QCD and the incomplete links between mass and gauge

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Abstract

The topic of 'mass and gauge' in QCD is taken up implying that despite much effort since early 'beginnings' there remain analytically unsolved questions. The work covered is unfinished and concentrates on the renewed analysis of some of these questions.

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1 - Conditions of enveloping local gauge invariance, integrability of field strengths from connections and boundary values in eventual conflict with lattice QCD

Connections and associated field strengths pertaining to a local, compact and semi-simple gauge group $\mathcal{G}$ $(\rightarrow SU3_c$ for QCD $)$ shall be called complete, if extendable to the full ring of representations of potential matter fields , irrespective of the actual presence of such fields in the full gauge theory . The requirement of continuity with respect to space time derivatives shall apply to classical configurations, as substrate of path integrals.

Let a general irreducible unitary representation of $\mathcal{G}$ of dimension $dim \, (\, \mathcal{D} \,)$ be denoted $\mathcal{D}$ with

(1)
$$
\begin{aligned}
& (\, D \, (\, g \,) \,)_{\alpha\beta} \in \mathcal{D} \; : \; \alpha \, , \, \beta \, = \, 1 \, , \cdots , \, dim \, (\, \mathcal{D} \,) \;\; ; \;\; g \, \in \, \mathcal{G} \\
& (\, D \, (\, g \,) \,)_{\alpha\beta} \, \rightarrow \, D \, (\, g \,)
\end{aligned}
$$

Then the $Lie \, (\, \mathcal{D} \,)$ associated connection is represented by the connection one form

(2)
$$
\begin{aligned}
& \left(W^{(1)} \, (\, \mathcal{D} \,) \right)_{\alpha\beta} \, = \, W^{\, r}_{\, \mu} \, (\, x \,) \, (\, d_{\, r} \, (\, \mathcal{D} \,) \,)_{\alpha\beta} \, d \, x^{\, \mu} \\
& W^{\, r}_{\, \mu} \, (\, x \,) \; : \; \textbf{real} \;\; ; \;\; r \, = \, 1, \, \cdots, \, dim \, (\, \mathcal{G} \,) \\
& \left(W^{(1)} \, (\, \mathcal{D} \,) \right)_{\alpha\beta} \; \rightarrow \; W^{(1)} \, (\, \mathcal{D} \,) \; \rightarrow \; W^{(1)} \, (\, | \, _{\mathcal{D}} \,) \\
& (\, d_{\, r} \, (\, \mathcal{D} \,) \,)_{\alpha\beta} \; \rightarrow \; d_{\, r} \, (\, \mathcal{D} \,) \; \rightarrow \; d_{\, r} \, = \, - \, d^{\, \dagger}_{\, r} \;\; ; \;\; r \, = \, 1 \, , \cdots , \, dim \, \mathcal{G}
\end{aligned}
$$

The antihermitian matrices $d_{\, r}$ in eq. 2 form a basis of $Lie \, (\, \mathcal{D} \,)$. →

$Lie\,(\,\mathcal{D}\,)$ **is aligned with the adjoint representation** $Lie\,(\,\mathcal{G}\,)$ **as explained below, but is conceived in an apparently simpler context all by itself through the exponential mapping and its inverse**

(3)
$$D\,(\,g\,)\,D\,(\,h\,)\;=\;D\,(\,g\,.\,h\,)\;\;;\;\;D\,(\,g\,)\,D^{\,\dagger}\,(\,g\,)\;=\;\P\,|_{\,dim\,(\,\mathcal{D}\,)\,\times\,dim\,(\,\mathcal{D}\,)}$$
$$D\,(\,g\,)\;\rightarrow\;D\;\;;\;\;Det\,(\,D\,)\;=\;1$$

The unimodularity of the matrices D follows from the requirement that $\mathcal{G}$ **be semi-simple , i.e. be a direct product of simple factor groups, none of which contain any continuous normal subgroups .**
The exponential mapping associates the linear space of antihermitian matrices $\equiv\;Lie\,(\,\mathcal{D}\,)$ **with the set of representation matrices** $\{\,D\,\}\;\equiv\;\mathcal{D}$

(4)
$$(\,\omega\,)_{\,\alpha\beta}\;\rightarrow\;\widehat{\omega}\;\in\;Lie\,(\,\mathcal{D}\,)\;\;;\;\;\widehat{\omega}^{\,\dagger}\;=\;-\,\widehat{\omega}$$
$$D\;=\;\exp\,\widehat{\omega}\;\;;\;\;\widehat{\omega}\;=\;\omega^{\,r}\,d_{\,r}\;\;,\;\;\omega^{\,r}\;:\;\textbf{real}\;\;;\;\;r\;=\;1\,,\cdots,\,dim\,\mathcal{G}$$

The precise definition of the matrix valued quantity $\widehat{\omega}$ **introduced in eq. 4 is given in eq. 6 below.**
The exponential mapping $Lie\,(\,\mathcal{D}\,)\;\rightarrow\;\mathcal{D}$ **as defined in eq. 4 is embedded into the one parameter abelian subgroup of** $\mathcal{D}$ **as represented restricting to** $\mathcal{D}$ **by associating first the adjoint representation and then also** $\mathcal{D}$ **with the ’notion of motion’**

(5)
$$\omega\;\rightarrow\;\tau\,\omega\;\;;\;\;\tau\;:\;\textbf{real}\;\;\rightarrow\;\;D\,(\,\tau\,;\,\omega\,)\;=\;exp\,(\,\tau\,\widehat{\omega}\,)$$
$$D\,(\,\tau_{\,1}+\,\tau_{\,2}\,;\,\omega\,)\;=\;D\,(\,\tau_{\,1}\,;\,\omega)\,D\,(\,\tau_{\,2}\,;\,\omega\,)\;=\;D\,(\,\tau_{\,2}\,;\,\omega)\,D\,(\,\tau_{\,1}\,;\,\omega\,)\qquad\rightarrow$$

Interpreting the variable τ as representing the time development of the group element $g \in \mathcal{G}$ a differential eq. follows from eq. 5

$$\frac{d}{d\tau} D(\tau; \omega) = \left. \frac{D(d\tau; \omega) - ¶}{d\tau} \right|_{d\tau \to 0} D(\tau; \omega) \equiv \widehat{\omega} D(\tau; \omega)$$

(6) $$\frac{D(dt; \omega) - ¶}{dt} \equiv \left. \frac{d}{dt} D(t; \omega) \right|_{t=0} \equiv \widehat{\omega} = \omega^{r} d_{r} \in Lie(\mathcal{D})$$

$$\widehat{\omega} = \widehat{\omega}(\mathcal{D}) = \dot{D}(0; \omega) \quad ; \quad \dot{} = \frac{d}{d\tau \text{ or } dt}$$

The relations in eqs. 4 - 6 define the real coordinates ω^{r} and antihermitian base matrices d_{r} – both independent of τ – the latter forming the matrix valued structure $\widehat{\omega}(\mathcal{D})$, identified for shortness of notation with $Lie(\mathcal{D})$. The solution to the differential equation in eq. 6 with initial condition $D(0; \omega) = ¶$ is given in eq. 5 .

1 a - Derivations from continuous coordinate transformation groups representing $\mathcal{G}$, fibre manifolds and irreducible submanifolds at the origin of conditions on complete connections

This section shall contain a most concise résumé of those notions inherent to the mathematics underlying compact semi-simple Lie groups as is necessary to infer conditions on field theoretical connections as announced in points 3) and 4) of the introduction, whereby most derivations are omitted. A minimum of historical and textbook references shall be given here [11-1951-1961] - [14-1963] . The results presented below are based on my treatment in ref. [15-2010] .

3 a-1 - general fibres $\longrightarrow$ irreducible ones $\simeq$ homogenous spaces [13-1962]

A general fibre manifold $\mathcal{F}$ (called $\mathcal{B}$ in ref. [15-2010]) with structure group $\mathcal{G}$ is required to allow a continuous representation of $\mathcal{G}$ by coordinate transformations of $\mathcal{F} \longrightarrow \mathcal{F}$. The set of these coordinate transformations shall be denoted $\mathcal{T}_{\mathcal{F}} = \left\{ \bigcup_{a} T_{a} \, ; \, a \in \mathcal{G} \right\}_{\mathcal{F}}$

$$\phi = \left(\phi^{1} , \cdots , \phi^{F} \right) \quad ; \quad F = dim \left(\mathcal{F} \right) \quad : \quad \text{coordinates on } \mathcal{F}$$

(7)
$$T_{a} \, \phi = \psi \left(\phi ; a \right) \quad ; \quad \psi^{j} = \psi^{j} \left(\phi^{1} , \cdots \phi^{F} ; a^{1} , \cdots , a^{G} \right)$$

$$j = 1 , \cdots , F \quad ; \quad G = dim \, \mathcal{G}$$

with suitable continuity / differentiability requirements for the functions $\psi \left(\phi ; a \right)$ defined in eq, 7 . $\longrightarrow$

The group property of $\mathcal{T}_{\mathcal{F}}$ then translates to, by the assciative property of coordinate transformations

(8) $$T_b\,(T_a\,\phi) = (T_b\,T_a)\,\phi\;;\;T_b\,T_a = T_{b\,a} \text{ with } b\,a = b\,.\,a \;:\; \text{group multiplication} \in \mathcal{G}$$

$$\text{in coordinates : } \psi\,(\psi\,(\phi\,;\,a)\,;\,b) = \psi\,(\phi\,;\,b\,.\,a)\;;\;\psi\,,\,\phi \in \mathcal{F}\;;\;b\,,\,a \in \mathcal{G}$$

The group transformation properties on $\mathcal{G}$ enter implicitely into the $\mathcal{T}_{\mathcal{F}}$ ones, as shown in eq. 8

(9) $$b\,.\,a = @\,(b\,,a)\;;\quad @^{\,\nu} = @^{\,\nu}\,(b^{\,1}\,,\cdots b^{\,G}\,;\,a^{\,1}\,,\cdots\,,\,a^{\,G})$$
$$\nu = 1\,,\cdots\,,\,G$$

We use the symbol @ (instead of c) to denote the G functions $@^{\,\nu}\,,\;\nu = 1\,,\cdots\,,\,G$ determining multiplication on $\mathcal{G}$ in order to freely use the symbols $a\,,\,b\,,\,c\,,\,\cdots$ for group elements $\in \mathcal{G}$.

We display here the eligibility of $\mathcal{G}$ as a special fiber manifold using left multiplication first, renaming the fiber variable h and the variables $a\,,\,b\,,\,c\,\cdots$ for $T \in \mathcal{T}_{\mathcal{G}\,L}$

$$T_L \rightarrow T \in \mathcal{T}_{\mathcal{G}\,L} \rightarrow T_a\,h = @\,(a\,;\,h)$$

(10) $$T_b\,(T_a\,h) = (T_b\,T_a)\,h = @\,(b\,;\,@\,(a\,;\,h))$$

$$@\,(b\,;\,@\,(a\,;\,h)) = b\,.\,@\,(a\,;\,h) = b\,.\,(a\,.\,h) = (b\,.\,a)\,.\,h$$

$\longrightarrow$

Clearly the choice $\mathcal{F} = \mathcal{G}$ is singled out, since there exists in this case, independently of the left multiplication transfomation group – $\mathcal{T}_{\mathcal{G}\,L}$ – also the right multiplication one – $\mathcal{T}_{\mathcal{G}\,R}$ with the associations

(11)
$$
\begin{aligned}
& T_R \rightarrow T \in \mathcal{T}_{\mathcal{G}\,R} \rightarrow T_a\, h = @\left(h \,;\, a^{-1} \right) \\
& T_b \left(T_a\, h \right) = \left(T_b\, T_a \right) h = @\left(@\left(h \,;\, a^{-1} \right) ;\, b^{-1} \right) \\
& @\left(@\left(h \,;\, a^{-1} \right) ;\, b^{-1} \right) = @\left(h \,;\, a^{-1} \right) . \, b^{-1} \\
& \qquad\qquad = \left(h \,.\, a^{-1} \right) . \, b^{-1} = h \,.\, \left(b \,.\, a \right)^{-1}
\end{aligned}
$$

Requiring an inversion symmetry on the tangent spaces of $\mathcal{F}$ – following ref. [13-1962] – allows to identify the irreducible parts of the fibre manifolds on which $\mathcal{T}_{\mathcal{F}}$ acts transitively to the (right or left -) coset spaces

(12)
$$
\mathcal{F} \rightarrow \mathcal{F}_{irr} \rightarrow \mathcal{G} / \mathcal{H} \;\; ; \;\; \mathcal{H} \; : \; \textbf{Lie subgroup of } \; \mathcal{G}
$$

With the identification $\mathcal{F} \rightarrow \mathcal{F}_{irr}$ in eq. 12 and choosing right cosets for definitensess $\mathcal{T}_{\mathcal{F}}$ becomes

(13)
$$
\begin{aligned}
& \mathcal{F} \ni \phi = h \sim h \,.\, h_R \;\forall\, h_R \in \mathcal{H} \subset \mathcal{G} \\
& \mathcal{T}_{\mathcal{F}} \ni T_a \;\; , \;\; \mathcal{T}_{\mathcal{G}\,L} \ni T_a^{'} \; : \; T_a\, \phi = \left(T_a^{'}\, h \right) \sim a \,.\, h \,.\, h_R \;\forall\, h_R \in \mathcal{H}
\end{aligned}
$$

$\rightarrow$

The classification of fibre manifolds according to cosets $\mathcal{G} \, / \, \mathcal{H}$ as specified in eq. 12 allows a graduation of fibres $\mathcal{F}$:
regularity conditions for connections as described in eq. 2 compatible with $\mathcal{F}_1 = \mathcal{G} \, / \, \mathcal{H}_1$ are not less restrictive than relative to $\mathcal{F}_2 = \mathcal{G} \, / \, \mathcal{H}_2$ for $\mathcal{H}_1 \subseteq \mathcal{H}_2$. We denote this graduation as

(14) $$W_{\mathcal{F}_1}^{(1)} (\mathcal{D}) \succeq W_{\mathcal{F}_2}^{(1)} (\mathcal{D}) \text{ for } \mathcal{F}_1 \succeq \mathcal{F}_2 \equiv \mathcal{H}_1 \subseteq \mathcal{H}_2$$

At this stage the connection 1-forms are still defined for a given matrix representation $Lie (\mathcal{D})$. The direct anchoring of connections to the fibre manifolds $\mathcal{F}$ as considered in this subsection will be defined after its conclusion .
While $\mathcal{H}_0 = \{ ¶ \}$ – i.e. consisting only of the unit element of $\mathcal{G}$ – is strictly speaking not a Lie (sub)group of $\mathcal{G}$, we adjoin the group $\mathcal{G}$ as the maximal fibre manifold, with both tranformation groups $\mathcal{T}_{\mathcal{G} \, L}$ and $\mathcal{T}_{\mathcal{G} \, R}$, as defined in eqs. 9 - 11 . Accordingly we adjoin as unique discrete subgroup $\mathcal{H}_0 = \{ ¶ \}$ to the set of genuine Lie subgroups $\bigcup (\mathcal{H})$. Then eq. 14 takes the form

(15) $$\mathcal{F}_{max} \to \mathcal{G} \; ; \; \mathcal{F} = \mathcal{G} \, / \, \mathcal{H} \longrightarrow W_{\mathcal{G}}^{(1)} (\mathcal{D}) \succeq W_{\mathcal{F}}^{(1)} (\mathcal{D}) \; \forall \; \mathcal{F}$$

This subsection $($1 a-1$)$ serves to define the selection of fibre manifolds and among them the maximal one according to eqs. 12 - 15 characterizing complete connection as introduced in eqs. 1 - 2 .

1 b - Ordered differentials : Killing fields on the group fibre manifold $\mathcal{G}$ with transformation group(s) $\mathcal{T}_{\mathcal{G}\ L(R)}$ and as derivations on associated induced representations

1 b-1 - adjoint representation from infinitesimal group coordinates and the Lie algebra

Coordinates of group elements of $\mathcal{G}$ in the sense of a classical manifold shall be denoted with the same symbol as the group elements as such.

(16) $$\mathcal{G} \rightarrow \mathcal{M} \,|_{\mathcal{G}} \quad \sim \quad h \in \mathcal{G} \rightarrow h = (h^{\nu}) = \left(h^{1} , \cdots , h^{G} \right) \ \Big|_{\mathcal{M}}$$

The suffix $|_{\mathcal{M}}$ in eq. 16 signals that h as coordinates on $\mathcal{M}$ are not unique. It shall be suppressed for simplicity with exceptions granted to avoid confusion .
The unit elements $e \rightarrow T_{e} \left(\doteq \P \right)$ – with $T \in \mathcal{T}$ arbitrary – have the property

(17) $$e \, . \, a = a \, . \, e = a \quad \leftrightarrow \quad T_{a} T_{e} = T_{e} T_{a} = T_{a} \ (\forall \, a)$$

It is no loss of generality to assign the neutral element e the coordinates in $\mathcal{G}$

(18) $$e = \left(e^{1} , \cdots , e^{G} \right) \ ; \ e^{\nu} = 0 \ , \ \nu = 1 , \cdots , G$$

The infinitesimal neighbourhood of e forms the substrate of tangent space at e . Here we anchor the exponential mapping discussed with respect to a finite dimensional linear representation $\mathcal{D} (\mathcal{G}) \rightarrow Lie (\mathcal{D})$ defined in eqs. 3 - 6 in $Lie (\mathcal{G})$ $\longrightarrow$

with the substitutions

$$\omega \; = \; (\, \omega^{\,\nu} \,) \; \in \; \mathcal{G} \quad \longrightarrow \quad d\,\tau\,\omega \; \in \; \mathcal{G} \, |_{\,\mathcal{M}} \; ; \; d\,\tau\,\omega \quad \longrightarrow \quad T_{\,d\,\tau\,\omega} \; \in \; \mathcal{T} \; (\, \textbf{arbitrary} \,)$$

(19) $$\left. \frac{T_{\,d\,\tau\,\omega} \; - \; \P}{d\,\tau} \right|_{d\tau \rightarrow 0} \quad \longrightarrow \quad \widehat{\omega} \; = \; \widehat{\omega} \; \left(\, \mathcal{T}_{\,(1)} \, \right) \; = \; \omega^{\,\nu} \, \widehat{I}_{\,\nu} \quad ; \quad \widehat{I}_{\,\varrho} \; \in \; \mathcal{T}_{\,(1)}$$

$$\widehat{\omega} \quad \longrightarrow \quad \omega \; \in \; Lie \, (\, \mathcal{G} \,) \, |_{\,\mathcal{M}^{\,(1)} \, (\, e \,)}$$

In eq. 19 we have introduced two notions

1) $\mathcal{T}_{\,(1)}$: the set of differential quotients of transformations $T_{\,h} \; \in \; \mathcal{T}_{\,\mathcal{G}}$.

2) $\mathcal{M}^{\,(1)} \, (\, h \,) \, ; \, h \; \in \; \mathcal{G} \, |_{\,\mathcal{M}}$: the tangent spaces of $\mathcal{M}$ at $h \; \in \; \mathcal{G}$.

All differential operations on the ordered fibre manifolds admitted as described in subsection 1 a-1 in particular on the group manifold itself are allowed by the condition of differentiablity – once for $\mathcal{T}_{\,(1)}$ and $\bigcup_{\,h} \; \mathcal{M}^{\,(1)} \, (\, h \,)$ – imposed 'eo ipso' on fibres and Lie groups .
With the definitions given in eqs. 16 - 19 we set out next to find the adjoint representation . $\longrightarrow$

The finite adjoint matrix representation denoted $\mathcal{A}$ arises directly from the group tranformations on $\mathcal{G} = \mathcal{M}$

$$d\,c = a\,.\,d\,\tau\,\omega\,.\,a^{-1}\,|_{\mathcal{M}} \quad ; \quad d\,c = d\,\tau\,\Big(\omega^{\,\prime}\,(\,a\,;\,\omega\,)\Big)$$

(20) $$\omega^{\,\prime\,\nu}\,(\,a\,,\,\omega\,) = Ad\,(\,a\,)^{\nu}{}_{\mu}\,\omega^{\,\mu} \quad ; \quad {}^{\nu},\,{}_{\mu} = 1\,,\,\cdots\,,\,G \quad \rightarrow \quad \omega^{\,\prime} = Ad\,(\,a\,)\,\omega$$

$$Ad\,(\,b\,)\,Ad\,(\,a\,) = Ad\,(\,b\,.\,a\,)\,|_{\mathcal{M}} \quad ; \quad \mathcal{A} = \left\{\bigcup_{a\,\in\,\mathcal{G}}\,Ad\,(\,a\,)\right\}$$

The real matrices $Ad\,(\,h\,)$ depend on the coordinates chosen on $\mathcal{M}$ and are for general choices not unitary, i.e. not orthogonal , but depend continuously on the coordinates h . The condition that $\mathcal{G}$ be semi-simple implies

(21) $$Det\,Ad\,(\,h\,) = 1 \;\; \forall\,h \in \mathcal{G}$$

At this point we extend the order of differentials to 2 which allows to introduce the Lie algebra .
To this end we use the exponential mapping for the operator valued quantities defined on the second line of eq. 19 ordering them as a pair

(22) $$\widehat{\omega}_{\,(k)} = (\,\omega^{\,\nu}\,)_{\,(k)}\,\widehat{I}_{\,\nu}\,,\,k = 1,2 \quad ; \quad \widehat{I}_{\,\varrho} \in \mathcal{T}_{(1)}$$

We associate an initially finite 'time like' quantity $\tau_{\,1}$ with $\widehat{\omega}_{\,(1)}$ and a first order differential $d\,\tau_{\,2}$ with $\widehat{\omega}_{\,(2)}$ $\longrightarrow$

and consider the equivalence exponential mapping and the differential equation with respect to τ_1 , valid for finite τ_1

$$\widehat{\Omega}(\tau_1) = \exp(\tau_1 \widehat{\omega}_{(1)}) \left(¶ + d\tau_2 \widehat{\omega}_{(2)} \right) \exp(-\tau_1 \widehat{\omega}_{(1)})$$

$$\text{(23)} \qquad \frac{d}{d\tau_1} \widehat{\Omega}(\tau_1) = \exp(\tau_1 \widehat{\omega}_{(1)}) \, d\tau_2 \left[\widehat{\omega}_{(1)}, \widehat{\omega}_{(2)} \right] \exp(-\tau_1 \widehat{\omega}_{(1)})$$

$$\left[\widehat{\omega}_{(1)}, \widehat{\omega}_{(2)} \right] = \widehat{\omega}_{(1)} \widehat{\omega}_{(2)} - \widehat{\omega}_{(2)} \widehat{\omega}_{(1)}$$

In eq. 23 the operator valued commutator $\left[\widehat{\omega}_{(1)}, \widehat{\omega}_{(2)} \right]$ naturally appears .
Next we let also τ_1 become infinitesimal $\tau_1 \rightarrow d\tau_1$ and expand $\widehat{\Omega}(d\tau_1)$ up to second order in the differentials $d\tau_k \, ; \, k = 1, 2$

$$\text{(24)} \quad \widehat{\Omega}(d\tau_1) = \left\{ \begin{array}{c} \left(¶ + d\tau_1 \widehat{\omega}_{(1)} + \frac{1}{2} (d\tau_1)^2 \widehat{\omega}^{\,2}_{(1)} \right) \times \\ \times \left(¶ + d\tau_2 \widehat{\omega}_{(2)} \right) \times \\ \times \left(¶ - d\tau_1 \widehat{\omega}_{(1)} + \frac{1}{2} (d\tau_1)^2 \widehat{\omega}^{\,2}_{(1)} \right) \end{array} \right\} = \left\{ \begin{array}{c} ¶ + \\ d\tau_1 \, d\tau_2 \left[\widehat{\omega}_{(1)}, \widehat{\omega}_{(2)} \right] \end{array} \right\}$$

The absence of a term $\propto (d\tau_1)^2$ on the right hand side of eq. 24 is due to the exponential mapping . $\longrightarrow$

It follows that the finite commutator $\left[\, \widehat{\omega}_{\,(1)} \, , \, \widehat{\omega}_{\,(2)} \, \right]$ upon the combined exponential mapping, defined in eq. 25 below, generates a commuting one parameter family of transformations . by the semi-simple condition on $\mathcal{G}$ a one parameter commutative subgroup of $\mathcal{G}$

(25)

$$T\,(\,\vartheta\,) \;=\; T_{\,h\,(\,\vartheta\,)} \;=\; \exp\,\left(\,\vartheta\,\left[\,\widehat{\omega}_{\,(1)}\,,\,\widehat{\omega}_{\,(2)}\,\right]\,\right) \quad ; \quad h(\,\vartheta\,) \,\in\, \mathcal{G}\,\forall\,\vartheta \quad \text{with}$$

$$h(\,\vartheta_{\,1} \,+\, \vartheta_{\,2}\,) \;=\; h\,(\,\vartheta_{\,1}\,)\,.\,h\,(\,\vartheta_{\,2}\,) \;=\; h\,(\,\vartheta_{\,2}\,)\,.\,h\,(\,\vartheta_{\,1}\,)$$

The Lie algebra relation follows, first on the level of all transformation groups $\mathcal{T}_{\,\mathcal{F}}$ defined in section 1 a

(26)

$$\left[\,\widehat{\omega}_{\,(1)}\,,\,\widehat{\omega}_{\,(2)}\,\right] \;=\; \omega^{\,\varrho}_{\,(1)}\,\omega^{\,\sigma}_{\,(2)}\,\left[\,\widehat{I}_{\,\varrho}\,,\,\widehat{I}_{\,\sigma}\,\right] \quad ; \quad \widehat{I}_{\,\varrho}\,,\,\widehat{I}_{\,\sigma} \,\in\, \mathcal{T}_{\,(1)} \;\leftrightarrow\; \text{eq. 19}$$

$$\downarrow$$

$$\left[\,\widehat{I}_{\,\varrho}\,,\,\widehat{I}_{\,\sigma}\,\right] \;=\; f^{\,\chi}{}_{\,\varrho\,\sigma}\,\widehat{I}_{\,\chi}$$

$$f^{\,\chi}{}_{\,\varrho\,\sigma} \;=\; -\,f^{\,\chi}{}_{\,\sigma\,\varrho} \quad : \quad \text{structure constants of } \mathcal{G} \quad ; \quad \chi\,,\,\varrho\,,\,\sigma \;=\; 1\,,\,\cdots\,,\,G$$

Clearly, in order to implement the operator relations on the level of $\mathcal{T}_{\,\mathcal{G}\;L\,(R)}$ to the coordinate multiplication as given by the functions $@\,(\,b\,;\,a\,) \;=\; b\,.\,a$ for $b\,,a\,\in\,\mathcal{G}$ as defined in eq. 9 at least twofold partial differentiability of $@$ is necessary . $\longrightarrow$

This further becomes threefold partial differentiability to safeguard the operator Jacobi identity a necessary condition to ensure regular convergence of nested exponential mappings of all allowed transformation groups $\mathcal{T}_{\mathcal{F}}$

$$\left[\widehat{I}_{\alpha} , \left[\widehat{I}_{\beta} , \widehat{I}_{\gamma} \right] \right] + (\alpha\beta\gamma \rightarrow \gamma\alpha\beta) + (\alpha\beta\gamma \rightarrow \beta\gamma\alpha) = 0 \;\; \forall \;\; \alpha, \beta, \gamma$$

(27) $$\longrightarrow \quad f^{\sigma}{}_{\alpha\,\varrho} f^{\varrho}{}_{\beta\,\gamma} + f^{\sigma}{}_{\gamma\,\varrho} f^{\varrho}{}_{\alpha\,\beta} + f^{\sigma}{}_{\beta\,\varrho} f^{\varrho}{}_{\gamma\,\alpha} = 0$$

$$f^{\sigma}{}_{\alpha\,\gamma} \doteq (\xi_{\alpha})^{\sigma}{}_{\gamma}$$

With the matrix substitutions $(\xi_{\alpha})^{\sigma}{}_{\gamma} \rightarrow \xi_{\alpha} \;\; ; \;\; {}^{\sigma} , \alpha , \gamma = 1 , \cdots , G$ **the relation on the second line of eq. 27 can be cast into the form**

(28) $$\left([\xi_{\alpha} , \xi_{\beta}] = f^{\chi}{}_{\alpha\,\beta} \xi_{\chi} \right)^{\sigma}{}_{\gamma} \sim \left[\widehat{I}_{\alpha} , \widehat{I}_{\beta} \right] = f^{\chi}{}_{\alpha\,\beta} \widehat{I}_{\chi} \;\; \leftrightarrow \;\; \textbf{eq. 26}$$

The equivalence relation in eq. 28 $\xi_{\alpha} \sim \widehat{I}_{\alpha}$ **allows to reconstruct the adjoint representation** $\mathcal{A}$ **defined in eqs. 20 - 21 by the finite dimensional exponential mappings**

$$(Ad (h (\tau ; \omega)) = \exp (\tau \, \omega^{\alpha} \xi_{\alpha}))^{\sigma}{}_{\gamma} \;\; ; \;\; h (\tau ; \omega) \in \mathcal{G}$$

(29) $$h (\tau_{1} + \tau_{2} ; \omega) = h (\tau_{1} ; \omega) \, . \, h (\tau_{2} , \omega)$$

$\omega \in$ **tangent space of** $\mathcal{G}$ **at** e

→

Coordinates on $\mathcal{G}$, for compact, semi-simple $\mathcal{G}$ can be chosen such, that all finite dimensional representations $\mathcal{D}\,(\,\mathcal{G}\,)$ are unitary, unimodular as shown (e.g.) in refs. [11-1951-1961] , [15-2010] .
From the present (sub-) perspective we then infer from eq. 29

(30)
$$\begin{array}{l} h \;=\; h\,(\,\tau\,;\,\omega\,) \;\rightarrow\; Ad\,(\,h\,)\;(\,Ad\,(\,h\,)\,)^{\,T} \;=\; \P_{\,G \times G} \;\longrightarrow \\ (\,\xi_{\,\alpha}\,)^{\,\gamma}_{\;\;\beta} \;=\; -\;(\,\xi_{\,\alpha}\,)^{\,\beta}_{\;\;\gamma} \quad ; \quad (\,\xi_{\,\alpha}\,)^{\,\gamma}_{\;\;\beta} \;=\; f^{\,\gamma}_{\;\;\alpha\,\beta} \end{array}$$

In eq. 30 $^{\,T}$ denotes matrix transposition.
The identifications in the relations on the second line of eq. 30 justify the substitutions – after unitarity of all finite dimensional representations of $\mathcal{G}$ is achieved –

(31)
$$(\,\xi_{\,\alpha}\,)^{\,\gamma}_{\;\;\beta} \rightarrow (\,\xi_{\,\alpha}\,)_{\,\gamma\,\beta} \quad \textbf{and} \quad f^{\,\gamma}_{\;\;\alpha\,\beta} \rightarrow f_{\,\gamma\alpha\beta}$$

With the substitutions defined in eq. 31 we rewrite eq. 30

(32)
$$\begin{array}{rcl} (\,\xi_{\,\alpha}\,)_{\,\gamma\,\beta} \;=\; -\;(\,\xi_{\,\alpha}\,)_{\,\beta\,\gamma} \;\longleftrightarrow\; f_{\,\gamma\alpha\beta} & = & -\,f_{\,\beta\alpha\gamma} \\ & & -\,f_{\,\gamma\beta\alpha} \end{array}$$

The commutator definition in eq. 26 yields the second relation for the structure constants in eq. 32 .
As a consequence of the two independent antisymmetry conditions for the structure constants $f_{\,\alpha\beta\gamma}$ the latter are totally antisymmetric with respect to their three indices . $\longrightarrow$

Recapitulation of characteristics of the Lie algebra and adjoint representation $\left(\text{eqs. 30 - 32} \right)$

The orthogonal, adapted adjoint representation is rewritten in 3 equations below, using the (standard) symbols or definitions

$$\left(\xi_{\alpha} \right)_{\gamma \beta} \doteq \left(ad_{(\alpha)} \right)_{\gamma \beta} = f_{\gamma \alpha \beta}$$

$$ad \left(\omega \right) = \omega^{\varrho} \, ad_{(\alpha)} \; ; \; \omega = \left(\omega^{1} , \cdots , \omega^{G} \right) \in \text{ tangent space of } \mathcal{G} \text{ at } e$$

(33) $$\widehat{\omega} = \omega^{\varrho} \, \widehat{I}_{\varrho} \; ; \; \widehat{\omega} \in \mathcal{T}_{(1)} \text{ relative to general } \mathcal{T}_{\mathcal{G}} \text{ as defined in eq. 19}$$

$$\mathcal{D} : d \left(\omega \right) = \omega^{\varrho} \, d_{(\varrho)} \in Lie \left(\mathcal{D} \right) \; d_{\varrho} \longrightarrow d_{(\varrho)} \text{ from eq. 2}$$

$$d \left(\omega \right) = \left(d \left(\omega \right) \right)_{rs} \; ; \; r, s = 1 , \cdots , dim \left(\mathcal{D} \right)$$

Lie algebra relations depend for all representations on the universal structure constants of $\mathcal{G}$

$$\left[ad_{(\alpha)} , ad_{(\beta)} \right] = f_{\alpha\beta\,\gamma} \, ad_{(\gamma)} \quad , \quad \left[d_{(\alpha)} , d_{(\beta)} \right] = f_{\alpha\beta\,\gamma} \, d_{(\gamma)}$$

(34) $$\left[\widehat{I}_{\alpha} , \widehat{I}_{\beta} \right] = f_{\alpha\beta\gamma} \, \widehat{I}_{\gamma}$$

$$f_{\alpha\beta\gamma} \; : \; \text{totally antisymmetric with respect to } \alpha\beta\,\gamma$$

$\longrightarrow$

Respective exponential mapping on the adjoint representation $\mathcal{A}$, on a general finite dimensional unitary representation $\mathcal{D}$ as well as on all operator valued transformation group representation $\mathcal{T}_{\mathcal{G}}$ determine on $\mathcal{G}$ a commuting one parameter family of group elements $h\,(\,\tau\,;\,\omega\,)\,\in\,\mathcal{G}$

(35) $$\left\{\begin{array}{c}\mathcal{A}\\ \mathcal{D}\\ \mathcal{T}_{\mathcal{G}}\end{array}\right\}\quad h\,(\,\tau\,;\,\omega\,)\,\leftarrow\,\left\{\begin{array}{lcl}\exp\,(\,\tau\,ad\,(\,\omega\,)\,) & = & Ad\,(\,h\,(\,\tau\,;\,\omega\,)\,)\\ \exp\,(\,\tau\,d\,(\,\omega\,)\,) & = & D\,(\,h\,(\,\tau\,;\,\omega\,)\,)\\ \exp\,(\,\tau\,\widehat{\omega}\,) & = & T_{h\,(\,\tau\,;\,\omega\,)}\end{array}\right\}$$

$$h\,(\,\tau_1\,+\,\tau_2\,;\,\omega\,)\,=\,h\,(\,\tau_1\,;\,\omega\,)\,.\,h\,(\,\tau_2\,;\,\omega\,)$$

The construction of this family – $h\,(\,\tau\,;\,\omega\,)\,\in\,\mathcal{G}$ – indicated as $h\,(\,\tau\,;\,\omega\,)\,\leftarrow$ in eq. 35 corresponds to a system of G first order differential equations with respect to τ , subject of the subsection 1 b-2 below .

We recall that up to this point threefold partial differentiability is required as regularity condition on the group transformation functions $(\,b\,.\,a\,)^{\,\nu}\,=\,@^{\,\nu}\,\left(b^{\,1}\,,\,\cdots\,b^{\,G}\,;\,a^{\,1}\,,\,\cdots\,,\,a^{\,G}\right)$.

This ends the present subsection $($ 1 b-1 $)$.

1 b - Killing fields on the group fibre manifold $\mathcal{G}$ with transformation group(s) $\mathcal{T}_{\mathcal{G}\ L(R)}$ – continued

An excellent exposition can be found in ref. [10-1941-1986] .

Here we follow the thread laid out in ref. [15-2010] starting from eqs. 9 , 10 defining the action of $T_a \in \mathcal{T}_{\mathcal{G}\ L}$ on the group manifold $\mathcal{G}$, repeated below

(36)
$$\begin{aligned}
& T_a \in \mathcal{T}_{\mathcal{G}\ L} \ ; \ a , h = (a^{\nu}) , (h^{\nu}) \ ; \ \nu = 1, \cdots, G \in \mathcal{G} \ \rightarrow \\
& T_a h = a . h = @ (a ; h) \ \longleftrightarrow \ (a . h)^{\nu} = @^{\nu} (a^{\chi} ; h^{\varrho}) \\
& T_b (T_a h) = (T_b T_a) h = @ (b ; @ (a ; h)) \\
& @ (b ; @ (a ; h)) = b . @ (a ; h) = b . (a . h) = (b . a) . h
\end{aligned}$$

The differential quotient in eq. 19 for $\mathcal{T}_{(1)} (\mathcal{T}_{\mathcal{G}\ L})$ defines left multiplication Killing fields on $\mathcal{G}$

$$\omega = (\omega^{\nu}) \in \mathcal{G} \ \longrightarrow \ d\tau\, \omega \in \mathcal{G} |_{\mathcal{M}} \ ; \ d\tau\, \omega \ \longrightarrow \ T_{d\tau\,\omega} \in \mathcal{T}_{\mathcal{G}\ L}$$

(37)
$$\left. \frac{T_{d\tau\,\omega} - ¶}{d\tau} \right|_{d\tau \rightarrow 0} \ \longrightarrow \ \widehat{\omega} = \widehat{\omega} \left(\mathcal{T}_{(1)} \right) = \omega^{\nu} \widehat{I}_{\nu} \ ; \ \widehat{I}_{\varrho} \in \mathcal{T}_{(1)} (\mathcal{T}_{\mathcal{G}\ L})$$

$$\widehat{\omega} \ \rightarrow \ \omega \in Lie (\mathcal{G}) |_{\mathcal{M}^{(1)} (e)}$$

$$\left(\widehat{I}_{\varrho} h \right)^{\chi} = (u_{(\varrho)})^{\chi} (h) = \partial_{a^{\varrho}} @^{\chi} (a ; h) |_{a = 0} \qquad \rightarrow$$

We use up to second order differentials as given in eq. 22 in the left multiplication order

(38)
$$\widehat{\omega}_{(k)} = (\omega^{\nu})_{(k)} \widehat{I}_{\nu}, \; k = 1,2 \; ; \; \widehat{I}_{\varrho} \in \mathcal{T}_{(1)} (\mathcal{T}_{\mathcal{G}\,L})$$
$$d\tau_2 \, \omega_{(2)} = d\tau_2 \, (\omega^{\nu})_{(2)} \rightarrow d\tau_1 \, \omega_{(1)} = d\tau_1 \, (\omega^{\nu})_{(1)}$$

The last relation in eq. 36 then becomes , upon identifying the first order differentials

$b \leftrightarrow d\,b \,; \; a \leftrightarrow d\,a$

(39)
$$@(b; @(a; h)) = b \,.\, @(a; h) \longrightarrow b = d\tau_2 \, \omega_{(2)} \; ; \; a = d\tau_1 \, \omega_{(1)}$$
$$\begin{aligned}(b \,.\, a \,.\, h)^{\nu} &= @^{\nu} \Big(d\tau_2 \, (\omega_{(2)})^{\beta} ; h^{\alpha} + d\tau_1 \, \omega^{\chi}_{(1)} \, (u_{(\chi)}(h))^{\alpha} \Big) \\ &= h^{\nu} + d\tau_1 \, \omega^{\chi}_{(1)} \, u^{\nu}_{(\chi)}(h) \\ &\qquad + d\tau_2 \, \omega^{\chi}_{(2)} \, u^{\nu}_{(\chi)} \Big(h + d\tau_1 \, \omega^{\psi}_{(1)} \, u_{(\psi)}(h) \Big) \\ &= h^{\nu} + \big(d\tau_1 \, \omega_{(1)} + d\tau_2 \, \omega_{(2)} \big)^{\chi} \, u^{\nu}_{(\chi)}(h) \\ &\qquad + \Big(d\tau_2 \, d\tau_1 \, \omega^{\chi}_{(2)} \, \omega^{\psi}_{(1)} \Big) \, u^{\varrho}_{(\psi)}(h) \; \partial_{\xi^{\varrho}} \, u^{\nu}_{(\chi)}(\xi) \Big|_{\xi = h}\end{aligned}$$

$\longrightarrow$

We rewrite the last relation in eq. 39 to render the structure of sequential differential orders, denoted $[\,.\,]_{0,1,\cdots}$, up to second order more transparent

$$\left(b \sim d\,b = d\,\tau_{2}\,\omega_{(2)} \right)^{\beta} \;,\; \left(a \sim d\,a = d\,\tau_{1}\,\omega_{(1)} \right)^{\alpha} \;;\; {}^{\beta},{}^{\alpha} = 1 , \cdots , G$$

$$(b \,.\, a \,.\, h)^{\nu} = (b \,.\, a \,.\, h)^{\nu}_{0} + (b \,.\, a \,.\, h)^{\nu}_{1} + (b \,.\, a \,.\, h)^{\nu}_{2} + \cdots$$

---

(40)
$$\begin{array}{lll}
(b \,.\, a \,.\, h)^{\nu}_{0} & = & h^{\nu} \\
(b \,.\, a \,.\, h)^{\nu}_{1} & = & \left(d\,\tau_{1}\,\omega_{(1)} + d\,\tau_{2}\,\omega_{(2)} \right)^{\chi} u^{\nu}_{(\chi)} (h) \\
(b \,.\, a \,.\, h)^{\nu}_{2} & = & d\,\tau_{2}\,d\,\tau_{1} \left(\omega_{(2)} \right)^{\beta} \left(\omega_{(1)} \right)^{\alpha} L^{\nu}{}_{\beta\,\alpha} (h)
\end{array}$$

$$L^{\nu}{}_{\beta\,\alpha} (h) = \left(u_{(\alpha)} \right)^{\varrho} \partial_{\xi\,\varrho} \left(u_{(\beta)} \right)^{\nu} (\xi) \Big|_{\xi = h}$$

It follows from eq. 40 that the zeroth and first orders of $(b \,.\, a \,.\, h)^{\nu}$ are symmetric under the exchange $\omega_{(2)} \leftrightarrow \omega_{(1)}$, whereas the second order is not .

We further note the relation

(41)
$$\begin{array}{l}
X^{\nu}{}_{\varrho\,\beta} = \partial_{\xi\,\varrho} \left(u_{(\beta)} \right)^{\nu} (\xi) \Big|_{\xi = h} = \partial_{\xi\,\varrho}\,\partial_{\xi\,\beta}\,@^{\nu} (\xi) \Big|_{\xi = h} = X^{\nu}{}_{\beta\,\varrho} \\
X^{\nu}{}_{\varrho\,\beta} = X^{\nu}{}_{\varrho\,\beta} (h)
\end{array}$$

$\longrightarrow$

From eq. 40 we infer the Lie algebra relation for $\mathcal{T}_{\mathcal{G}\,L}$

$$L^{\nu}{}_{\beta\,\alpha}(h) - L^{\nu}{}_{\alpha\,\beta}(h) = f^{\gamma}{}_{\beta\alpha}\left(u_{(\gamma)}(h)\right)^{\nu} \quad \longrightarrow$$

(42) $$\left(u_{(\alpha)}(\xi)\right)^{\varrho}\partial_{\xi\,\varrho}\left(u_{(\beta)}(\xi)\right)^{\nu} - (\alpha \leftrightarrow \beta) = f_{\gamma\beta\alpha}\left(u_{(\gamma)}(\xi)\right)^{\nu}$$

upon the substitutions : $h \rightarrow \xi \;\; ; \;\; f^{\gamma}{}_{\beta\alpha} \rightarrow f_{\gamma\beta\alpha}$

We assume that coordinates on $\mathcal{G}$ are adapted such that the structure constants allow the substitution $f^{\gamma}{}_{\beta\alpha} \rightarrow f_{\gamma\beta\alpha}$ in eq. 42 and become totally antisymmetric – as discussed in subsection 1 b-1 previously $($ eqs. 30 - 32 $)$.
The induced representation [12-1953] operating on a collection of functions $\varphi(\xi)\,;\,\xi \in \mathcal{G}$ relative to $\mathcal{T}_{\mathcal{G}\,L}$ shall be denoted $\blacktriangleright\,\mathcal{T}_{\mathcal{G}\,L}$ and the transformations forming the induced representation as $\blacktriangleright\,T_a$

$$\blacktriangleright\,T_a \in \blacktriangleright\,\mathcal{T}_{\mathcal{G}\,L} \quad \longleftrightarrow \quad T_a \in \mathcal{T}_{\mathcal{G}\,L} \;\; ; \;\; a\,,\,\xi \in \mathcal{G} \quad \rightarrow$$

$\blacktriangleright\,T_a\,\varphi(\xi) = \varphi(a^{-1}\,.\,\xi)$; **with the multiplication**

$$\blacktriangleright\,T_b\,(\blacktriangleright\,T_a\,\varphi)(\xi) = \varphi\left(a^{-1}\,.\,b^{-1}\,.\,\xi\right) = \varphi\left((b\,.\,a)^{-1}\,.\,\xi\right) = \blacktriangleright\,T_{b\,.\,a}\,\varphi(\xi)$$

---

(43) $$\longrightarrow \quad \blacktriangleright\,T_b \;\; \blacktriangleright\,T_a = \blacktriangleright\,T_{b\,.\,a}$$

$\rightarrow$

Then , using eq. 37 to define

(44) $$\widehat{\omega} = \widehat{\omega} \left(\blacktriangleright \mathcal{T}_{(1)} \left(\blacktriangleright \mathcal{T}_{\mathcal{G}\ L} \right) \right) = \omega^{\nu} \widehat{I}_{\nu} \ ; \ \widehat{I}_{\varrho} \in \blacktriangleright \mathcal{T}_{(1)} \left(\blacktriangleright \mathcal{T}_{\mathcal{G}\ L} \right)$$

we obtain $\widehat{I}_{\varrho}$ **as a derivation with respect to the Killing fields pertaining to** $\mathcal{T}_{\mathcal{G}\ L}$ **multiplied with** -1

(45) $$\widehat{I}_{\varrho} = - \left(u_{(\varrho)} (\xi) \right)^{\nu} \partial_{\xi^{\nu}} \ ; \ \left[\widehat{I}_{\alpha} , \widehat{I}_{\beta} \right] = f_{\alpha\beta\gamma} \widehat{I}_{\gamma}$$
$$\left[\widehat{I}_{\alpha} , \widehat{I}_{\beta} \right] = \left[\ \left(u_{(\alpha)} \right)^{\varrho} \left(\partial_{\xi^{\varrho}} \left(u_{(\beta)} \right)^{\nu} \right) - \left(\alpha \leftrightarrow \beta \right) \ \right] \partial_{\nu}$$

In eq. 45 the derivation operators are implied to act from the left on a function $\varphi (\xi)$ **, which is suppressed for simplification of notation .**

The expression in brackets in the last relation in eq. 45 agrees with the one derived with respect to $\mathcal{T}_{\mathcal{G}\ L}$ **given in eq. 42 , verifying the universal nature of the structure constants .**

The collection of functions $\varphi (\xi)$ **is understood to form the Hilbert space over the left and right invariant Haar measure [7-1933] with respect to** $\mathcal{T}_{\mathcal{G}\ L\ \&\ R}$

$$\left\{ \bigcup_{\varphi} \varphi (\xi) \right\} \sim \mathcal{H} (\mathcal{G}) = L_{2} \left[\mathcal{G} ; (d\mu)_{Haar} (\xi)) \right]$$

(46) **allowing both induced representations :** $\blacktriangleright \mathcal{T}_{\mathcal{G}\ L}$ **and** $\blacktriangleright \mathcal{T}_{\mathcal{G}\ R}$

$$L \rightarrow R : u_{(\varrho)} \rightarrow v_{(\varrho)} \text{ with } \left(v_{(\varrho)} (\xi) \right)^{\nu} = - \partial_{a^{\varrho}} @^{\nu} (\xi , a) |_{a=0}$$

In eq. 46 $v_{(\varrho)}$ **denote the Killing vector fields pertaining to** $\mathcal{T}_{\mathcal{G}\ R}$ **.** $\rightarrow$

Invariance of the Haar measure [7-1933] with respect to $\mathcal{T}_{\mathcal{G}\ L\ \&\ R}$

(47) $$(d\mu)_{Haar}(\xi) = (d\mu)_{Haar}(a.\xi) = (d\mu)_{Haar}(\xi.b) \ ; \ a, b \in \mathcal{G}$$

sets the stage for the operators $T_a \in \blacktriangleright \mathcal{T}_{\mathcal{G}\ L\ \&\ R}$ to become unitary ones in $\mathcal{H}$ as defined in eq. 46 and in turn through the invere exponential mapping using eq. 44 and the definitions in eq. 35

(48) $$\begin{aligned} & T_{h(\tau;\omega)} = \exp(\tau\,\widehat{\omega}) \in \blacktriangleright \mathcal{T}_{\mathcal{G}\ L\ \textbf{or}\ R} \\ & \widehat{\omega} = \omega^{\nu}\,\widehat{I}_{\nu} \ ; \ \widehat{I}_{\nu} \in \blacktriangleright \mathcal{T}_{(1)}\,(\blacktriangleright \mathcal{T}_{\mathcal{G}\ L\ \textbf{or}\ R}) \longrightarrow \\ & T_h\,T_h^{\dagger} = ¶\,|_{\mathcal{H}} \ ; \ \widehat{I}_{\nu} = -\widehat{I}_{\nu}^{\dagger} \ , \ \textbf{with}\ ^{\dagger} : \textbf{self adjoint conjugation in}\ \mathcal{H} \end{aligned}$$

In eq. 48 self adjoint conjugation refers to the hermitian scalar product in $\mathcal{H}$

(49) $$\zeta, \varphi \in \mathcal{H} : \langle \zeta, \varphi \rangle = \int (d\mu)_{Haar}(\xi)\,[\zeta^{*}(\xi)\,\varphi(\xi)]$$

Distinguishing explicitely $T_{a\ L}$ and $T_{a\ R}$, $\widehat{I}_{\nu\ L}$ and $\widehat{I}_{\nu\ R}$ as defined for the induced respective representations in $\mathcal{H}$, eq. 48 is extended below to include their Lie algebra relations

(50) $$\begin{aligned} & T_{h\ L}\,T^{\dagger}_{h\ L} = T_{h\ R}\,T^{\dagger}_{h\ R} = ¶\,|_{\mathcal{H}} \ ; \ \widehat{I}_{\nu\ L} = -\widehat{I}^{\dagger}_{\nu\ L} \ ; \ \widehat{I}_{\nu\ R} = -\widehat{I}^{\dagger}_{\nu\ R} \\ & \left[\widehat{I}_{\alpha\ K}, \widehat{I}_{\beta\ K}\right] = f_{\alpha\beta\gamma}\,\widehat{I}_{\gamma\ K} \ ; \ K = L, R \ ; \ \left[\widehat{I}_{\alpha\ L}, \widehat{I}_{\beta\ R}\right] = 0 \end{aligned}$$

In eqs. 43 - 50 the properties of the induced representations, denoted $\blacktriangleright \mathcal{T}_{\mathcal{G}\ L\ \&\ R}$, relevant for the following are made explicit. $\longrightarrow$

The apparent digression from the derivation of the differential equations determining the abelian one parameter subgroups as shown in eq. 35 in subsection 1 b-1 serves to make intrinsic use of the theorem by Peter and Weyl [6-1927] , which holds precisely for the induced representations ▶ $\mathcal{T}_{\mathcal{G}\ L\ \&\ R}$.
Now these representations were constructed with the regular representation of finite groups as guideline, the latter yielding in a clear deductive way to a complete reduction involving all irreducible linear representations of the finite group $\mathcal{G}_{\ finite}$ in question with the dimension of the group beeing equal to the multiplicity of its appearances within the regular representation. So a similar construction was sought and found for compact semi-simple Lie groups, indeed analogous to the regular representation in what the theorem of Peter and Weyl asserts : the reprentations ▶ $\mathcal{T}_{\mathcal{G}\ L\ \&\ R}$, for $\mathcal{G}$ a compact semi-simple Lie group can be fully reduced with respect to finite dimensional irreducible representations, if each simple factor subgroup and in this way all such representations are recovered with each representation appearing with a multiplicity equal to its dimension , allowing both $\mathcal{T}_{\mathcal{G}\ L}$ and $\mathcal{T}_{\mathcal{G}\ R}$ to be represented – commuting according to eq. 50 .
This property is thus characteristic or self consistent for complete connections within the ordered sequence of fibre manifolds considered here, as discussed in subsection 1a-1 , to which we will return in the next section .

1 b-2 – construction of one parameter abelian subgroups on $\mathcal{G}$

We return to the construction of one parameter abelian subgroups of $\mathcal{G}$ using the transformation group $\mathcal{T}_{\mathcal{G}\ L}$ to be specific , using the notions introduced in eqs. 35 and 37 . →

The exponential mapping induced by the left multiplication operators $T_a \in \mathcal{T}_{\mathcal{G}\,L}$

(51)
$$\widehat{\omega} = \omega^{\varrho}\,\widehat{I}_{\varrho} \;\;;\;\; \widehat{\omega}\,,\,\widehat{I}_{\varrho} \in \mathcal{T}_{(1)}\,(\,\mathcal{T}_{\mathcal{G}\,L}\,) \;\;\longrightarrow$$
$$\exp\,(\,\tau\,\widehat{\omega}\,)\;e = h\,(\,\tau\,;\,\omega\,) \in \mathcal{G} \;\;\textbf{with}\;\; e = \textbf{unit element} \in \mathcal{G}$$

defines a set of one parameter abelian subgroups of $\mathcal{G}$ **with the initial condition**

(52)
$$h\,(\,\tau\,;\,\omega\,)\,|_{\,\tau=0} = e$$

which involves higher order differentials than third for the group multiplication functions as specified in eqs. 9 and 10

(53)
$$(\,b\,.\,a\,)^{\,\nu} = @^{\,\nu}\,\left(\,b^{\,1}\,,\,\cdots\,b^{\,G}\,;\,a^{\,1}\,,\,\cdots\,,\,a^{\,G}\,\right)$$

Using the first order differentials (eqs. 39 - 41) we infer the system of differential equations

(54)
$$\begin{aligned} \frac{d}{d\,\tau}\;h^{\,\nu}\,(\,\tau\,;\,\omega\,) &= \omega^{\,\varrho}\,\left[\,\partial_{\,a\,\varrho}\,@^{\,\nu}\,(\,a\,,h\,(\,\tau\,,\,\omega\,)\,|_{\,a\,=\,0}\,\right] \\ &= \omega^{\,\varrho}\,\left[u_{\,(\varrho)}^{\;\nu}\;(\,h\,(\,\tau\,;\,\omega\,)\,)\right] \end{aligned}$$

$$u_{\,(\varrho)}^{\;\nu}\,(\,0\,(\leftrightarrow\;e\,)\,) = \delta_{\,(\varrho)}^{\;\nu}\;\;;\;\;h^{\,\nu}\,(\,\tau = 0\,;\,\omega\,) = 0\,(\leftrightarrow\;e\,)$$

In eq. 54 $\left[\,\partial_{\,a\,\varrho}\,@^{\,\nu}\,(\,a\,,\xi\,)\,|_{\,a\,=\,0}\,\right] = u_{\,(\varrho)}^{\;\nu}\,(\,\xi\,)$ **denote the Killing vector fields on** $\mathcal{G}$ **generated by the first order differentials** $\mathcal{T}_{(1)}\;(\,\mathcal{T}_{\mathcal{G}\,L}\,)$ **as defined in eqs. 36 and following .** $\rightarrow$

We rewrite the differential equation (eq. 54) suppressing the argument ω of the one parameter family coordinates $h^{\nu}(\tau;\omega)$, which plays a parametric important role, for simplicity of notation

$$h^{\nu}(\tau;\omega) \to h^{\nu}(\tau) \quad \longrightarrow$$

(55) $$\frac{d}{d\tau} h^{\nu}(\tau) = \omega^{\varrho} \, u_{(\varrho)}{}^{\nu}(h(\tau)) \quad ; \quad \frac{d}{d\tau} h^{\nu}(0) = \omega^{\nu} \quad ; \quad h^{\nu}(0) = 0$$

Conversely let $h^{\nu}(\tau)$ satisfy the differential equations and initial conditions in eq. 55 .
Then we consider the function associated with the group product $p(\tau,\vartheta) \sim (h(\tau).h(\vartheta))$

$$p^{\nu}(\tau,\vartheta) = @^{\nu}(h(\tau);h(\vartheta)) \longrightarrow$$

(56) $$\left[\begin{array}{l} p^{\nu}(\tau + d\tau, \vartheta) \\ -p^{\nu}(\tau,\vartheta) \end{array} \right] \begin{array}{l} = @^{\nu}(d\tau\,\omega; p(\tau,\vartheta)) \\ = d\tau\,\omega^{\varrho} \; u_{(\varrho)}{}^{\nu}(p(\tau,\vartheta)) \\ = d\tau\,\omega^{\varrho} \; u_{(\varrho)}{}^{\chi}(h(\tau)) \; \partial_{a\,\chi} @^{\nu}(a;h(\vartheta)) \,|_{\,a=h(\tau)} \end{array}$$

$\longrightarrow$

Eq. 56 gives rise to the differential equations for $p^{\nu}(\tau, \vartheta)\ (= (h(\tau).h(\vartheta))^{\nu})$

(57) $$\frac{d}{d\tau}\ p^{\nu}(\tau, \vartheta) = \omega^{\varrho}\ {u_{(\varrho)}}^{\nu}(p(\tau, \vartheta))$$

$$= \omega^{\varrho}\ {u_{(\varrho)}}^{\chi}(h(\tau))\ \partial_{a\,\chi} @^{\nu}(a; h(\vartheta))\,|_{a=h(\tau)} \longrightarrow$$

$$\frac{d}{d\tau}\ p^{\nu}(0, \vartheta) = \omega^{\varrho}\ {u_{(\varrho)}}^{\nu}(h(\vartheta)) \quad ; \quad p(0, \vartheta) = h(\vartheta)$$

We compare differential equations and initial conditions for $p^{\nu}(\tau, \vartheta)$ **in eq. 57 with the ones for**

$$q^{\nu}(\tau, \vartheta) = h^{\nu}(\tau + \vartheta)$$

(58) $$\frac{d}{d\tau}\ q^{\nu}(\tau, \vartheta) = \omega^{\varrho}\ {u_{(\varrho)}}^{\nu}(q(\tau, \vartheta))$$

$$\frac{d}{d\tau}\ q^{\nu}(0, \vartheta) = \omega^{\varrho}\ {u_{(\varrho)}}^{\nu}(h(\vartheta)) \quad ; \quad q(0, \vartheta) = h(\vartheta)$$

which are both identical . It follows $\longrightarrow$

by the uniqueness of solutions to a system of first order differential equations subject to the same initial conditions

(59)
$$p\,(\,\tau\,,\,\vartheta\,)\;=\;q\,(\,\tau\,,\,\vartheta\,)\quad\longleftrightarrow\quad h\,(\,\tau\,)\,.\,h\,(\,\vartheta\,)\;=\;h\,(\,\tau\,+\,\vartheta\,)$$

in the domain of validity of solutions to eqs. 55 - 58 .

From the structure of all one parameter families $h^{\,\nu}\,(\,\tau\,;\,\omega\,)$ **and the first order differentsal equations they satisfy – eqs. 55 - 59 – at least for small values of** τ **it follows that first in the neighbourhood of the unit element group elements can be uniquely parametrized by the tangent vectors** $\omega\;=\;(\,\omega^{\,1}\,,\,\cdots\,,\,\omega^{\,G}\,)$ **and using these 'normal' coordinates the convolution functions** $@^{\,\nu}\,(\,\omega_{\,(1)}\,;\,\omega_{\,(2)}\,)\;=\;\left(\,h\,(\,\omega_{\,(1)}\,)\,.\,h\,(\,\omega_{\,(2)}\,)\,\right)$ **once threefold differentiability is assumed – become (real -) analytic functions of the special tangent vector variables** $\omega_{\,(1)}\,,\,\omega_{\,(2)}$ **through the exponential mapping**

(60)
$$@^{\,\nu}\,(\,\omega_{\,(1)}\,;\,\omega_{\,(2)}\,)\;=$$
$$\sum_{m_1\,\cdots\,m^G\,;\,n_1\,\cdots\,n^G=0}^{\infty}\left(\,a_{\,m_1,\cdots,m_G\,;\,n_1,\cdots,n_G}\,\right)\,\times$$
$$\times\,\left(\omega^1_{(1)}\right)^{m_1}\cdots\left(\omega^G_{(1)}\right)^{m_G}\;\left(\omega^1_{(2)}\right)^{n_1}\cdots\left(\omega^G_{(2)}\right)^{n_G}$$ **: convergent in a neighbourhood of e**

From here $\mathcal{G}$ **emerges as its universal covering group .**

This ends the present subsection (1 b-2) and section (1 b) .

1 c - Complete connections : regularity conditions from the full collection of fibre manifolds as defined in subsection (1 a-1)

With the complete structure of analytic coordinates and the property of universal covering group with respect to 'path homotopy' of semisimple compact Lie groups $\mathcal{G}$ fully specified

in sections (1 a) , (1 b) and subsections (1 b-1) , (1 b-2)
and the selction of graded fibre manifolds in subsection (1 a-1)

we return to the properties of complete connections , introduced at the beginning of section 1 . The words used in modern mathematics to describe complete connections are : connections regular for the complete ring of representations of $\mathcal{G}$ [18-1968] .
In the following $\mathcal{G}$ shall be generically a compact simple Lie group, specifically $SU3_{\,c}$ and $\mathcal{D}$ a general irreducible representation of $\mathcal{G}$

(61) $$\mathcal{G} \; = \; \left\{ \begin{array}{c} \text{compact simpe Lie group} \\ \\ SU3_{\,c} \; \text{specifically} \end{array} \right. \quad ; \quad \mathcal{D} \; = \; \mathcal{D}\,|_{\,\mathcal{G}} \; \in \; \mathcal{R} \; = \; \mathcal{R}|_{\,\mathcal{G}}$$

$$\mathcal{R} \; = \; \text{complete ring generated by finite dimenional, unitary , irreducible representations of } \mathcal{G}$$

Following the notation of eqs. 1 - 6 $\longrightarrow$

we denote by $Lie\,(\,\mathcal{D}\,)$ a basis of antihermtian $dim\,(\,\mathcal{D}\,)\,\times\,dim\,(\,\mathcal{D}\,)$ matrices satisfying the Lie algebra commutation relations pertaining to $\mathcal{D}$

$$(\,d_{\,r}\,(\,\mathcal{D}\,)\,)_{\,\alpha\beta}\;\rightarrow\;d_{\,r}\,(\,\mathcal{D}\,)\;\rightarrow\;d_{\,r}\;=\;-\,d_{\,r}^{\,\dagger}\;;\;r\;=\;1\,,\cdots,\,dim\,\mathcal{G}\;\in\;Lie\,(\,\mathcal{D}\,)$$

(62) $$[\,d_{\,p}\,,\,d_{\,q}\,]\;=\;f_{\,pqr}\,d_{\,r}\;\;:\;\;f_{\,pqr}\;=\;\left\{\begin{array}{l}\textbf{real, totally antisymmetric}\\ \textbf{structure constants of }\mathcal{G}\end{array}\right.$$

$$\alpha\,,\,\beta\;=\;1\,,\cdots,\,dim\,(\,\mathcal{D}\,)$$

Now lets assume the situation as defined in eq. 51 , where the one parameter subgroup $h\,(\,\tau\,;\,\omega\,)\;\in\;\mathcal{G}$ was constructed through the exponential mapping

(63) $$\begin{array}{l}\widehat{\omega}\;=\;\omega^{\,r}\,\widehat{I}_{\,r}\;\;;\;\;\widehat{\omega}\,,\,\widehat{I}_{\,r}\;\in\;\mathcal{T}_{(1)}\,(\,\mathcal{T}_{\mathcal{G}\;L}\,)\;\;;\;\;\left[\,\widehat{I}_{\,p}\,,\,\widehat{I}_{\,q}\,\right]\;=\;f_{\,pqr}\,\widehat{I}_{\,r}\\ \longrightarrow\;\exp\;(\,\tau\,\widehat{\omega}\,)\;e\;=\;h\,(\,\tau\,;\,\omega\,)\;\in\;\mathcal{G}\;\;\textbf{with}\;\;e\;=\;\textbf{unit element}\;\in\;\mathcal{G}\end{array}$$

of which the tangent vector at e is $\omega\;=\;(\,\omega^{1}\,,\,\cdots\,,\,\omega^{\,G}\,)$, as shown again in eq. 63 , with the transformation group $\mathcal{T}_{\mathcal{G}\;L}$, i.e. left multiplication on $\mathcal{G}$. Then the finite transformation matrix $D\;(\,h\,(\,\tau\,;\,\omega\,)\,)\;\in\;\mathcal{D}$ is obtained through the exponential mapping

(64) $$D\;(\,h\,(\,\tau\,;\,\omega\,)\,)\;=\;\exp\;(\,\tau\,\omega^{\,r}\,d_{\,r}\,)\;\in\;\mathcal{D}\;\;\forall\;\;\mathcal{D}\;\in\;\mathcal{R}$$

This anchors the meaning of eqs. 3 - 6 as is necessary to analyze complete connections , next . $\rightarrow$

1 c-1 – Complete connections in detail

At this stage we come back to the connection one form as defined in eq. 2 , expanding on the regularity conditions implied by the adopted complete fibre manifolds – in subsection 1 a-1 – as they give rise to the complete ring of representations $\cup_{\mathcal{D}} = \mathcal{R}$, defined in eq. 61

(65)
$$\begin{array}{l} \left(W^{(1)} (\mathcal{D}) \right)_{\alpha\beta} = W_{\mu}^{r} (x) \left(d_{r} (\mathcal{D}) \right)_{\alpha\beta} \, d x^{\mu} \\ W_{\mu}^{r} (x) \; : \; \textbf{real} \; ; \; r = 1 , \cdots , G = dim (\mathcal{G}) \\ \left(W^{(1)} (\mathcal{D}) \right)_{\alpha\beta} \; \rightarrow \; \alpha , \beta = 1 , \cdots , D = dim (\mathcal{D}) \\ W^{(1)} (\mathcal{D}) \; \rightarrow \; W^{(1)} \left(|_{\mathcal{D}} \right) \; ; \; \mathcal{D} \in \mathcal{R} \; \left(\textbf{unrestricted} \right) \\ \left(d_{r} (\mathcal{D}) \right)_{\alpha\beta} \; \rightarrow \; d_{r} (\mathcal{D}) \; \rightarrow \; d_{r} \in Lie (\mathcal{D}) \end{array}$$

The brackets around $\mathcal{D}$ specifying the chosen representation forming the connection 1-form $W^{(1)} \left(|_{\mathcal{D}} \right)$ in eqs. 2 and 65 shall indicate that this suffix may be suppressed in the following for simplicity of notation.

The argument x and the differentials $d x^{\mu} \; ; \; ^{\mu} = 0, 1, 2, 3$ refer to 1 + 3 dimensional uncurved space-time as base-space of the connection 1-form . Eventually the base space can be continued to its Euclidean version . The conditions in eq. 65 define a complete classical connection together with its regularity conditions . $\longrightarrow$

In order to maintain exact local gauge invariance , imposed here, complete connections are understood to form a collection of such, denoted $\mathcal{C}$

(66) $$\mathcal{C} \ = \bigcup_{W} \ \left(W^{(1)} \right) \ \text{complete connections}$$

invariant under local gauge transformations to which we turn next.

Let $F(x)$ be a classical field quantity – with arbitrary spin, untouched by charge like gauges – transforming according to the local irreducible representation $\Omega(x) \in \mathcal{D}(\mathcal{G})$

(67)
$$\Omega(x) \ : \ F(x) \ \rightarrow \ F^{\Omega}(x) \ = \ \Omega(x) F(x)$$
$$F_{\alpha} \ \rightarrow \ F_{\alpha}^{\Omega} \ = \ \Omega_{\alpha\beta} F_{\beta} \quad ; \quad \Omega \in \mathcal{D}$$

In eq. 67 and subsequently we use matrix notation and also suppress the local argument x , whenever clarity of notation allows .

Then the $\left(\mathcal{D} \text{ adapted} \right)$ covariant derivative 1-form is determined from the relations, derivatives meant to act on F , F^{Ω} from the left

(68)
$$d x^{\mu} \delta_{\alpha\beta} \partial_{\mu} \ = \ d x^{\mu} \P_{D \times D} \partial_{\mu} \ = \ \partial$$
$$\Omega \ : \ D^{(1)} \ = \ \partial + W^{(1)} \ \rightarrow \ D^{\Omega (1)} \ = \ \partial + W^{\Omega (1)} \quad \text{with}$$
$$D^{\Omega (1)} F^{\Omega} \ = \ \Omega D^{(1)} F \ = \ \Omega D^{(1)} \left(\Omega^{-1} F^{\Omega} \right)$$

In eq. 68 the usual derivative symbol d is replaced by ∂ to distinguish it from the matrices d_{r} . $\longrightarrow$

From eq. 68 the inhomogeneous transformation law for connections follows

(69) $$W^{\,\Omega\,(1)} = \Omega\,\left(\,\partial + W^{\,(1)}\,\right)\,\Omega^{\,-1} = \Omega\,D^{\,(1)}\,\Omega^{\,-1}$$

Performing the inverse of the exponential mapping or equivalently for infinitesimal $\Omega\,-\,¶$ we obtain

(70) $$\delta_{\,\omega}\,W^{\,(1)} = -\,\left(\,\partial\,\omega + \left[\,W^{\,(1)}\,,\,\omega\,\right]\,\right) \quad \text{in components} \quad \longrightarrow$$
$$-\,\delta_{\,\omega}\,W_{\,\mu}^{\,r}\,d_{\,r} = \partial_{\,\mu}\,\omega^{\,r}\,d_{\,r} + f_{\,pqr}\,W_{\,\mu}^{\,p}\,\omega^{\,q}\,d_{\,r} \quad \forall\,\mathcal{D} \in \mathcal{R}$$

Dealing with complete connections, the common factor $d_{\,r}$ can be projected out and eq. 70 becomes

(71) $$-\,\delta_{\,\omega}\,W_{\,\mu}^{\,r} = \partial_{\,\mu}\,\omega^{\,r} + f_{\,rpq}\,W_{\,\mu}^{\,p}\,\omega^{\,q} = \left(\,\partial_{\,\mu}\,\delta_{\,rq} + \left(\,W_{\,\mu}^{\,ad}\,\right)_{\,rq}\,\right)\,\omega^{\,q}$$
$$\left(\,W_{\,\mu}^{\,ad}\,\right)_{\,rq} = W_{\,\mu}^{\,p}\,\left(\,ad_{\,p}\,\right)_{\,rq} \quad ; \quad \left(\,ad_{\,p}\,\right)_{\,rq} = f_{\,rpq}$$

In turn if we identify $\omega^{\,r}\,(\,x\,) \,\sim\, F_{r} \in Ad\,(\,\mathcal{G}\,)$ as a field quantity transforming under the adjoint representation of $\mathcal{G}$ the right hand side of the first relation in eq. 71 represents its covariant derivative for a given connection $W^{\,(1)}$ with respect to this representation . The above local apparent reduction of both gauge connections and field strengths , to allow their full characterization through the adjoint representation is in particular for complete connections incorrect, as a consequence of nonlocal gauge co- and invariance emerging through finite distance parallel transport, as discussed in the next subsection (1 c-2) . Here we continue defining local field strengths . →

From a complete connection relative to the irreducible local gauge group representation $\mathcal{D} (\mathcal{G})$ as defined in eqs. 65 (- 71) the $\mathcal{D}$ associated field strengths obtain through the two form

(72)
$$
\begin{array}{l}
W^{(2)} (\mathcal{D}) \rightarrow W^{(2)} = \partial W^{(1)} + \left(W^{(1)} \right)^{2} \\
\left(W^{(2)} \right)_{\alpha\beta} = \frac{1}{2} W^{r}_{\mu\nu} \left(d_{r} \right)_{\alpha\beta} d x^{\mu} \wedge d x^{\nu} \; ; \; d_{r} \in Lie (\mathcal{D}) \rightarrow \\
W^{(2)}_{\mu\nu} = \partial_{\mu} W^{(1)}_{\nu} - \partial_{\nu} W^{(1)}_{\mu} + \left[W^{(1)}_{\mu} , W^{(1)}_{\nu} \right] \\
W^{r}_{\mu\nu} = - W^{r}_{\nu\mu} = \partial_{\mu} W^{r}_{\nu} - \partial_{\nu} W^{r}_{\mu} + f_{rpq} W^{p}_{\mu} W^{q}_{\nu} \\
\hline
W^{(2)} (\mathcal{D}) \equiv B^{(2)} (\mathcal{D}) \; ; \; W^{r}_{\mu\nu} \equiv B^{r}_{\mu\nu}
\end{array}
$$

components of field strengths independent of $\mathcal{D}$

In the last relation in eq. 72 we have identified the curvature two form and its $\mathcal{D} -$ independent components with the letter B for field strengths pertaining to charge like gauges. $d x^{\mu} \wedge d x^{\nu}$ denotes the antisymmetric de Rham wedge product for $(2) \longrightarrow (k) -$ forms [19-1931] .
We note the local, covariant gauge transformation properties of $F^{(2)}$ following from eqs. 69 and 72

(73)
$$
\begin{array}{l}
W^{(1)} \rightarrow W^{\Omega (1)} = \Omega \left(\partial + W^{(1)} \right) \Omega^{-1} = \Omega D^{(1)} \Omega^{-1} \big|_{\mathcal{D}} \longrightarrow \\
B^{(2)} \rightarrow B^{\Omega (2)} = \Omega B^{(2)} \Omega^{-1} \big|_{\mathcal{D}} \; ; \; \Omega (x) \in \mathcal{D}
\end{array}
$$

$\longrightarrow$

The Bianchi identity, only local consequence of a system of local connections and field strengths

In a system as specified in the (sub-) title above one identity follows, generalising the homogeneous Maxwell equations in QED

$$
\begin{aligned}
B^{(3)} &= \partial B^{(2)} + \left[W^{(1)} , B^{(2)} \right] \Big|_{\mathcal{D}} \equiv 0 \\
&= \partial \, \partial W^{(1)} + \left[\begin{array}{c} \left[\left(\partial W^{(1)} \right) , W^{(1)} \right] + \left[W^{(1)} , \left(\partial W^{(1)} \right) \right] \\ + \left[W^{(1)} , \left(W^{(1)} \right)^{2} \right] \end{array} \right]
\end{aligned}
\tag{74}
$$

$$
\longleftrightarrow \; B^{(2)} (\mathcal{D}) \rightarrow W^{(2)} = \partial W^{(1)} + \left(W^{(1)} \right)^{2}
$$

The (Bianchi -) identity in eq. 74 holds independently of whether other matter fields, e.g. $q \, , \, \overline{q}$ are included with finite masses or not . However the identity only holds if $F^{(2)} (\mathcal{D})$ actually derives from connections , and furthermore if allowed connections do satisfy regularity conditions, here those appropriate for complete such .

This ends subsection 1 c-1 and we turn towards nonlocal properties of complete connections in the next subsection 1 c-2 .

1 c-2 – Parallel transport with complete connections

Here I follow the outline in ref. [20-2004] [a] .

We first establish the relation of notation between ref. [20-2004], Appendix A.3 op.cit.: hep-ph/0405032v1, pp. 42 - 52 , eqs. 93 - 127 , to the present work : subsection 1 c-1 , eqs. 65 - 74 .

In ref. [20-2004] only the adjoint representation is discussed which implies

(75)

present work	ref. [20-2004]
$\mathcal{D} \; = \; Ad \, (\, \mathcal{G} \,)$	$ABC \; \longrightarrow \; qpr$
	$\frac{1}{i} \, \mathcal{F}_{A=r \; B=p \; C=q} \; = \; f_{\, rpq} \; = \; (\, ad_{\; p} \,)_{\; rq}$
	$(\, \mathcal{W}_{\, \mu} \,)_{\; A=r \; B=q} \; = \; \left(\, W_{\, \mu}^{\, ad} \, \right)_{\; rq} \; = \; W_{\, \mu}^{\, p} \; (\, ad_{\; p} \,)_{\; rq}$
	$V_{\, \mu} \, (\, x \, , \; D = p \,) \; = \; - \, W_{\, \mu}^{\, p} \, (\, x \,)$
	$X \, (\, x \, , \; B = r \,) \; = \; F^{\, r} \, (\, x \,)$
	$F_{\, [\, \mu\nu \,]} \, (\, x \, , \; D = r \,) \; = \; W_{\, \mu\nu}^{\, r} \, (\, x \,)$

---

a In ref. [20-2004] I called instead of 'complete connections over the complete ring generate by finite dimenional, unitary , irreducible representations of $\mathcal{G}$ ' as defined here in eq. 61 : 'universal bundle' . However the notion of 'universal bundle' in the mathematical literature is reserved for another structure. I thank Martin Lüscher, for pointing this error out to me . It applies to my nomenclature , not to the derivations . $\longrightarrow$

The identifications in eq. 75 shall suffice here, they allow to deduce all further such.

We return to general complete connections based on a given representation $\mathcal{D}\,(\,\mathcal{G}\,)$ **as specified in subsection 1 c-1 .**

Parallel transport – a priori along a general curve C **– in the base space** $\mathcal{B} \;=\; \{\; x\;\}$ **– leads to the path ordered exponential integral of a given connection 1-form denoted** $W^{\,(1)}(\,x\,)\,\big|_{\,\mathcal{D}\,(\,\mathcal{G}\,)}$ **as defined in eq. 65 out of the collection** $\mathcal{C}$ **of complete connections (eq. 66) denoted** $U \;=\; U_{\,\alpha\beta} \;\in\; \mathcal{D}$

$$U\,\left(\,x \;\overset{C}{\leftarrow}\; y\,\right) \;=\; P\,\exp\,\left(\,-\int_{C} W^{\,(1)}\,(\,\overline{x}\,)\,\right) \;\rightarrow\; (\,U\,(\,x\,,C\,,y\,)\,)_{\,\alpha\beta} \;\in\; \mathcal{D}\,(\,\mathcal{G}\,)$$

(76) $$C \;=\; C\;\{\,\overline{x}\,\} \quad : \quad \overline{x} \;=\; \overline{x}\,(\,s\,) \quad ; \quad \tau \;\geq\; s \;\geq\; 0 \;\;;\;\; s \;:\; \text{path parameter}$$

$$\overline{x}\,(\,s \;=\; \tau\,) \;=\; x \;\;;\;\; \overline{x}\,(\,s \;=\; 0\,) \;=\; y$$

In eq. 76 P denotes matrix ordering along the path C since $W^{\,(1)}$ **is matrix valued**

(77) $$\left(\,W^{\,(1)}\,(\,\mathcal{D}\,)\,\right)_{\,\alpha\beta} \;=\; W_{\,\mu}^{\,r}\,(\,\overline{x}\,)\;(\,d_{\,r}\,(\,\mathcal{D}\,)\,)_{\,\alpha\beta}\; d\,\overline{x}^{\,\mu}$$

However since we consider here classical configurations the quantities $W_{\,\mu}^{\,r}\,(\,\overline{x}\,)$ **are commuting for arbitrary** $\overline{x}$ **. We will only use straight line curves C , which would leave local field variables commuting only for space like straight line paths , except for the derivation of the differential equation for** $U\,(\,x\,,C\,,y\,)$ **, which would also be true for noncommuting field variables, next .** $\longrightarrow$

The differential equation derives from the parallel transport and has its roots in expanding on the meaning of eq. 70. We do not explicitely do so here . We assume that the general path C , as specified by the functions $\overline{x}^{\,\mu}(s)$ (eq. 76) , which respect all regularity conditions and are known functions also beyond the specific boundary value τ described in eq. 76 .
Then it follows for $\tau \rightarrow \tau + d\tau$

$$U \rightarrow U(\tau)$$

$$U(\tau) + d\tau \frac{d}{d\tau} U = \left(¶ - d\tau \, \mathbf{v}^{\mu}(\tau) W_{\mu}^{\,r}(y(\tau)) \, d_r \right) U \longrightarrow$$

---

(78)

$$\frac{d}{d\tau} U(\tau) = - \left(\mathbf{v}^{\mu} \mathcal{W}_{\mu} \right)(\tau) U(\tau) \;\; ; \;\; \left[\begin{array}{l} \mathbf{v}^{\mu} = \frac{d}{d\tau} \overline{x}^{\,\mu}(\tau) \\ \mathcal{W}_{\mu} = W_{\mu}^{\,r} d_r = \mathcal{W}_{\mu}(\tau) \end{array} \right]$$

$$U(\tau = 0) = U_y = ¶$$

The regularity conditions imposed on complete connections go over $\longrightarrow$

to the quantities $U(\tau) \rightarrow U\left(x \overset{C}{\leftarrow} y\right)$ if and only if all paths are chosen to respect these conditions forming a network, i.e. the functions $\overline{x}(s)$ are chosen accordingly, e.g. to be real analytic functions or n-fold differentiable ones of the argument s .

If two paths C_1 and C_2 can be joined without reduction of imposed regularity conditions individually to a combined path C_{2+1} then the associated unitary operators can be combined and obey the composition law

$$U\left(x_2 \overset{C_2}{\leftarrow} x_1\right) U\left(x_1 \overset{C_1}{\leftarrow} y\right) = U\left(x_2 \overset{C_{2+1}}{\leftarrow} y\right) \tag{79}$$

in a natural way. The best known case, whereby parallel transport is called holonomy (whence applied to a field F^{β} at y) for a closed path $x \rightarrow x_{end} = y$.

But we leave the paths open and consider a local gauge transformation as defined in eq. 69

$$W^{\Omega(1)}(x) = \Omega(x)\left(\partial_x + W^{(1)}(x)\right)\Omega^{-1}(x) = \Omega D^{(1)} \Omega^{-1}(x) \tag{80}$$

Then it follows

$$\begin{array}{ccc} U\left(x \overset{C}{\leftarrow} y \,;\, W^{\Omega(1)}\right) & = & P \exp\left(-\int_C W^{\Omega(1)}\right) \\ || & & \\ \Omega(x)\, U\left(x \overset{C}{\leftarrow} y \,;\, W^{(1)}\right) \Omega^{-1}(y) & & \end{array} \tag{81}$$

$\longrightarrow$

The vertical relation in eq. 81 is repeated below

$$U\left(x \overset{C}{\leftarrow} y\,;\, W^{\,\Omega\,(1)}\right) \;=\; \Omega\,(\,x\,)\,U\left(x \overset{C}{\leftarrow} y\,;\, W^{\,(1)}\right)\,\Omega^{\,-1}\,(\,y\,)$$

(82)

$$W^{\,\Omega\,(1)}\,(\,z\,) \;=\; \Omega\,(\,z\,)\,\left(\partial_{\,z} \,+\, W^{\,(1)}\,(\,z\,)\right)\,\Omega^{\,-1}\,(\,z\,) \;=\; \Omega\,D^{\,(1)}\,\Omega^{\,-1}\,(\,z\,)$$

$$\Omega\,(\,x\,)\,,\;\Omega\,(\,y\,)\,,\;\Omega\,(\,z\,)\;\in\;\mathcal{D}\,(\,\mathcal{G}\,)\quad\forall\;\;\mathcal{D}\quad\text{and}\quad\forall\quad\text{complete}\quad W^{\,(1)}\,\Big|_{\,\mathcal{D}}$$

The relation on the first line of eq. 82 demonstrates that complete connections – by splitting the arguments of $\Omega\,(\,x\,)$ from $\Omega\,(\,y\,)$ for all $\mathcal{D}\,(\,\mathcal{G}\,)$ – allow to reconstruct the full global structure of $\mathcal{G}$. Hence they do not admit singularities at finite discrete normal subgroups of $\mathcal{G}$ a (semi-) simple compact group , as e.g. the center $Z_{\,3}$ of $SU3_{\,c}$.

The eventual conflicts with regularity conditions respected in lattice formulations of QCD are based on the properties derived for complete connections, maintaining associated extended gauge invariance, in sections 1 a - 3 c and are the subject of the next section 1 d .

This concludes subsection 1 c-2 and section 1 c .

1 d - Complete connections , regularity conditions in potential conflict with lattice QCD – selected specific points

We shall enumerate specific points below

1) the Bianchi identity $\left(\right.$ here defined in eq. 74 $\left.\right)$ repeated below

$$\begin{aligned} B^{(3)} &= \partial B^{(2)} + \left[W^{(1)} , B^{(2)} \right] \Big|_{\mathcal{D}} \equiv 0 \\ &= \frac{1}{6} \begin{pmatrix} \partial_{\nu} \mathcal{B}_{\varrho\sigma} + \left[\mathcal{W}_{\nu} , \mathcal{B}_{\varrho\sigma} \right] \\ + \partial_{\sigma} \mathcal{B}_{\nu\varrho} + \left[\mathcal{W}_{\sigma} , \mathcal{B}_{\nu\varrho} \right] \\ + \partial_{\varrho} \mathcal{B}_{\sigma\nu} + \left[\mathcal{W}_{\sigma} , \mathcal{B}_{\nu\varrho} \right] \end{pmatrix}_{\alpha\beta} d x^{\nu} \wedge d x^{\varrho} \wedge d x^{\sigma} \end{aligned}$$

---

(83)

$$W^{(1)} = \mathcal{W}_{\mu} \, d x^{\mu} \quad ; \quad B^{(2)} = \frac{1}{2} \mathcal{B}_{\mu\nu} \, d x^{\mu} \wedge d x^{\nu}$$

$$B^{(3)} = \frac{1}{6} \mathcal{B}_{\mu\nu\varrho} \, d x^{\mu} \wedge d x^{\nu} \wedge d x^{\varrho}$$

$$\begin{pmatrix} \mathcal{W}_{\mu} \\ \mathcal{B}_{\mu\nu} \\ \mathcal{B}_{\mu\nu\varrho} \end{pmatrix}_{\alpha\beta} = \begin{pmatrix} W_{\mu}^{\;r} \\ B_{\mu\nu}^{\;r} \\ B_{\mu\nu\varrho}^{\;r} \end{pmatrix} \left(d_{r} \right)_{\alpha\beta} \in Lie \left(\mathcal{D} \right)$$

$\longrightarrow$

1) (continued)

From eq. 83 we infer

$$\frac{1}{2}\,\varepsilon^{\,\mu\nu\varrho\sigma}\,(\,\partial_{\,\nu}\,\mathcal{B}_{\,\varrho\sigma}\;+\;[\,\mathcal{W}_{\,\nu}\,,\,\mathcal{B}_{\,\varrho\sigma}\,]\,)\;\equiv\;0\quad;\quad\widetilde{\mathcal{B}}^{\,\mu\,\nu}\;\doteq\;\frac{1}{2}\,\varepsilon^{\,\mu\nu\varrho\sigma}\,\mathcal{B}_{\,\varrho\sigma}$$

(84)

$$\longrightarrow\quad\partial_{\,\nu}\,\widetilde{\mathcal{B}}^{\,\mu\,\nu}\;+\;\left[\,\mathcal{W}_{\,\nu}\,,\,\widetilde{\mathcal{B}}^{\,\mu\,\nu}\,\right]\;\equiv\;0\quad\forall\;\;\mathcal{D}\,(\,\mathcal{G}\,)$$

It is worth noting that for the time-space signatures $(+ - - -)$ **as well as** $(- + + +)$ **and independent of the sign of** $\varepsilon_{\,0123}\;\pm\;1$ **we have contrary to d=4 Euclidean space**

(85)
$$\varepsilon^{\,\mu\nu\varrho\sigma}\;=\;-\,\varepsilon_{\,\mu\nu\varrho\sigma}\quad\longrightarrow\quad\widetilde{\widetilde{\mathcal{B}}}_{\,\mu\nu}\;=\;-\,\mathcal{B}_{\,\mu\nu}$$

The structure of eq. 75 implies that the Bianchi identity is indeed an identity, provided the field strengths are derived from a connection , and complete connections then imply regularity conditions for all Lie algebra valued such associated with any one out of all representations $\mathcal{D}\,(\,\mathcal{G}\,)$ **, not to be obstructed by singular connections for which the identity may be violated minimally in one singular point . It is this regularity feature, which is at least not clearly satisfied in lattice QCD or versions thereof e.g. without quark flavors.** [a]

$\longrightarrow$

[a] **I am indepted to Uwe-Jens Wiese for his patience and many discussions, in which he brought up the question as to fulfilment of Bianchi identities in lattice QCD .**

2) complete connections and boundary conditions for finite time $\Delta t = \beta = 1 / T$ thermal path integrals with respect to gauge bosons

Subsection 1 c-2 is conceived particularly to assess the boundary conditions between two spacelike parallel planes with constant time each, in the rest system of a thermal equilibrium ensemble, set apart by $\Delta t = \beta = 1 / T$ with T denoting the temperature, and at finite but asymptotically large space volume V . The Gibbs potential is then associated with the generating functional for fixed intensive variables $\beta , \chi_a = \mu_a \beta ; \mu_a$: chemical potentials

(86)
$$Z (\beta , \mu_a , V) \simeq |_{V \to \infty} \; tr \left(\exp \left[- \beta \mathbf{H} + \sum_b \chi_b \mathbf{N}_b \right] \right)$$
$$\text{with } \chi_a = \mu_a / T \; ; \; Z \sim e^{g V} \; ; \; g = g (\beta , \chi_a) = \beta p$$

In eq. 86 H , $\mathbf{N}_a$; $a = 1, \cdots , n_{fl}$ denote the conserved operators for energy and net charges respectively , and p the pressure .

Just in order to keep most simple, precise and correct notions, the number of exactly conserved charges in the absence of all electroweak interactions and leptons and scalar elementary fields , for n_{fl} of quarks (and antiquarks) with nondegenrate and nonzero masses is equal to n_{fl} .

A formally equivalent way to calculate Z as defined in eq. 86 is to use imaginary time and perform a path integral weighted as $\exp (- S_\beta)$ using the Euclidean form of the time rstricted action in the associated Euclidean field theory . An excellent exposé of thermodynamic notions in the environment

→

of local fields can be found in ref. [21-2002] .

Now we go back to subsection 1 c-2 and recall eq. 82 below

$$U \left(x \overset{C}{\leftarrow} y \, ; \, W^{\,\Omega\,(1)} \right) \; = \; \Omega \, (\, x \,) \, U \left(x \overset{C}{\leftarrow} y \, ; \, W^{\,(1)} \right) \Omega^{\,-1} \, (\, y \,)$$

(87)

$$W^{\,\Omega\,(1)} \, (\, z \,) \; = \; \Omega \, (\, z \,) \, \left(\partial_{\,z} \, + \, W^{\,(1)} \, (\, z \,) \right) \Omega^{\,-1} \, (\, z \,) \; = \; \Omega \, D^{\,(1)} \, \Omega^{\,-1} \, (\, z \,)$$

$\Omega \, (\, x \,) \, , \, \Omega \, (\, y \,) \, , \, \Omega \, (\, x \,) \, \in \, \mathcal{D} \, (\, \mathcal{G} \,) \;\; \forall \;\; \mathcal{D}$ **and** $\forall$ **complete** $W^{\,(1)} \, \big|_{\,\mathcal{D}}$

The relation on the first line of eqs. 82 , 87 demonstrates that complete connections – by splitting the arguments of $\Omega \, (\, x \,)$ from $\Omega \, (\, y \,)$ for all $\mathcal{D} \, (\, \mathcal{G} \,)$ – allow to reconstruct the full global structure of $\mathcal{G}$. Hence ensuing regularity conditions, based on the properties derived for complete connections, maintaining associated extended gauge invariance can only tolerate periodic connections modulo gauge transformations , from the space time points $x \, = \, (\, 0 \, , \, \vec{x} \,) \, \rightarrow \, x \, + \, \Delta \, t \, = \, (\, \Delta \, t \, , \, \vec{x} \,)$

(88)

$$W^{\,(1)} \, (\, x \, + \, \Delta \, t \,) \; = \; W^{\,\Omega(1)} \, (\, x \,) \; = \; \Omega \, \left(\partial \, + \, W^{\,(1)} \right) \Omega^{\,-1} \, (\, x \,)$$

for a suitably general set of gauge transformations $\Omega \, (\, x \,) \, \in$ any $\mathcal{D} \, (\, \mathcal{G} \,)$

$W^{\,\Omega\,(1)}$ is defined in eq. 69 . The generalized boundary conditions in eq. 88 maintain periodicity of (fully) gauge invariant local quantities . →

Furthermore the quantity $U\left(x+\Delta t \overset{C}{\leftarrow} x\,;W^{(1)}\right)$ **transforms under a local gauge tranformation as**

$$\begin{aligned} & U\left(x+\Delta t \overset{C}{\leftarrow} x\,;W^{(1)}\right) \rightarrow \\ & U\left(x+\Delta t \overset{C}{\leftarrow} x\,;W^{\Omega\,(1)}\right) = \\ & = \Omega\,(x+\Delta t\,)\,U\left(x+\Delta t \overset{C}{\leftarrow} x\,;W^{(1)}\right)\Omega^{-1}(x) \end{aligned} \tag{89}$$

with $\Omega\,(x+\Delta t\,) \neq \Omega\,(x\,)$ **for general allowed gauge transformations**

As a consequence what is known as the trace of the Polyakov loop

$$\begin{aligned} & Tr\,U\left(x+\Delta t \overset{C}{\leftarrow} x\,;W^{\Omega\,(1)}\right) = \\ & = Tr\,\Omega^{-1}(x)\left(\Omega\,(x+\Delta t\,)\;U\left(x+\Delta t \overset{C}{\leftarrow} x\,;W^{(1)}\right)\right) \\ & \neq Tr\,U\left(x+\Delta t \overset{C}{\leftarrow} x\,;W^{(1)}\right) \end{aligned} \tag{90}$$

is not gauge invariant within the conditions imposed by complete connections .

The consequences from eqs. 88 - 90 are in conflict with the conditions imposed on lattice QCD applied to the thermodynamical environment .

Specific non-complete fiber spaces (not all manifolds) were defined and discussed in ref. [22-1991] .

The general option therein can be adapted to complete connections , notwithstanding the alternative conjectures suggested by the authors . This ends section 1 d and all of chapter 1 .

2 - Outlook

1) Gauge boson pair condensation

Some time ago I set out to study the consequences of gauge boson pair condensation as a coherence phenomenon of pairs and eventual multiples of gauge bosons in QCD [23-2010] , [24-1979] . This involved also collaborations with Harald Fritzsch, Luzi Bergamin, Wolfgang Ochs and Sonia Kabana . With the latter we investigate the phase structure of QCD especially at vanishing chemical potentials [25-2010] and thereby the relevance of respecting regularity conditions reveals an eventual disagreement with results of thermal studies in lattice QCD .

2) The study presented here contains recent results as characteristic for ’complete connections’ .

3) This part of the discussion is more or less final, but there is some way to go to an analytical derivation of the phases of QCD .

— Thank you —

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