \Longrightarrow \qquad b_\mu{}^a = \delta_\mu{}^a \tag{50}$$

Then the mass term

$$U(e) = \det\bigl(e_\mu{}^a + \lambda \delta_\mu{}^a\bigr) \tag{51}$$

Define the standard form of the vierbein:

$$\hat{e}_\mu{}^a = \begin{matrix} {\scriptstyle \mu=0} \\ {\scriptstyle \mu=i} \end{matrix} \begin{pmatrix} N & N^i e_i^a \\ 0 & e_i^a \end{pmatrix} \Longleftarrow \quad \text{fix 3 d.o.f. out of 6 for LL} \tag{52}$$

then the general vierbein can be parametrized as

$$e_\mu^a = \hat{e}_\mu{}^b \cdot \underbrace{\Lambda(\boldsymbol{p})_b{}^a}_{\text{Lorentz boost 3}} = \hat{e}_\mu{}^b \cdot \begin{pmatrix} \gamma \equiv \sqrt{1+\boldsymbol{p}^2} & \boldsymbol{p}^a \\ \boldsymbol{p}_b & \delta_b^a + \frac{1}{\gamma+1}\boldsymbol{p}_b\boldsymbol{p}^a \end{pmatrix}$$

$$= \begin{matrix} \mu=0 \\ \mu=i \end{matrix} \begin{pmatrix} N\gamma + N^i e_i^a \boldsymbol{p}_a & N\boldsymbol{p}^a + N^i e_i^b \left(\delta_b^a + \frac{1}{\gamma+1}\boldsymbol{p}_b\boldsymbol{p}^a\right) \\ e_i^a \boldsymbol{p}_a & e_i^b \left(\delta_b^a + \frac{1}{\gamma+1}\boldsymbol{p}_b\boldsymbol{p}^a\right) \end{pmatrix} \tag{53}$$

Even in this general form, the lapse N and shift N^i appear only linearly in $e^0_{\mu=0}$ and $e^a_{\mu=0}$ alone.

The mass term is clearly at most linear in $e^*_{\mu=0}$ so that

$$U(e) = N\mathcal{C}^{\mathrm{m}}(e,\boldsymbol{p}) + N^i\mathcal{C}_i^{\mathrm{m}}(e,\boldsymbol{p}) + \mathcal{H}(e,\boldsymbol{p}) \tag{54}$$

On the other hand, the canonical form for the $\mathcal{L}_{\mathrm{EH}}$ part is: (a: only space)

$$\int d^4x \left[\pi_a^i \dot{e}_i^a - N\,\mathcal{C}(e,\pi) - N^i\mathcal{C}_i(e,\pi) - \frac{1}{2}\ \lambda^{ab}\ \underbrace{\mathcal{P}_{ab}(e,\pi)}_{\text{spacial LL generator}} \right] \tag{55}$$

So the canonical form for the total system is:

$$\int d^4x \Big[\pi^i_a \dot{e}^a_i - \mathcal{H}(e, \boldsymbol{p}) - \frac{1}{2}\lambda^{ab}\mathcal{P}_{ab}(e, \pi) \\ -N\left(\mathcal{C}(e,\pi) + \mathcal{C}^{\mathrm{m}}(e,\boldsymbol{p})\right) - N^i\left(\mathcal{C}_i(e,\pi) + \mathcal{C}^{\mathrm{m}}_i(e,\boldsymbol{p})\right)\Big] \tag{56}$$

$$\frac{\delta}{\delta N^i} = 0 \quad \Rightarrow \quad \mathcal{C}_i(e,\pi) + \mathcal{C}^{\mathrm{m}}_i(e,\boldsymbol{p}) = 0 \quad \Rightarrow \quad \boldsymbol{p}^a = \boldsymbol{p}^a(e,\pi)$$

Now the counting of degrees of freedom becomes:

spacial vierbein e^a_i and its conjugate momentum π^i_a: $3^2 \times 2$

spacial LL constraint $\mathcal{P}_{ab}$ + secondary constraint: -3×2

N constraint + its secondary constraint: -1×2

thus,

$$2 \times (3^2 - 3 - 1) = 2 \times 5 \quad = \# \text{ of canonical variables of massive spin 2!}$$

8 What we want to show

Instead of the unitary gauge, we want to use

$$\hat{b}_\mu^A = \delta_\mu^A + \left(\frac{1}{m} \partial_\mu A_\nu(x) - \frac{1}{m^2} \partial_\mu \partial_\nu \pi(x) \right) \eta^{\nu A} \tag{57}$$

with which GC and LL and U(1) gauge symmetry are manifest. Then we have only to show that the higher time derivatives do not appear for the scalar $\pi(x)$ on arbitrary background field $\langle e_\mu^a \rangle \equiv \bar{e}_\mu^a$.

But, this program has turned out to be misleading! Actually we see that higher time derivative terms appear when the background metric has non-vanishing shift $\langle N^i \rangle \neq 0$.

What we have to show is: when we define the Stückelberg 'vector' field ϕ_μ by

$$\hat{b}_\mu^A = \delta_\mu^A + \partial_\mu \phi_\nu(x) \eta^{\nu A} \tag{58}$$

or, by

$$f_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu \phi_\nu + \partial_\nu \phi_\mu + \partial_\mu \phi^a \eta_{ab} \partial_\nu \phi^b \tag{59}$$

then, they have a singular quadratic kinetic term on any background:

$$U = \frac{1}{2}\dot{\phi}_\mu \mathcal{A}_{\mu\nu}\dot{\phi}_\nu + \cdots \;\Rightarrow\; \det \mathcal{A} = 0 \tag{60}$$

The mass term can be rewritten in the form

$$\det(e^a_\mu + \lambda \hat{b}^a_\mu) \tag{61}$$

Then, the index μ of $\hat{b}^A_\mu$ is anti-symmetrized by the epsilon tensor, and so is for the index a. This structure claerly lead the $F^2_{\mu\nu}$ structure for ϕ_μ on the flat background case. But, when the background metric is general, such structure is no longer clear.

The difficulty is that the vierbein contains non-dynamical 6 components corresponding to the LL freedom, which, if solved, become non-trivial functions of $\partial_\mu\phi_\nu$ when the background is non-flat. The analysis of that structure is veru complicated.

The elimination of the 6 auxiliary components in the vierbein is equivalent to use directly the original mass term of dRGT:

$$\sqrt{-g}\,\det(1 + \lambda\sqrt{g^{-1}f}) \tag{62}$$

The source of the difficulty is that, on a general background $\langle g_{\mu\nu}\rangle \equiv \bar{g}_{\mu\nu}$, the expansion of the square root of the matrix $\sqrt{g^{-1}f}$ is very difficult. For some examples we can show that the kinetic term of ϕ_μ is singular. For example: for the background

$$ds^2 = -dt^2 + \delta_{ij}(dx^i + 2l^i dt)(dx^j + 2l^j dt) \tag{63}$$

the time derivative terms of the Stückelberg field are calculated to be

$$U = \frac{1}{2\sqrt{1-l^2}} \left[\frac{(\dot{\phi}_1 - l\dot{\phi}_0)^2}{1-l^2} + \dot{\phi}_2^2 + \dot{\phi}_3^2 \right] \tag{64}$$

where we rotated the direction of the shift l^i into $l^i = l\delta^{i1}$.

But we cannot see why such degeneracy of the kinetic term appear.

For

So, we cannot yet prove the absence of BD ghost for arbitrary background.