# Some Curious Consequences of the Minimal Length Uncertainty Relation

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January 11, 2012

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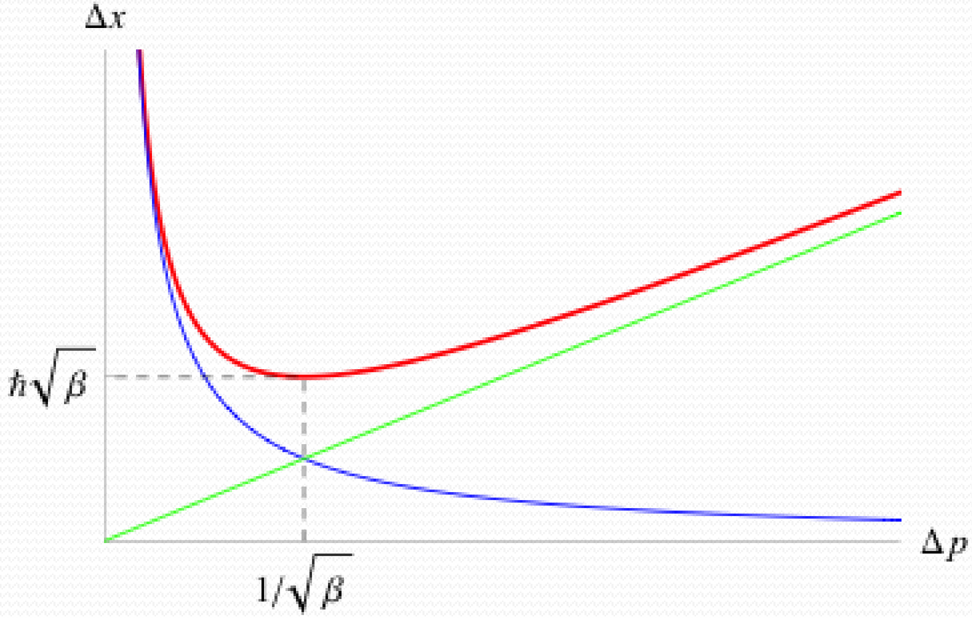
# The Minimal Length Uncertainty Relation

$$\Delta x \geq \frac{\hbar}{2}\left(\frac{1}{\Delta p} + \beta \Delta p\right) \quad \Rightarrow \quad \Delta x \geq \Delta x_{\min} = \hbar\sqrt{\beta}$$




Suggested by Quantum Gravity. Observed in perturbative String Theory.

# Deformed Commutation Relation

$$\frac{1}{i\hbar}\left[\hat{x},\ \hat{p}\right]=1+\beta\hat{p}^2 \qquad \Rightarrow \quad \Delta x \geq \frac{\hbar}{2}\left(\frac{1}{\Delta p}+\beta\Delta p\right)$$

The operators can be represented as:

$$\begin{cases} \hat{x} = i\hbar\left(1+\beta\hat{p}^2\right)\dfrac{d}{dp} \\ \hat{p} = p \end{cases}$$

and the inner product as

$$\langle f | g \rangle = \int_{-\infty}^{\infty} \frac{dp}{(1+\beta p^2)} f^*(p)\, g(p)$$

# Harmonic Oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} k \hat{x}^2$$

$$\Rightarrow \left[ -\frac{\hbar^2 k}{2} \left\{ (1 + \beta p^2) \frac{d}{dp} \right\}^2 + \frac{p^2}{2m} \right] \psi(p) = E \psi(p)$$

Can be solved exactly. Energy eigenvalues :

$$E_n = \frac{k}{2} \left[ \left( n + \frac{1}{2} \right) \sqrt{(\Delta x_{\min})^4 + 4a^4} + \left( n^2 + n + \frac{1}{2} \right) (\Delta x_{\min})^2 \right]$$

$$a = \sqrt[4]{\frac{\hbar^2}{km}} = \sqrt{\frac{\hbar}{m\omega}}$$

No longer evenly spaced. $n^2$ - dependence is introduced.

# Multi Dimensional Case

$$\frac{1}{i\hbar}\left[\hat{x}_i,\ \hat{p}_j\right]=\left(1+\beta\hat{p}^2\right)\delta_{ij}+\gamma\,\hat{p}_i\hat{p}_j$$

$$\left[\hat{p}_i,\ \hat{p}_j\right]=0$$

$$\frac{1}{i\hbar}\left[\hat{x}_i,\ \hat{x}_j\right]=-\left\{(2\beta-\gamma)+\beta(2\beta+\gamma)\hat{p}^2\right\}\hat{L}_{ij}, \qquad \hat{L}_{ij}=\frac{\hat{x}_i\hat{p}_j-\hat{x}_j\hat{p}_i}{1+\beta\hat{p}^2}$$

The operators can be represented as:

$$\begin{cases}\hat{x}_i=i\hbar\left[\left(1+\beta\hat{p}^2\right)\dfrac{\partial}{\partial p_i}+\gamma\,p_i p_j\dfrac{\partial}{\partial p_j}+\left\{\beta+\gamma\left(\dfrac{D+1}{2}\right)-\delta(\beta+\gamma)\right\}p_i\right]\\[2ex] \hat{p}_i=p_i\end{cases}$$

and the inner product as

$$\langle f|g\rangle=\int_{-\infty}^{\infty}\frac{dp}{[1+(\beta+\gamma)p^2]^{\delta}}\,f^*(p)\,g(p)$$

# Isotropic Harmonic Oscillator in D dimensions

$$\Psi_D(p_1, p_2, \ldots, p_D) = R(p)\, Y_{\ell\, m_{D-2} m_{D-3} \ldots m_2 m_1}(\Omega)$$

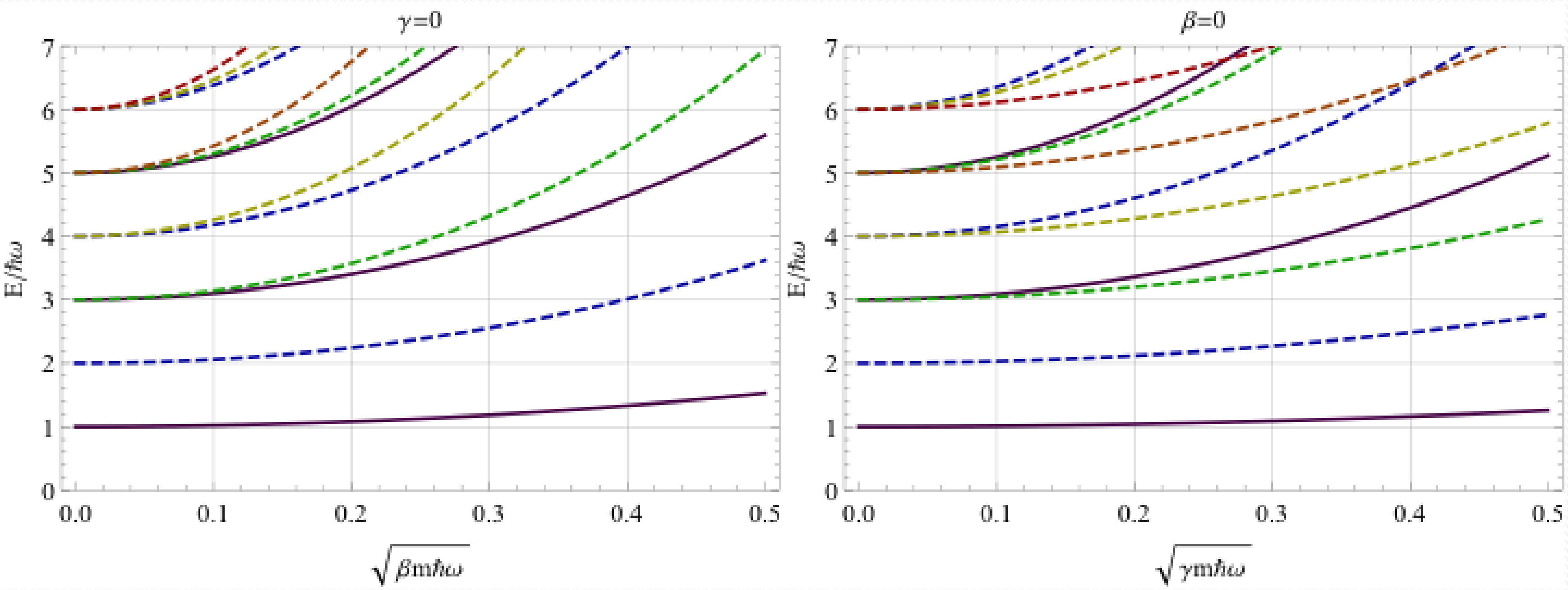
$$-\frac{\hbar^2 k}{2}\left[\left\{[1+(\beta+\gamma)p^2]\frac{\partial}{\partial p}\right\}^2 + \frac{(D-1)(1+\beta p^2)[1+(\beta+\gamma)p^2]}{p}\frac{\partial}{\partial p} - \frac{L^2(1+\beta p^2)^2}{p^2}\right]R(p) + \frac{p^2}{2m}R(p) = E R(p), \qquad L^2 = \ell(\ell + D - 2)$$

Can be solved exactly. Energy eigenvalues :

$$E_{n\ell} = \frac{k}{2}\left[\left(n+\frac{D}{2}\right)\sqrt{\Delta x_{\min}^4\left[4L^2+(D+\eta)^2\right]+4a^4} + \Delta x_{\min}^2\left\{(1+\eta)\left(n+\frac{D}{2}\right)^2 + (1-\eta)\left(L^2+\frac{D^2}{4}\right) + \eta\frac{D}{2}\right\}\right], \quad \Delta x_{\min} = \hbar\sqrt{\beta}, \quad \eta = \frac{\gamma}{\beta}.$$

Dependence on angular momentum introduced. SU(D) degeneracy is broken.

# 2D Case



# Some Details (1D case) :

Change variable to :

$$\rho = \frac{1}{\sqrt{\beta}} \arctan \sqrt{\beta} p \qquad -\frac{\pi}{2\sqrt{\beta}} < \rho < \frac{\pi}{2\sqrt{\beta}}$$

$$\begin{cases} \hat{x} = i\hbar \dfrac{d}{d\rho} \\ \hat{p} = \dfrac{1}{\sqrt{\beta}} \tan \sqrt{\beta} \rho \end{cases} \qquad \langle f | g \rangle = \int_{-\pi/2\sqrt{\beta}}^{\pi/2\sqrt{\beta}} d\rho \, f^*(\rho) \, g(\rho)$$

Schrodinger Equation :

$$\left[ -\frac{\hbar^2 k}{2} \frac{d^2}{d\rho^2} + \frac{1}{2m\beta} \tan^2 \sqrt{\beta} \rho \right] \psi(\rho) = E \, \psi(\rho)$$

Infinite square - well problem for large $n$, and also in the limit $m \to \infty$.

# Solution:

The solution is:

$$\psi_n^{(\lambda)}(\rho) = \sqrt[4]{\beta}\left[2^{\lambda}\Gamma(\lambda)\sqrt{\frac{n!(n+\lambda)}{2\pi\Gamma(n+2\lambda)}}\right]\left(\cos\sqrt{\beta}\rho\right)^{\lambda} C_n^{\lambda}\left(\sin\sqrt{\beta}\rho\right)$$

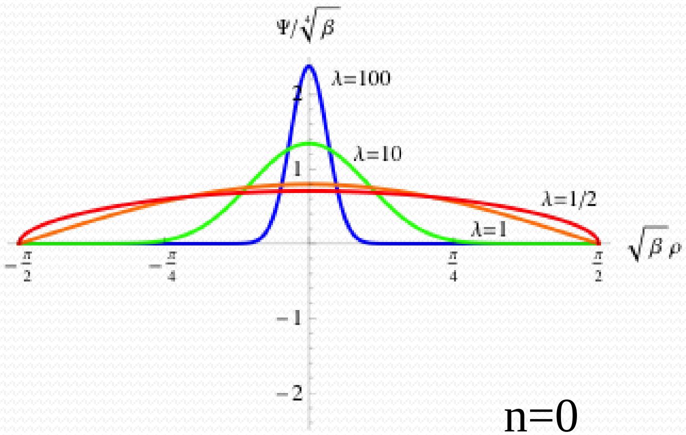
where:

$$\lambda = \frac{1}{2} + \sqrt{\frac{1}{4} + \left(\frac{a}{\Delta x_{\min}}\right)^4} \qquad (1 < \lambda)$$

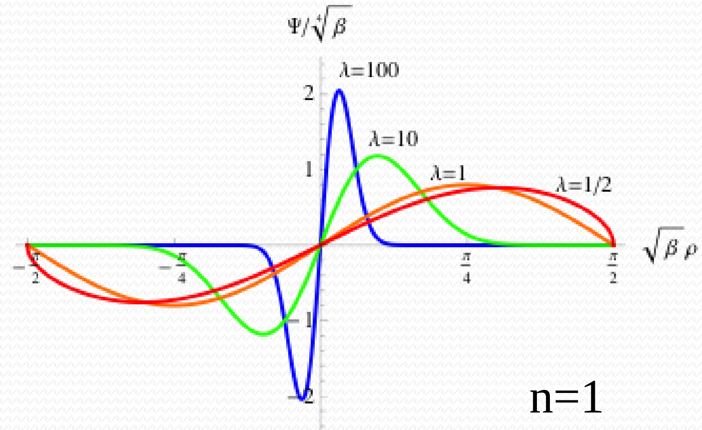
Uncertainties:

$$\Delta x = \Delta x_{\min}\sqrt{\frac{(\lambda+n)[(2\lambda-1)n+\lambda}{(2\lambda-1)}}, \qquad \Delta p = \frac{1}{\sqrt{\beta}}\sqrt{\frac{2n+1}{2\lambda-1}}$$

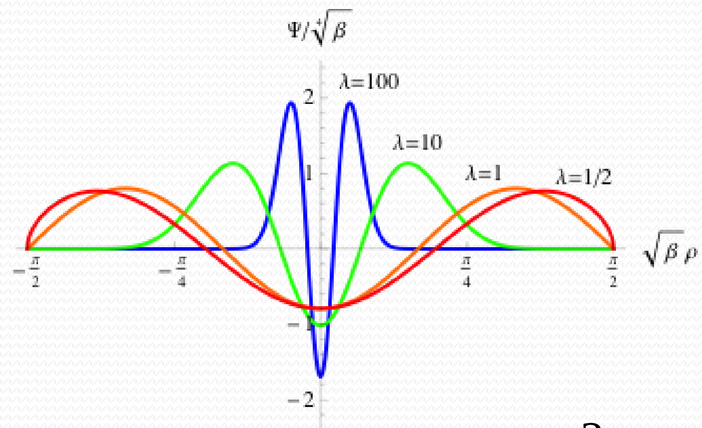
# Wave-functions:




n=0

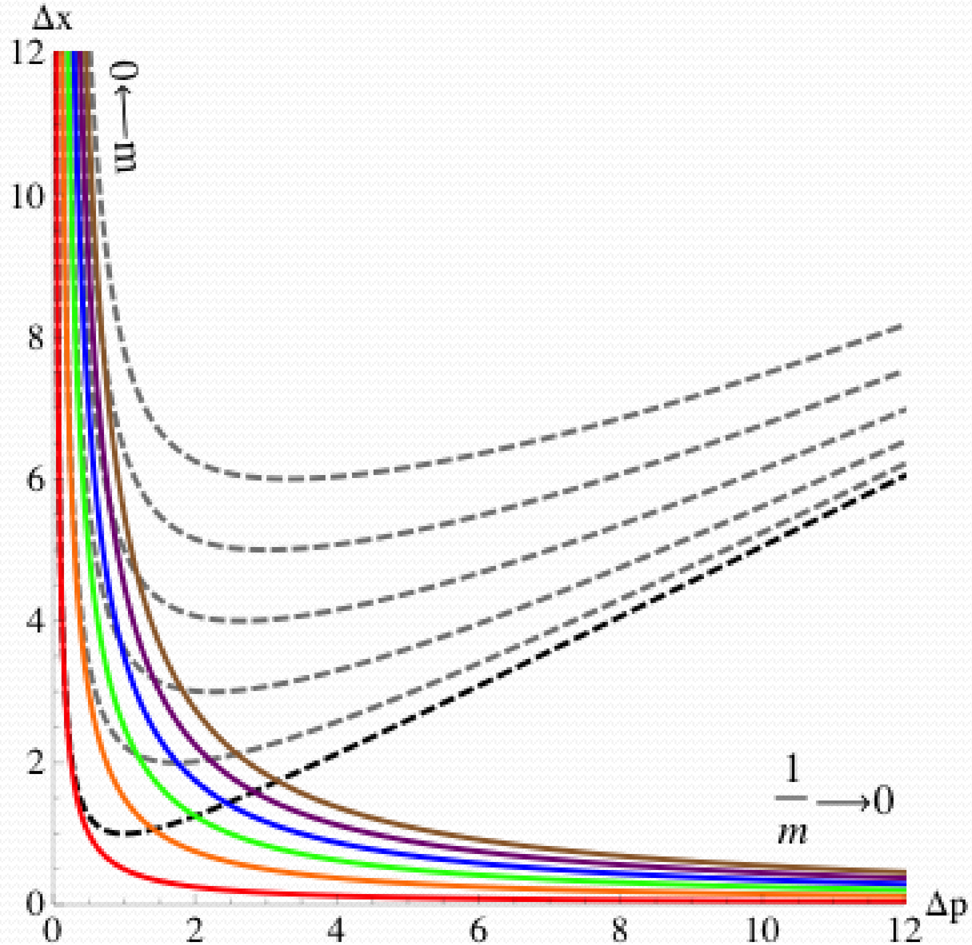



n=1

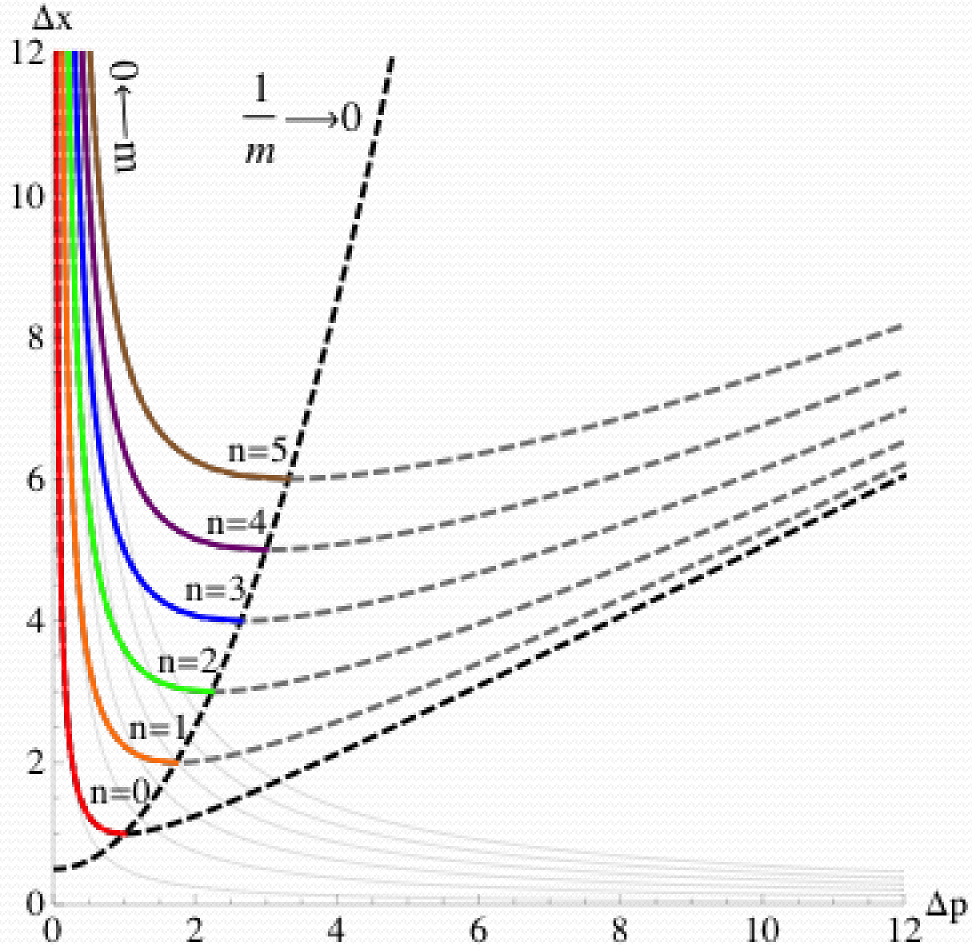



n=2

# Uncertainties of the Harmonic Oscillator: β=0



# Uncertainties of the Harmonic Oscillator: β≠0




How can we get onto the Δx~Δp branch?

# Harmonic Oscillator with negative mass:

$$\hat{H} = -\frac{\hat{p}^2}{2|m|} + \frac{1}{2}k\hat{x}^2$$

$$\Rightarrow \left[-\frac{\hbar^2 k}{2}\left\{(1+\beta p^2)\frac{d}{dp}\right\}^2 - \frac{p^2}{2|m|}\right]\psi(p) = E\psi(p)$$

Energy eigenvalues :

$$E_n = \frac{k}{2}\left[\left(n+\frac{1}{2}\right)\sqrt{(\Delta x_{\min})^4 - 4a^4} + \left(n^2+n+\frac{1}{2}\right)(\Delta x_{\min})^2\right]$$

$$a = \sqrt[4]{\frac{\hbar^2}{k|m|}}$$

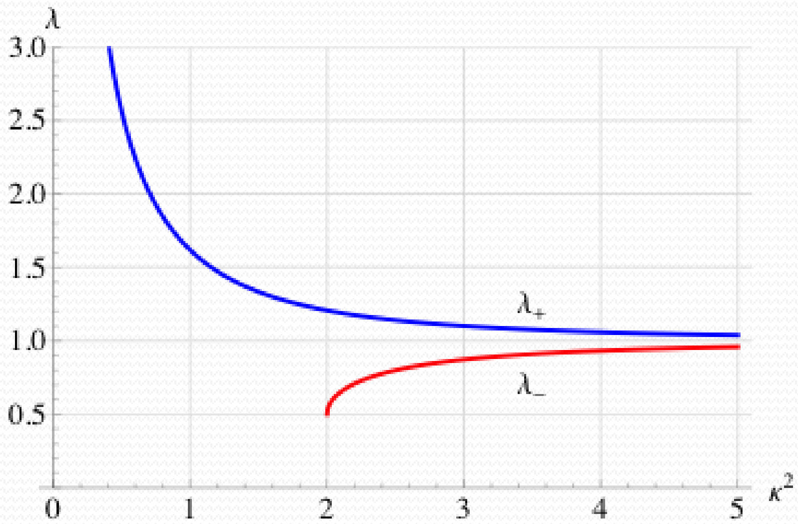
# Harmonic Oscillator with negative mass:

The solution is the same as before

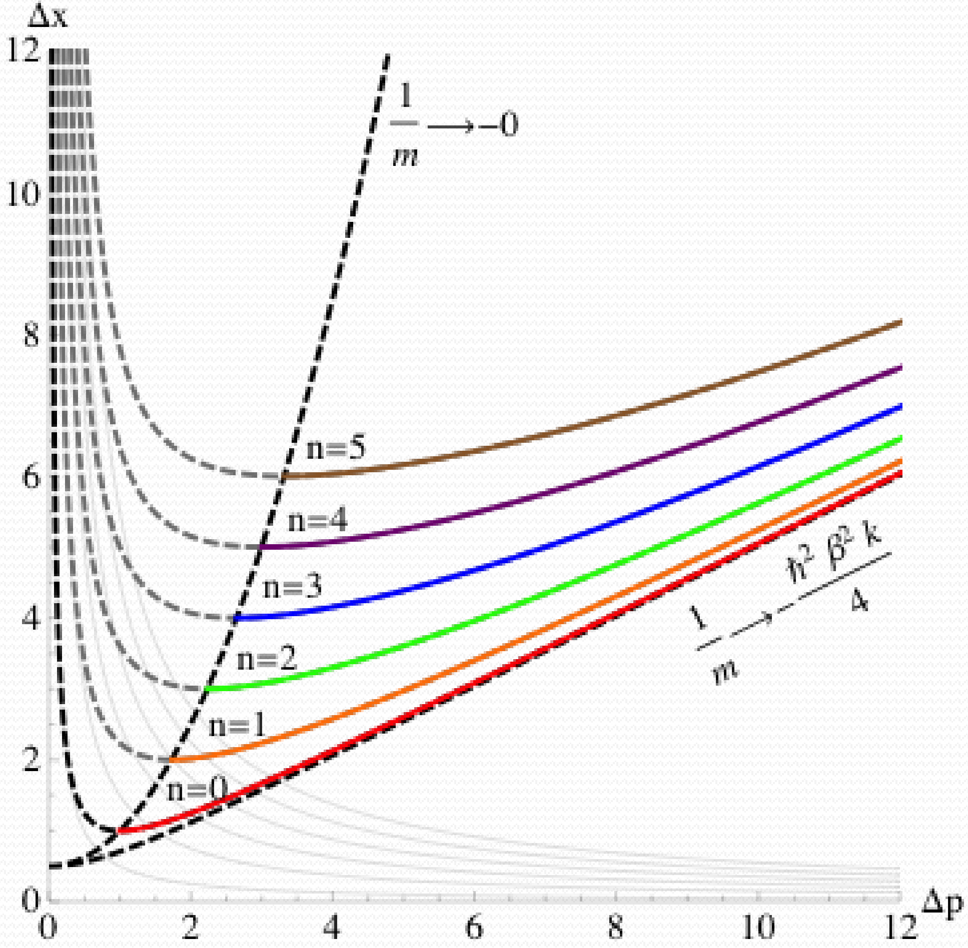
$$\psi_n^{(\lambda)}(\rho) = \sqrt[4]{\beta}\left[2^{\lambda}\Gamma(\lambda)\sqrt{\frac{n!(n+\lambda)}{2\pi\Gamma(n+2\lambda)}}\right]\left(\cos\sqrt{\beta}\rho\right)^{\lambda} C_n^{\lambda}\left(\sin\sqrt{\beta}\rho\right)$$

except :

$$\lambda = \frac{1}{2} + \sqrt{\frac{1}{4} - \left(\frac{a}{\Delta x_{\min}}\right)^4} \qquad \left(\frac{1}{2} < \lambda < 1, \quad \Delta x_{\min} > \sqrt{2}a\right)$$



# Uncertainties of the Harmonic Oscillator: β≠0, m<0



# Classical Limit:

$$\frac{1}{i\hbar}\left[\hat{x},\hat{p}\right]=\left(1+\beta\hat{p}^2\right) \quad\Rightarrow\quad \left\{x,p\right\}=\left(1+\beta p^2\right)$$

Classical Equations of Motion:

$$\dot{x}=\left\{x,H\right\}, \qquad \dot{p}=\left\{p,H\right\}$$

Liouville Theorem:

$$dx\wedge dp \qquad \rightarrow \qquad \frac{dx\wedge dp}{1+\beta p^2}$$

$\hbar\left(1+\beta p^2\right)$ can be considered a $p$-dependent effective $\hbar(p)$.

# Classical Harmonic Oscillator:

Hamiltonian :

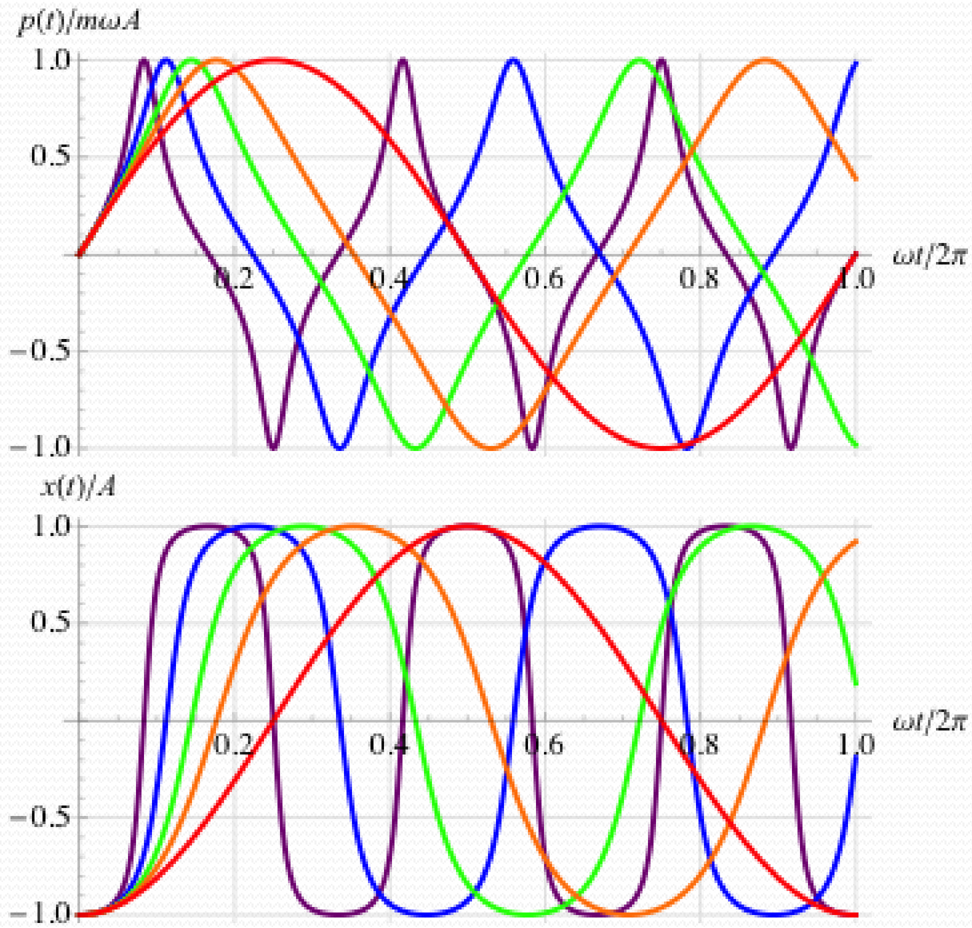
$$H = \frac{p^2}{2m} + \frac{1}{2} k x^2$$

Classical Equations of Motion :

$$\begin{cases} \dot{x} = \{x, H\} = \dfrac{1}{m}\left(1 + \beta p^2\right) p \\ \dot{p} = \{p, H\} = -k\left(1 + \beta p^2\right) x \end{cases}$$

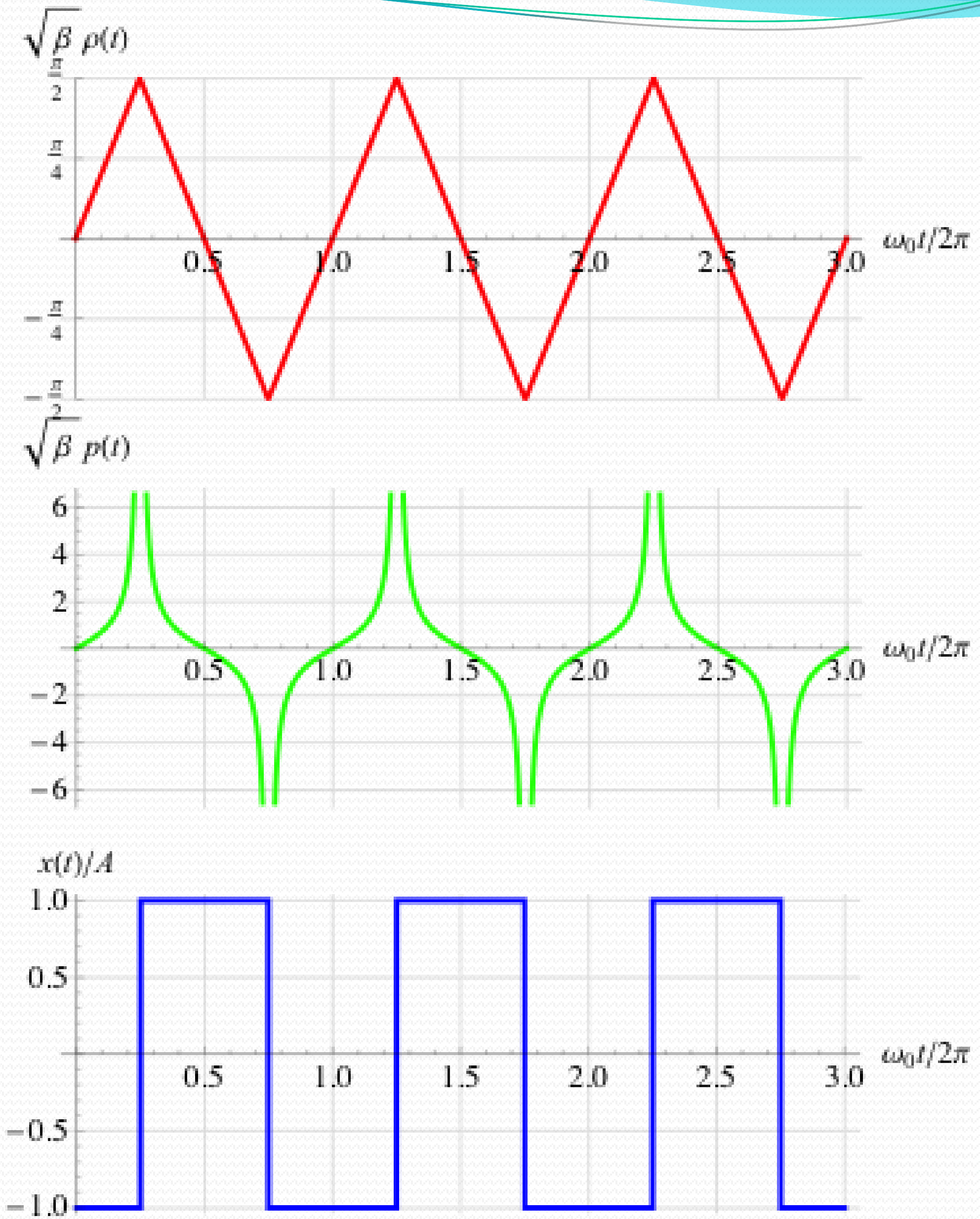
Time - dependence of $x$ and $p$ are different, but the trajectories in phase space are the same as the $\beta = 0$ case since the Hamiltonian is the same.

# m>0 case:



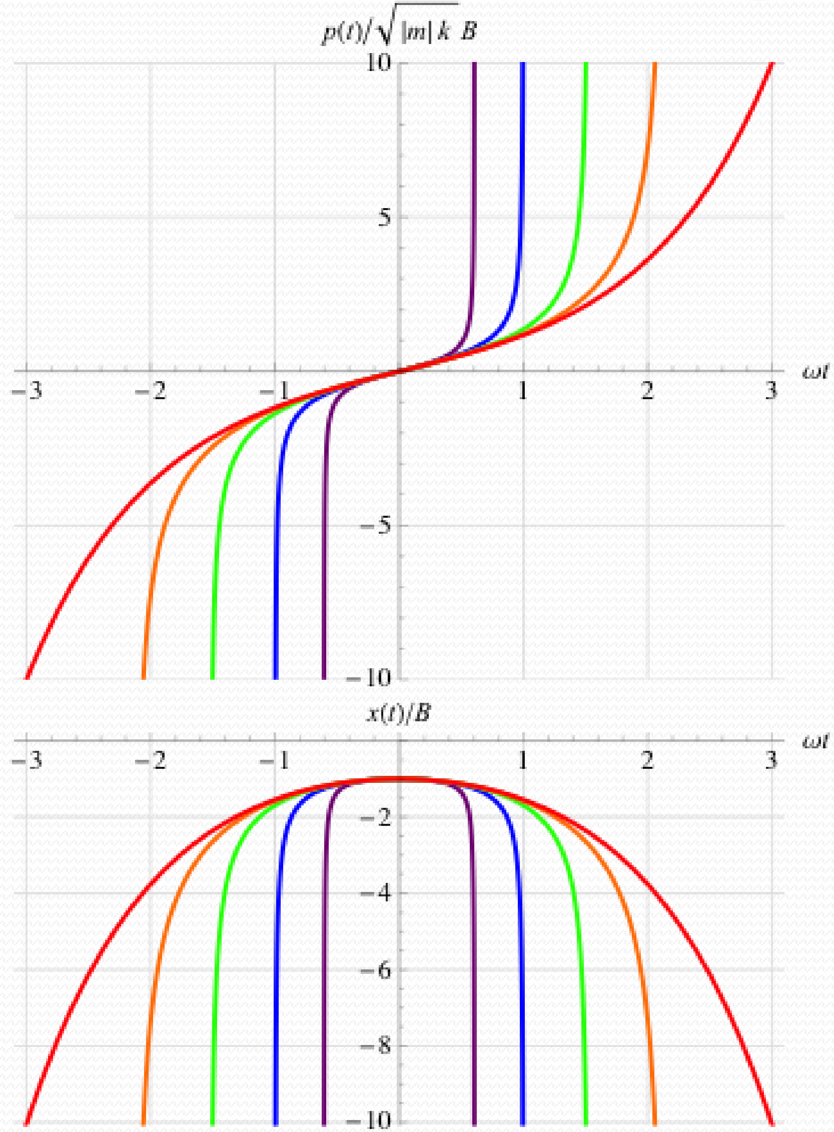
p(t)/mωA
1.0
0.5
−0.5
−1.0
0.2
0.4
0.6
0.8
1.0
ωt/2π
x(t)/A
1.0
0.5
−0.5
−1.0
0.2
0.4
0.6
0.8
1.0
ωt/2π

# m→limit:

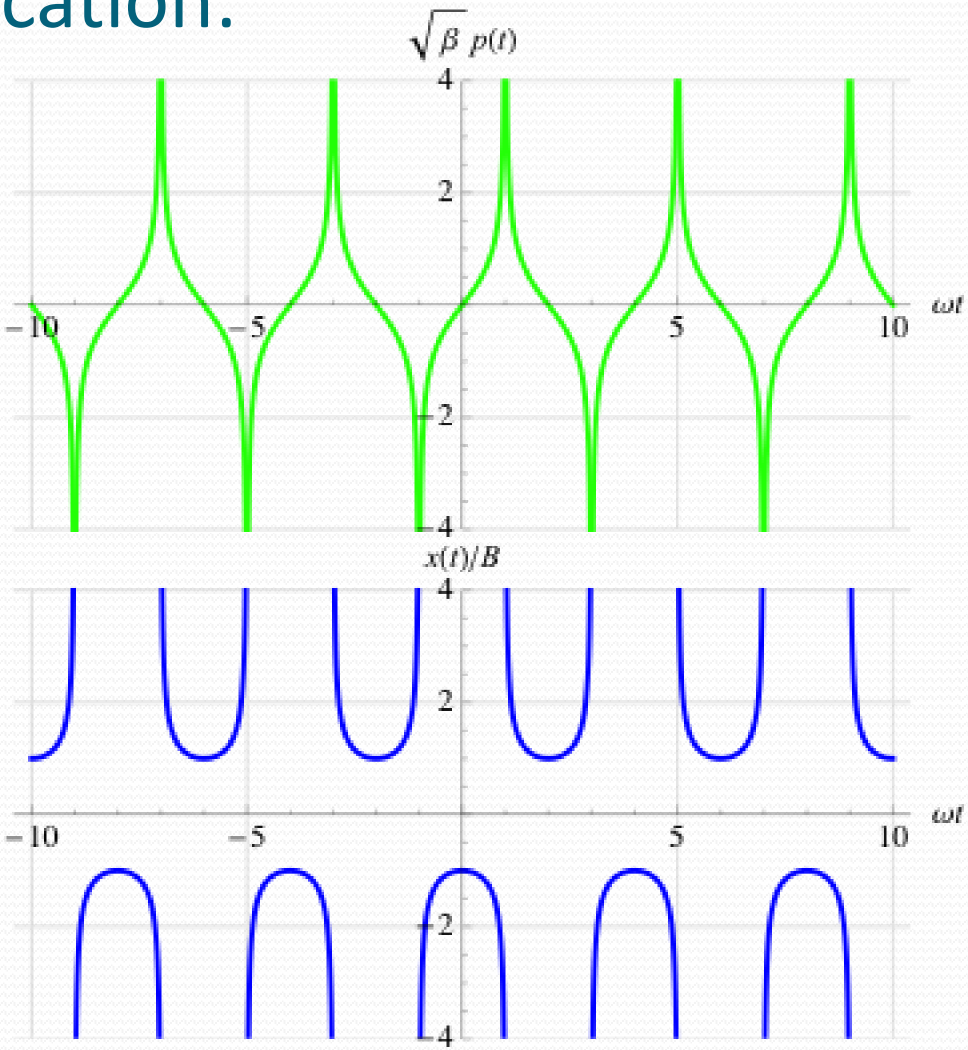


$\sqrt{\beta}\,\rho(t)$
$\frac{\pi}{2}$
$\frac{\pi}{4}$
$-\frac{\pi}{4}$
$-\frac{\pi}{2}$
$\omega_0 t/2\pi$
0.5
1.0
1.5
2.0
2.5
3.0
$\sqrt{\beta}\,p(t)$
6
4
2
−2
−4
−6
$\omega_0 t/2\pi$
$x(t)/A$
1.0
0.5
−0.5
−1.0
$\omega_0 t/2\pi$

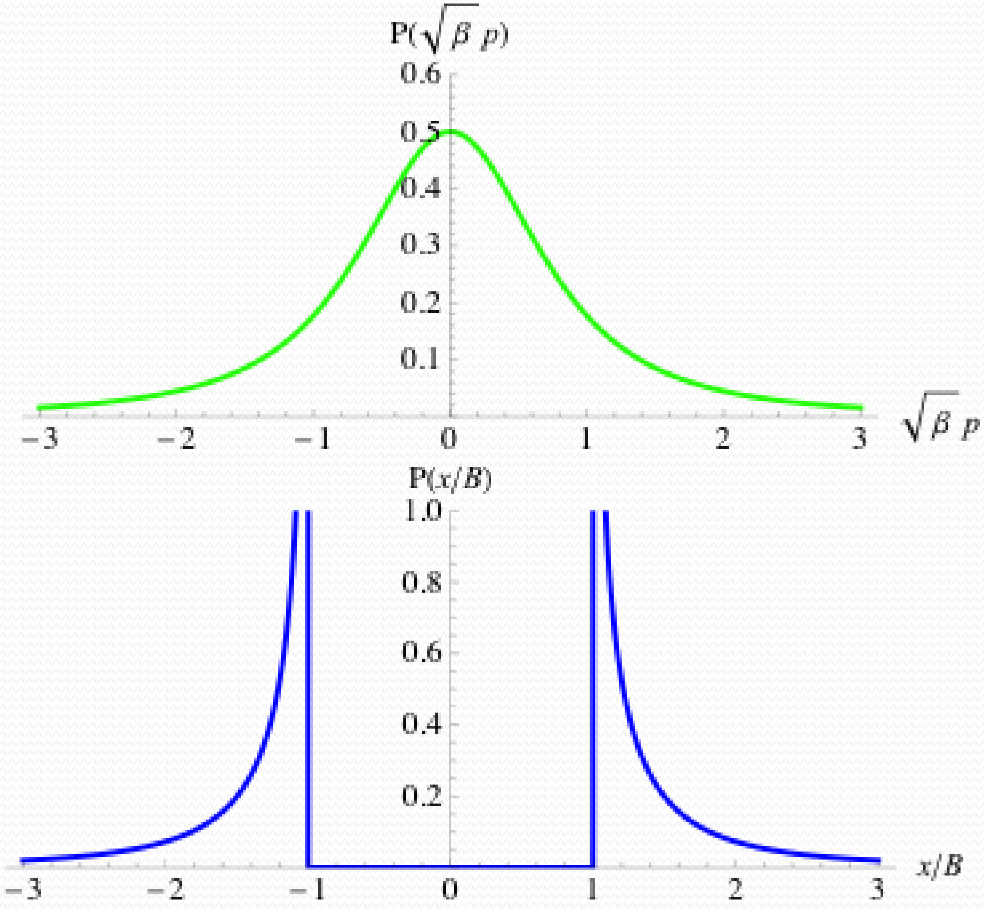
# m<0 case:



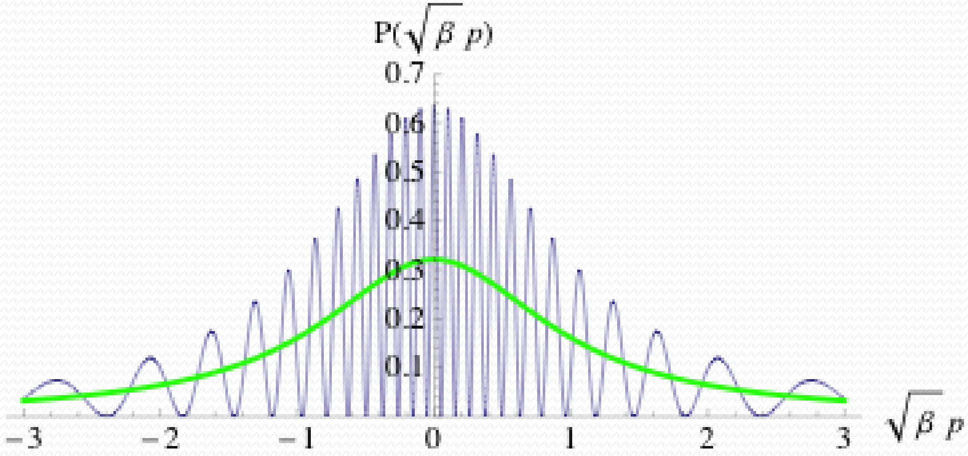
# Compactification:

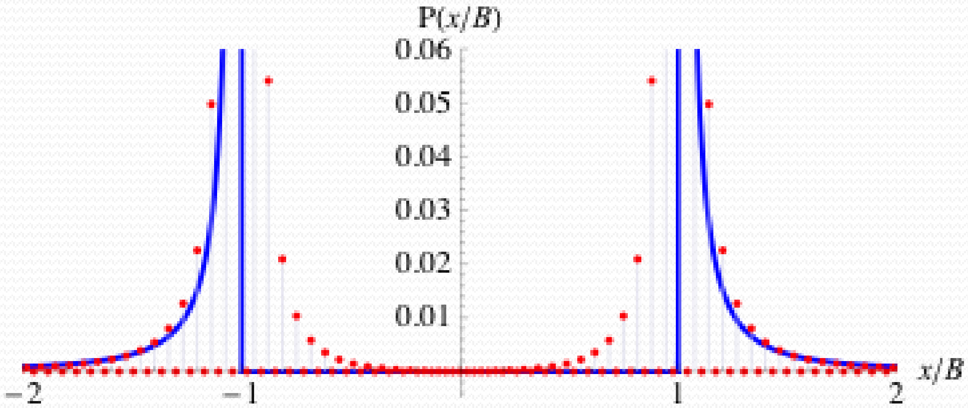


# Classical Probabilities:



# Comparison with Quantum Probabilities:



# Work in progress:

Other potentials:

$$V(x) = F|x|$$

$$V(x) = \begin{cases} Fx & (x > 0) \\ \infty & (x < 0) \end{cases}$$

- Discreate energy eigenstates have been found for the negative mass case.
- Uncertainties are difficult to calculate. What do we mean by $x=0$?

# Conclusions:

- The minimal length uncertainty relation allows discreate energy “bound” states for “inverted” potentials.
- In the classical limit, these “bound” states can be understood to be due to the finite time the particle spends near the phase-space origin.
- Particles move at arbitrary large velocites. Do the non-relativistic negative mass states correspond to relativistic imaginary mass tachyons?