Neutrino flavors within the left chiral environment of the standard model – easy to associate but difficult to see beyond

Peter Minkowski

Albert Einstein Center for Fundamental Physics - ITP, University of Bern

Abstract

In this lecture the general discussion of the seesaw mechanism will be complemented illustrating the difficulties of gaining a profound extension of knowledge from the experimental information behind the presently observed neutrino flavor oscillation patterns, without at the same time beeing driven to relatively large violations of leptonic numbers, not observed so far.

Kyoto Sangyo University , 11. March - 19. March, 2011

List of contents

Supplementary material

$\longrightarrow$

Appendices

1-1 There does not exist a symmetry – within the standard model including gravity and containing only chiral spin $\frac{1}{2}$ 16 families of SO (10) – which could enforce the vanishing of neutrino mass(es) .

The divergence of the current associated to the global charge B - L for three standard model families of 15 base fields – in the left chiral basis removing – to infinite mass – the 16-th components $(\mathcal{N})$ pertaining to one full 16-representation of SO (10) [spin (10)]

(1)
$$\left(\begin{array}{cccccccccc} u^{\,1} & u^{\,2} & u^{\,3} & \nu & | & \mathcal{N} & \widehat{u}^{\,3} & \widehat{u}^{\,2} & \widehat{u}^{\,1} \\ d^{\,1} & d^{\,2} & d^{\,3} & e^{\,-} & | & e^{\,+} & \widehat{d}^{\,3} & \widehat{d}^{\,2} & \widehat{d}^{\,1} \end{array} \right)^{\dot{\gamma} \rightarrow L} = (f)^{\,\dot{\gamma}}$$

and admitting a gravitational background field is in this minimal neutrino flavor embedding anomalous , i.e. the global symmetry is broken by winding gravitational fields [1-2001] . $\longrightarrow$

$$j_\varrho(B-L)|_{3\times 15} =$$

$$\sum_{fam}\left[\begin{array}{c} \frac{1}{3}\begin{pmatrix} (u^*)^{\alpha\,\dot{c}}\,(\sigma_\mu)_{\alpha\,\dot{\gamma}}\,(u)^{\dot{\gamma}\,c} - (\widehat{u}^*)^{\alpha\,c}\,(\sigma_\mu)_{\alpha\,\dot{\gamma}}\,(\widehat{u})^{\dot{\gamma}\,\dot{c}} \\ + (d^*)^{\alpha\,\dot{c}}\,(\sigma_\mu)_{\alpha\,\dot{\gamma}}\,(d)^{\dot{\gamma}\,c} - \left(\widehat{d}^*\right)^{\alpha\,c}\,(\sigma_\mu)_{\alpha\,\dot{\gamma}}\,\left(\widehat{d}\right)^{\dot{\gamma}\,\dot{c}} \end{pmatrix} \\ -\left(e^-\right)^{*\,\alpha}(\sigma_\mu)_{\alpha\,\dot{\gamma}}\left(e^-\right)^{\dot{\gamma}} + \left(e^+\right)^{*\,\alpha}(\sigma_\mu)_{\alpha\,\dot{\gamma}}\left(e^+\right)^{\dot{\gamma}} \\ -\,(\nu)^{*\,\alpha}\,(\sigma_\mu)_{\alpha\,\dot{\gamma}}\,(\nu)^{\dot{\gamma}} \end{array}\right] e^{\mu}_{\varrho}$$

$g_{\varrho\,\tau} = e^{\mu}_{\varrho}\,\eta_{\mu\,\nu}\,e^{\nu}_{\tau}$ **: metric** ; e^{μ}_{ϱ} **: vierbein** ; * **: hermitian operator conjugation**

$(u^*)^{\alpha\,\dot{c}} \equiv \left(u^{\dot{\alpha}\,c}\right)^*$; $\eta_{\mu\nu} = diag(1,-1,-1,-1)$ **: tangent space metric**

$^c\ (\,^{\dot{c}}\,)$ **: color and anticolor** ; $c = 1,2,3$

(2)

The contribution of charged fermion (pairs) $q\,,\widehat{q}\,;\ e^{\mp}$ **can be combined to vector currents – Dirac doubling –** $\overline{q}\,\gamma_\mu\,q\,;\ \overline{e}\,\gamma_\mu\,e$ **with** $q \rightarrow u,d,c,s,t,b\,;\ e \rightarrow e^-,\mu^-,\tau^-$ **.** $\longrightarrow$

The anomalous Ward identy for the B - L current (- density) defined in eq. 2 takes the form

$$d^4 x \sqrt{|g|}\, D^{\varrho}\, j_{\varrho}(B - L)|_{3\times 15} = 3\, \widehat{A}_1(X)$$

$$\widehat{A}_1(X) = -\frac{1}{24}\, trX^2 \;\; ; \;\; (X)^a{}_b = \frac{1}{2\pi}\frac{1}{2}\, dx^{\varrho} \wedge dx^{\tau}\, (R^a{}_b)_{\varrho\tau}$$

(3)

$$(R^a{}_b)_{\varrho\tau} : \begin{cases} \textbf{Riemann curvature tensor} \\ \textbf{mixed components : } {}^a{}_b \longrightarrow \textbf{tangent space} \\ \qquad\qquad\qquad\qquad\quad {}_{\mu\nu} \longrightarrow \textbf{covariant space} \end{cases}$$

$$D^{\varrho}\, j_{\varrho}(B - L)|_{3\times(16)} = 0 \qquad\qquad \longrightarrow$$

Before discussing the extension $j_{\varrho}(B - L)|_{3\times(15)} \rightarrow j_{\varrho}(B - L)|_{3\times(16)}$ **which renders the latter current conserved, lets define the quantities appearing in eq. 3 :**

$$(R^a{}_b)_{\varrho\tau} = e^a_{\mu}\, e_{b\,\nu}\, (R^{\mu}{}_{\nu})_{\varrho\tau} \;\; ; \;\; e_{b\,\nu} = \eta_{bb'}\, e^{b'}_{\nu}$$

(4)

$$(R^{\mu}{}_{\nu})_{\varrho\tau} = (\partial_{\varrho}\Gamma_{\tau} - \partial_{\tau}\Gamma_{\varrho} + \Gamma_{\varrho}\Gamma_{\tau} - \Gamma_{\tau}\Gamma_{\varrho})^{\mu}{}_{\nu}$$

$(\Gamma^{\mu}{}_{\nu})_{\tau}$ **: matrix valued** $(GL(4, \mathbb{R}))$ **connection ; minimal here**

$\longrightarrow$

For clarity eq. 3 is repeated below

$$d^4 x \sqrt{|g|}\, D^{\varrho}\, j_{\varrho}(B - L)|_{3\times 15} = 3\,\widehat{A}_1(X)$$

$$\widehat{A}_1(X) = -\frac{1}{24}\, trX^2 \;\; ; \;\; (X)^a{}_b = \frac{1}{2\pi}\frac{1}{2}\, dx^{\varrho} \wedge dx^{\tau}\, (R^a{}_b)_{\varrho\tau}$$

(3) $(R^a{}_b)_{\varrho\tau}$: { **Riemann curvature tensor** / **mixed components :** ${}^a{}_b$ $\longrightarrow$ **tangent space** / ${}_{\mu\nu}$ $\longrightarrow$ **covariant space**

$$D^{\varrho}\, j_{\varrho}(B - L)|_{3\times(16)} = 0 \qquad \longrightarrow$$

In eq. 3 $\widehat{A}(X \rightarrow \lambda) = \frac{1}{2}\lambda / \sinh(\frac{1}{2}\lambda)$ **denotes the Atiyah - Hirzebruch character or** $\widehat{A}$ **– genus [2-1966] with its integral over a compact , euclidean signatured closed manifold** M_4 **, capable of carrying on SO4 - spin structure , becomes the index of the associated** *elliptic* **Dirac equation**

(5) $$\int \widehat{A}(X_E) = n_R - n_L = \text{integer}$$

In eq. 5 $n_{R,L}$ **denote the numbers of right - and left - chiral solutions of the Dirac equation on** M_4 **. The index** ${}_E \longrightarrow X_E$ **shall indicate the euclidean transposed curvature 2 - form , and is** *adapted* **here to physical curved and uncurved space time .** $\longrightarrow$

For the latter case the first relation in eq. 3 yields the integrated form – in the limit of infinitely heavy $\mathcal{N}_F$ (eq. 1) –

(6)
$$\Delta_{R-L} \, n_{\nu} \; = \; \int d^4 x \, \sqrt{| \, g \, |} \, D^{\mu} \, j_{\mu}^{B \, - \, L \, (15)} \; = \; 3 \; \Delta \, n \, (\, \widehat{A} \,)$$

$$3 \; = \; \text{number of families = odd} \quad ; \quad m_{\nu_F} \; \rightarrow \; 0$$

In eq. 6 $\Delta_{R-L} \, n_{\nu}$ denotes the difference of right - chiral $(\, \widehat{\nu} \,)$[a] and left - chiral $(\, \nu \,)$ flavors between times $t \; \rightarrow \; \pm \infty$.

Here a subtlety arises *precisely* because the number of families on the level of G_{SM} is odd , and the light neutrino flavors are not 'Dirac - doubled' , which according to eq. 6 could potentially lead to a change in fermion number being odd , which violates the rotation by $2 \, \pi$ symmetry , equivalent to $\widehat{\Theta}^{\,2}$ $\left(\, CPT^{\,2} \, \right)$, *unless*[b]

(7)
$$\Delta \, n \, (\, \widehat{A} \,) \; = \; \text{even} \qquad (\, \surd \; \text{for } dim \; = \; 4 \text{ mod } 8 \,)$$

$\rightarrow$

---

[a] $\widehat{\nu}_{\alpha} \; \equiv \; \varepsilon_{\alpha \, \beta} \, (\, \nu^{\,*} \,)^{\,\gamma} \; ; \; \varepsilon \; = \; i \, \sigma_2 \; ;$ (2nd Pauli matrix) stands for the left-chiral neutrino fields transformed to the right-chiral basis .

[b] The obviously nontrivial relation between the compact Euclidean - and noncompact asymptotic and locality restricted form of the index theorem involves not clearly formulated *boundary conditions* .

We now turn to the SO (10) inspired cancellation of the gravity induced anomaly, giving rise to the completion of neutrino flavors to 3 families of 16-plets , sometimes called 'right-handed' neutrino flavors, denoted $\mathcal{N}$ in the left-chiral basis in eq. 1 [3-2007]

(1)
$$\left(\begin{array}{cccccccccc} u^{1} & u^{2} & u^{3} & \nu & \mid & \mathcal{N} & \widehat{u}^{3} & \widehat{u}^{2} & \widehat{u}^{1} \\ d^{1} & d^{2} & d^{3} & e^{-} & \mid & e^{+} & \widehat{d}^{3} & \widehat{d}^{2} & \widehat{d}^{1} \end{array} \right)^{\dot{\gamma} \rightarrow L} = (f)^{\dot{\gamma}}$$

(8)
$$j_{\varrho} (B - L) |_{3 \times 15} \rightarrow j_{\varrho} (B - L) |_{3 \times 16}$$

$$d^{4} x \sqrt{| g |} D^{\varrho} j_{\varrho} (B - L) |_{3 \times 15} = 3 \widehat{A}_{1} (X)$$

$$\widehat{A}_{1} (X) = - \frac{1}{24} tr X^{2} \; ; \; (X)^{a}{}_{b} = \frac{1}{2 \pi} \frac{1}{2} d x^{\varrho} \wedge d x^{\tau} \left(R^{a}{}_{b} \right)_{\varrho \tau}$$

(3)
$$\left(R^{a}{}_{b} \right)_{\varrho \tau} : \left\{ \begin{array}{l} \textbf{Riemann curvature tensor} \\ \textbf{mixed components : } {}^{a}{}_{b} \longrightarrow \textbf{tangent space} \\ \qquad\qquad\qquad\qquad\quad {}_{\mu \nu} \longrightarrow \textbf{covariant space} \end{array} \right.$$

$$D^{\varrho} j_{\varrho} (B - L) |_{3 \times (16)} = 0 \qquad \longrightarrow$$

$\longrightarrow$

$$j_{\varrho}(B-L)|_{3\times 15} \to j_{\varrho}(B-L)|_{3\times 16} =$$

$$\sum_{fam} \left[\begin{array}{l} \frac{1}{3}\left(\begin{array}{l} (u^{*})^{\alpha\,\dot c}\,(\sigma_{\mu})_{\alpha\,\dot\gamma}\,(u)^{\dot\gamma\,c} - (\widehat{u}^{*})^{\alpha\,c}\,(\sigma_{\mu})_{\alpha\,\dot\gamma}\,(\widehat{u})^{\dot\gamma\,\dot c} \\ + (d^{*})^{\alpha\,\dot c}\,(\sigma_{\mu})_{\alpha\,\dot\gamma}\,(d)^{\dot\gamma\,c} - \left(\widehat{d}^{*}\right)^{\alpha\,c}\,(\sigma_{\mu})_{\alpha\,\dot\gamma}\,\left(\widehat{d}\right)^{\dot\gamma\,\dot c} \end{array} \right) \\ - (e^{-})^{*\,\alpha}\,(\sigma_{\mu})_{\alpha\,\dot\gamma}\,(e^{-})^{\dot\gamma} + (e^{+})^{*\,\alpha}\,(\sigma_{\mu})_{\alpha\,\dot\gamma}\,(e^{+})^{\dot\gamma} \\ - (\nu)^{*\,\alpha}\,(\sigma_{\mu})_{\alpha\,\dot\gamma}\,(\nu)^{\dot\gamma} \quad + \underbrace{(\mathcal{N})^{*\,\alpha}\,(\sigma_{\mu})_{\alpha\,\dot\gamma}\,(\mathcal{N})^{\dot\gamma}} \end{array} \right] e^{\mu}_{\varrho}$$

$g_{\varrho\,\tau} = e^{\mu}_{\varrho}\,\eta_{\mu\,\nu}\,e^{\nu}_{\tau}$ **: metric** ; e^{μ}_{ϱ} **: vierbein** ; * **: hermitian operator conjugation**

$(u^{*})^{\alpha\,\dot c} \equiv \left(u^{\dot\alpha\,c}\right)^{*}$; $\eta_{\mu\nu} = diag\,(1,-1,-1,-1)$ **: tangent space metric**

$^{c}\,(\,^{\dot c}\,)$ **: color and anticolor** ; $c = 1,2,3$

$$D^{\varrho}\; j_{\varrho}(B-L)|_{3\times(16)} = 0 \qquad\qquad \to \tag{9}$$

Let me illustrate the triple-doubling inherent in the elimination of the anomaly in the covariant divergence of $j_{\varrho}(B - L)|_{3\times 15}$ in eq. 2 as seen through the left-chiral basis , repeating only the ν , $\mathcal{N}$ components of the B - L current in eq. 9

$$j_{\varrho}(B - L)|_{3\times 16} =$$

$$\sum_{fmlies} \left[\begin{array}{c} \cdots \\ -(\nu)^{*\alpha}(\sigma_{\mu})_{\alpha\dot{\gamma}}(\nu)^{\dot{\gamma}} \quad + \underbrace{(\mathcal{N})^{*\alpha}(\sigma_{\mu})_{\alpha\dot{\gamma}}(\mathcal{N})^{\dot{\gamma}}} \end{array} \right]$$

(10)

	$\nu_F^{\dot{\gamma}}$	$\mathcal{N}_F^{\dot{\gamma}}$
$B - L$	-1	$+1$

; $F = 1, 2, 3$ **family**

1-2 There does not exist a symmetry – within the standard model including gravity and containing only chiral 16 families of SO (10) – enforcing the vanishing of neutrino mass(es), yet there exist chiral extensions, which accomplish this .

Here I briefly describe one such extension. It consists of replacing in each family the SO (10) induced $\mathcal{N}_F$ flavors by four alternative (sterile) $\mathcal{X}_{J=2,3,4,5\,;\,F}$ flavors, singlets under the electroweak gauge group with genuinely chiral B - L charges, changing the structure in eq. 10 to

$$j_{\varrho}\,(B - L)|_{3\times 16} =$$

$$\sum_F \left[\begin{array}{c} \cdots \\ -\,(\nu)^{*\,\alpha}\,(\sigma_{\mu})_{\alpha\,\dot{\gamma}}\,(\nu)^{\dot{\gamma}} \quad + \underbrace{\sum_{J=2}^{5}(\chi)_J\,(\mathcal{X}_J)^{*\,\alpha}\,(\sigma_{\mu})_{\alpha\,\dot{\gamma}}\,(\mathcal{X}_J)^{\dot{\gamma}}} \end{array} \right]$$

	$\nu^{\dot{\gamma}}_F = \mathcal{X}^{\dot{\gamma}}_{1,F}$	$\mathcal{X}^{\dot{\gamma}}_{2,F}$	$\mathcal{X}^{\dot{\gamma}}_{3,F}$	$\mathcal{X}^{\dot{\gamma}}_{4,F}$	$\mathcal{X}^{\dot{\gamma}}_{5,F}$
$B - L = (\chi)_J$	-1	-5	-9	7	8

; $F = 1, 2, 3$ **family**

$J = 1, 2 \cdots, 5$

(11)

→

The genuinely chiral couplings $(\chi)_{\,J = 1, \cdots, 5} \; = \; [\, -1 \, , \; -5 \, , \; -9 \, ; \; 7 \, , \; 8 \,]$ for neutrino flavors as shown in eq. 11 with 5 chiral base flavors merit some comments :

1) a sequence of charges $(\chi)_{\,J} \, , \; J \; = \; 1, \cdots, N$ with respect to the left-chiral basis – to be specific – shall be called *genuinely* chiral , if none of the charges vanishes and no pairs of opposite charge $[\, \pm \, (\chi) \,]$ are admitted.

2) the absence of an anomaly of the associated chiral current , of the form given for neutrino flavors in eqs. 2 , 8 and 11 including also gravitational fields leads *in 4 dimensions* to the two conditions

$$\sum_{J}^{N} \, (\chi)_{\,J} \; = \; 0 \quad , \quad \sum_{J}^{N} \, [\, (\chi)_{\,J} \,]^{\,3} \; = \; 0 \tag{12}$$

3) there does not exist a genuinely chiral set $\{ \, (\chi)_{\,J} \, , \; J \; = \; 1, \cdots, N \, \}$ for $N \; < \; 5$.

For N = 3,4 it is equivalent to show that the two equations

$$\begin{array}{l} A \; + \; B \; = \; C \; + \; D \quad , \quad A^{\,3} \; + \; B^{\,3} \; = \; C^{\,3} \; + \; D^{\,3} \qquad \longrightarrow \\ A \; = \; x \; - a \, , \; B \; = \; x \; + \; a \, , \; C \; = \; x \; - \; b \, , \; D \; = \; x \; + \; b \longrightarrow \\ x \, a^{\,2} \; = \; x \, b^{\,2} \quad \longrightarrow \quad \left\{ \; x \; = \; 0 \;\; \text{or} \;\; x \; \neq \; 0 \, ; \, b \; = \; \pm \, a \right. \end{array} \tag{13}$$

have no solution, satisfying the conditions for genuine chirality .

$\longrightarrow$

4) There are infinitely many solutions for $N \geq 5$, with chiral charges relatively irrational as well as rational . For integer values and N = 5 with the norm $|\,(\chi)\,| = \sum |\,(\chi)_J\,|$ the solution with smallest norm is unique up to an overall change of sign [a]

$$(\chi)_J = [\,-1\,,\,-5\,,\,-9\,;\,7\,,\,8\,] \tag{14}$$

Some conclusions from sections 1-1 and 1-2 .

C1) The oscillation phenomena indicate clearly , that a *genuinely chiral* extension of B - L to a conserved, global symmetry, generating a *continous* U1 - group of tranformations, is not involved.

C2) On the other hand the binary code of a (minimally) supposed unifying gauge group SO or spin (10) could, if B - L is *not* gauged, equivalently generate a global symmetry of the vectorlike nature. The latter however would allow neutrino mass through the (electroweak doublet-singlet) pairing

$$-\mathcal{L}_{\mathcal{M}} = \mu_{FG} \mathcal{N}^{F}_{\dot{\gamma}} \nu^{\dot{\gamma}\,G} + h.c. \quad ; \quad F, G = 1, 2, 3 \ \ \text{family} \tag{15}$$

without symmetry restrictions on the mass matrix μ_{FG} in eq. 15 .

$\longrightarrow$

[a] It is due to Paul Frampton , on a beautiful morning in 1993 , along the coastal range above the mediterranean sea near Cassis, France .

C3) Then however the question arises, why the mass matrix μ , involving the scalar doublet(s) within the electroweak gauge group, also generating masses of charged spin $\frac{1}{2}$ fermions, gives rise to very small physical neutrino masses. Thus we follow the *hypothesis* that SO (10) *is* gauged and that it is the *large* mass scale of the gauge boson associated with B - L in particular, which distinguishes neutrino flavors [4-1975], [5-1975], [6-1976] .

2-1 The Majorana logic [7-1994] and mass from mixing –
setting within the ’tilt to the left’ or ’seesaw’ of type I ($\cdots$)

Within the subgroup decompositions of SO (10) the ’tilt to the left’ does not appear obvious

$$\begin{array}{ccccc} & & \text{spin } (10) & & \\ & \swarrow & & \searrow & \\ \text{spin } (6) \equiv \text{SU4} & & \times & & \text{spin } (4) \equiv \text{SU2}_{L} \times \text{SU2}_{R} \end{array}$$

lepton number as 4th color [8-1974]

$$\begin{array}{ccccc} \downarrow & & & & \downarrow \\ \text{SU3}_{c} \times \text{U1}_{B-L} & & \times & & \text{SU2}_{L} \times \text{U1}_{I_{3R}} \\ & \searrow & & \swarrow & \\ & & \text{SU3}_{c} \times \text{U1}_{Q_{e.m.}} & & \end{array}$$

$$Q_{e.m.} / e = I_{3L} + I_{3R} + \frac{1}{2} (B - L) \tag{16}$$

$\longrightarrow$

In eq. 16 the conserved charge-like gauges are marked in red.

The large scale breaking of *gauged* B - L or 'tilt to the left' was not assumed essential in refs. [4-1975] - [6-1976] and brings about a definite 'mass from mixing' scenario [9-1977] , [10-1979/80] to which we turn below.

The Majorana logic characterized by $\mathcal{N}_F$

Here we consider the alternative subgroup decomposition

(17)
$$\textbf{spin}\ (10) \ \longrightarrow\ \textbf{SU5}\ \times\ \textbf{U1}_{\ J_{\ 5}}$$

Among the 3 generators of spin (10) commuting with SU3 $_c$, $I_{\ 3\ L}$, $I_{\ 3\ R}$, $B\ - L$ and forming part of the Cartan subalgebra of spin (10) there is one combination, denoted $J_{\ 5}$ in eq. 17, commuting with its largest unitary subgroup SU5 .

The 16 representation in the left-chiral basis displays the charges pertinent to $J_{\ 5}$ normalized to integer values *modulo an overall sign* – as in the discussion of genuinely chiral U1-charges in eq. 14 – but here referring to N = 16

While the Majorana logic indeed opens a 'path' to trace the origin of the 'tilt to the left' , the origin of three families remains unexplained at this stage. $\longrightarrow$

b5a

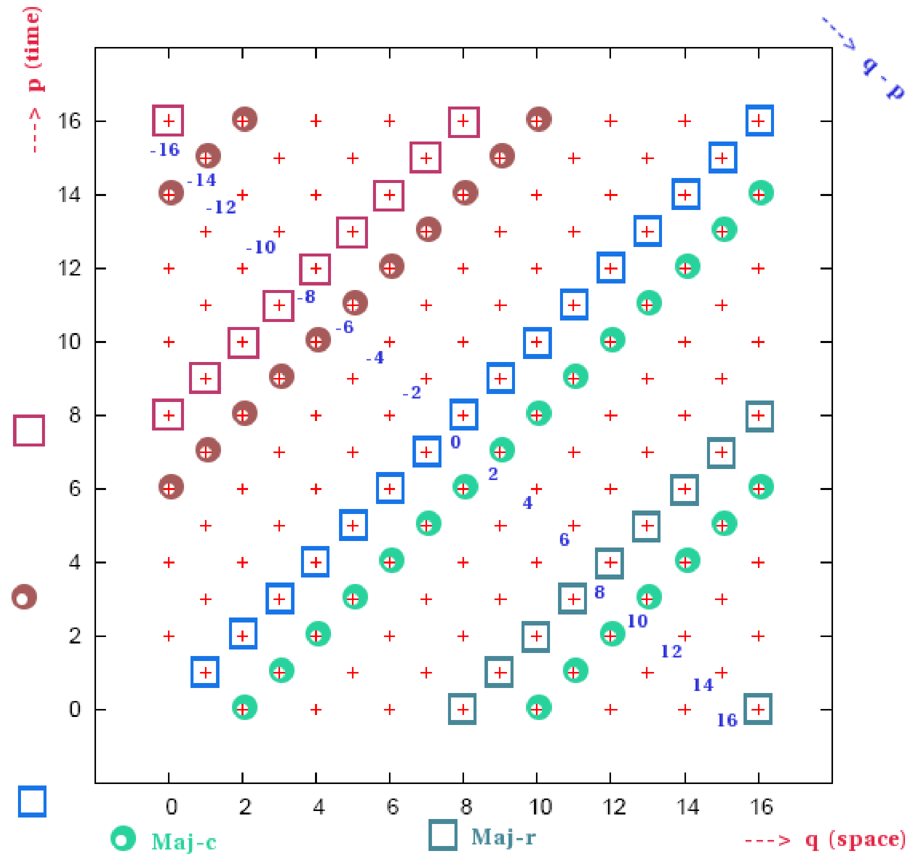

Fig B1 : The complex and real Majorana representations

MajCR $(\, p \, , \, q \,) \qquad \longleftrightarrow$

The associative Clifford algebras $\{\, \Gamma_{\, p\, ,\, q\, ;\, \mathbb{C}} \,\} \ \supset \ \Big\{\, \Gamma_{\, \widetilde{p}\, ,\, \widetilde{q}\, ;\, \mathbb{R}} \,\Big\}$ are constructed in sections 4-1a $\longrightarrow$ 4-1c , 4-2 and Appendices A , B forming the complementary material to the present outline .

p , q denote time like (p) and spacelike (q) dimensions of space-time .

Fig. B1 shows the repartition of real (Maj-r) and complex (Maj-c) character of irreducible *associative* , real (Majorana) Clifford algebras with their characteristic mod 8 property relative to q - p [11-1982].

These representations form the roots of the 'Majorana logic' discussed below . $\longrightarrow$

$$(f)^{\dot{\gamma}} = \begin{pmatrix} u^1 & u^2 & u^3 & \nu & | & \mathcal{N} & \widehat{u}^3 & \widehat{u}^2 & \widehat{u}^1 \\ d^1 & d^2 & d^3 & e^- & | & e^+ & \widehat{d}^3 & \widehat{d}^2 & \widehat{d}^1 \end{pmatrix}^{\dot{\gamma} \to L} \tag{18}$$

$$J_5 \to \begin{pmatrix} 1 & 1 & 1 & -3 & | & 5 & 1 & 1 & 1 \\ 1 & 1 & 1 & -3 & | & 1 & -3 & -3 & -3 \end{pmatrix}$$

The assignment of $J_5 -$ charges in eq. 18 follows from the fermionic oscillator representation of the spin (2n) associated Γ algebra through n such oscillators and the associated embedding spin (10) $\supset$ SU5 [12-1974] for n = 5 here [13-1980]

$$\left\{ a_s , a_t^\dagger \right\} = \delta_{st} \; ; \; s,t = 1,2 \cdots, n \; ; \; \{ a_s , a_t \} = 0 = \left\{ a_s^\dagger , a_t^\dagger \right\} \quad \to \tag{19}$$

$$J_n = \sum_{s=1}^n \begin{pmatrix} a_s^\dagger a_s \\ - a_s a_s^\dagger \end{pmatrix} = 2 \widehat{n} - n ¶_{2^n \times 2^n} \; ; \; \widehat{n} = \sum_{s=1}^n a_s^\dagger a_s$$

$\to$

The eigenvalues (X) *and* multiplicities (#) of J_n

$$\begin{array}{c|cccccc} (X) & n & n-2 & n-4 & \cdots & -n+2 & -n \\ \hline (\#) & \binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n-1} & \binom{n}{n} \end{array} \tag{20}$$

The orthogonal series for n even $\leftrightarrow$ real (spin (8) , spin(12) $\cdots$) has another decompostion within the associated Γ algebra , than the one with n odd $\leftrightarrow$ complex (spin (10) , spin (14) $\cdots$) . We give here the explicit numbers according to eq. 20 for n = 5 , i.e. spin (10)

$$\begin{array}{c|cccccc} (X) & 5 & 3 & 1 & -1 & -3 & -5 \\ \hline (\#) & \binom{5}{0} & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \binom{5}{4} & \binom{5}{5} \\ \textbf{SU5} & \{1\} & \{5\} & \{10\} & \{\overline{10}\} & \{\overline{5}\} & \{\overline{1}\} \end{array} \tag{21}$$

The subset of states in blue in eq. 21 $(X) = \{ 5 , 1 , -3 \}$ forms the 16 representation of spin 10, while those in red $(X) = \{ 3 , -1 , -5 \}$ the complex conjugate $\overline{16}$. $\rightarrow$

This opens the 'path' of linking the 'tilt to the left' with a substructure based on the primary in strength breakdown of the local gauged chargelike symmetry associated with

(22)
$$J_5 = -4\, I_{3\,R} + 3\,(B - L)$$

J_5 **as defined through integer eigenvalues** (X) **given in eqs. 18 and 21 is normalized differently from the other Cartan subalgebra charges** $I_{3\,L}$, $I_{3\,R}$, $B - L$

(23)
$$|\, Q_C \,|^2 = \sum_{\{16\}} (\, Q_C\,(\, f\,)\,)^2 \quad , \quad |\, I_{3\,L} \,|^2 = 2 \quad , \quad |\, I_{3\,R} \,|^2 = 2$$
$$|\, B - L \,|^2 = \frac{16}{3} \quad , \quad |\, J_5 \,|^2 = 80$$

The consequence as far as neutrino-mass and mixing is concerned follows from identifying the J_5 **direction with a major axis of primary spontaneous gauge-symmetry breaking , bringing about the 'tilt to the left' from eq. 15** $\longrightarrow$

$$\mathcal{H}_{\mathcal{M}} = \mu_{FG} \mathcal{N}^{F}_{\dot\gamma} \nu^{\dot\gamma G} + h.c. + \mathcal{H}_{M}$$

(24) $$\mathcal{H}_{M} = \frac{1}{2} M_{FG} \mathcal{N}^{F}_{\dot\gamma} \mathcal{N}^{\dot\gamma G} + h.c. \quad ; \quad F, G = 1, 2, 3$$

$M_{FG} = M_{GF}$: **complex arbitrary otherwise** ; $|M| \gg |\mu|$

It is the primary breakdown along the direction of J_5 which contrary to all 'mirror complexes' brings on the level of (pseudo-) scalar fields to the foreground the complex bosonic 126 and $\overline{126}$ representations of SO10

$$\mathcal{H}_{M} \longleftarrow$$

(25) $$\left(\Phi^{\overline{126}\, F G} \right)^{\overline{\xi}} \left(f_{a\, 16\, F} \right)_{\dot\gamma} \left(f_{b\, 16\, G} \right)^{\dot\gamma} C \left(\begin{array}{c|cc} 126 & 16 & 16 \\ \xi & a & b \end{array} \right) + h.c.$$

$\left(\Phi^{\overline{126}\, F G} \right)^{\overline{\xi}}$: **(pseudo-) scalar fields in the $\overline{126}$ representation of SO (10)**

In eq. 25 $C \left(\begin{array}{c|cc} 126 & 16 & 16 \\ \xi & a & b \end{array} \right)$ **denotes the coupling coefficients, projecting the symmetric product of two 16-representations of spin (10) to the 126 representation of SO (10) .** $\longrightarrow$

The 126 *complex* **representation of SO (10) is singled out by the value of** J_5 **of** $10 = 2 \times 5 \mathcal{N} \mathcal{N}$ **.**
The relatively complex conjugate representations $126 \oplus \overline{126}$ **are contained in the** *real* **,** *reducible* **fivefold antisymmetric tensor representation of SO (10) decomposing into the irreducible pair upon the duality conditions**

$$t^{[A_1 A_2 \cdots A_5]} \quad ; \quad A_{1 \ldots 5} = 1, 2, \cdots, 10$$

$$t^{[A_{\pi 1} A_{\pi 2} \cdots A_{\pi 5}]} = sgn \begin{pmatrix} 1 & 2 & \cdots & 5 \\ \pi_1 & \pi_2 & \cdots & \pi_5 \end{pmatrix} t^{[A_1 A_2 \cdots A_5]}$$

$$\frac{1}{5!} \varepsilon_{A_1 \cdots A_5 B_1 \cdots B_5} t_{\pm}^{[B_1 B_2 \cdots B_5]} = (\pm i) \, t_{\pm}^{[A_1 A_2 \cdots A_5]}$$

$$\varepsilon_{A_1 \cdots A_5 A_6 \cdots A_{10}} = sgn \begin{pmatrix} 1 & 2 & \cdots & 10 \\ \pi_1 & \pi_2 & \cdots & \pi_{10} \end{pmatrix} \varepsilon_{A_{\pi 1} \cdots A_{\pi 5} A_{\pi 6} \cdots A_{\pi 10}}$$

$$\varepsilon_{1 2 \ldots 10} = 1$$

(26)

→

Within the complex spin $(2\nu = 4\tau + 2)$, $\tau = 2, 3, \cdots$ **series –** $\tau = 2 \leftrightarrow$ **spin (10) – the relatively complex conjugate spinorial pair of representations with dimension** $4^{\tau} \leftarrow 16(64, \cdots)$ **and the complex selfdual-antiselfdual pair of representations with dimension**

$\frac{1}{2} \begin{pmatrix} 4\tau + 2 \\ 2\tau + 1 \end{pmatrix} \leftarrow 126\,(11.12.13 = 1716, \cdots)$ **are intrinsically related for** $\tau = 2, 3, 4, \cdots$ **.**

Some conclusions and questions from section 2-1 .

Q1) Is it enough to consider the primary breakdown and its characteristic, the 'tilt to the left' concerning 3 families, as due ***essentially*** **to spin (10) , which is the** ***lowest*** **simple spin group along the complex** ***orthogonal*** **chain ?**

It has been argued interestingly by Feza Gursey and collaborators [14-1975], that it is the chain of exceptional groups which encode intrinsically the number 3, which in turn underlies the 3 as the number of (left-chiral) families as well as the strong interaction gauge group SU3 $_{c}$ **.**

A1) I think the answer is to the affirmative, since all higher gauge groups , including the exceptional chain and especially E8 , but also spin (14) , (18) do ***not*** **explain the #3 of families , rather generate together with even the apparently correct 3 families – for E8 – also mirror families – 3 for E8 , and powers of 2 for the orthogonal chain with** $\tau \geq 3$ **.**

$\longrightarrow$

A1) *continued*

The tentative conclusion remains, that the structure of families has to be explained outside spin (10) and *also* **outside larger unifying gauge groups containing spin (10) , whereas the origin of neutrino mass is layed out by the lowest member of the complex orthogonal chain $\longrightarrow$ spin (10) .**

C4) The two apparently different phenomena of a) 'tilt to the left' and b) baryon number violation are intrinsically associated with the *unusual* **sequence of (pseudo)scalar fields generating primary breakdown . We use the notation (eq. 17)**

$$\text{spin}\ (10) \ \longrightarrow \ \text{SU5} \ \times \ \text{U1}_{\ J_{\ 5}} \ \longrightarrow \ \text{SU3}_{\ c} \ \times \ \text{SU2}_{\ L} \ \times \ \text{U1}_{\ \mathcal{Y}} \ = \ \text{G}_{\ s.m.}$$

(27)
$$\begin{array}{lll} [16] & = & \{1\}_{\ +5} \quad + \quad \{10\}_{\ +1} \quad + \quad \{\overline{5}\}_{\ -3} \\ [\overline{16}] & = & \{1\}_{\ -5} \quad + \quad \{\overline{10}\}_{\ -1} \quad + \quad \{5\}_{\ +3} \end{array}$$

$$\{\ \overline{5}\ \}_{\ -3} \ = \ \Big]\ (\ \overline{3}\ ,\ 1\ )_{\ +\frac{1}{3}}\ \Big[_{\ -3} \ + \ \Big]\ (\ 1\ ,\ 2\ )_{\ -\frac{1}{2}}\ \Big[_{\ -3}$$

$\longrightarrow$

C) *continued*

(p)scalar SO (10) representation	**active components**	**induced (a)symmetries**	**preserved gauge group**
$\left.\begin{array}{l}[\,126\,]_{\mathbb{C}}\\ [\,\overline{126}\,]_{\overline{\mathbb{C}}}\end{array}\right\}\rightarrow$	$\{\,1\,\}_{+10}$ $\{\,\overline{1}\,\}_{-10}$	P : **'tilt to the left'** $\mathcal{N}\mathcal{N}$ **mass** , $B\,-\,L$ $CP\ \downarrow$	**SU5**
$[\,45\,]_{\mathbb{R}}\ \}\ \nearrow\!\rightarrow$	$\{\,24\,\}_{0}\ \downarrow$ $]\,(\,1\,,\,1\,)_{0}\,[_{0}$	$B\,,\,L\,,\,\updownarrow\,CP$	**G** $_{s.m.}$
$[\,10\,]_{\mathbb{R}}\ \}\ \nearrow\!\rightarrow$	$]\,(\,1\,,\,2\,)_{-\frac{1}{2}}\,[_{-3}$ $]\,(\,1\,,\,\overline{2}\,)_{+\frac{1}{2}}\,[_{+3}$	$\mathcal{N}\nu$ **mass** $\uparrow$ $\overline{q}q$ **mass** CKM $CP\ \uparrow$	**SU3** $_{c}\times$ **U1** $_{e.m.}$

(28)

2-2 Mass from mixing for light ν flavors or 'seesaw'

Having outlined the 'fault-lines'

Fig F1 : Fault-lines of a quartz crystal (and carved lizard) [f1] $\longleftrightarrow$

$\longrightarrow$

of primary and secondary breakdown of charge-like gauge interactions , let me turn to some *general* consequence for neutrinos, light and heavy . To this end we take up eq. 24 repeated below

$$\mathcal{H}_{\mathcal{M}} = \mu_{FG} \mathcal{N}^{F}_{\dot{\gamma}} \nu^{\dot{\gamma} G} + h.c. + \mathcal{H}_{M}$$

(24) $$\mathcal{H}_{M} = \frac{1}{2} M_{FG} \mathcal{N}^{F}_{\dot{\gamma}} \mathcal{N}^{\dot{\gamma} G} + h.c. \; ; \; F, G = 1, 2, 3$$

$$M_{FG} = M_{GF} \quad : \quad \text{complex arbitrary otherwise} \quad ; \quad | M | \gg | \mu |$$

As the (p)scalar $\left[126 , \overline{126} \right]$ representations are singled out through their major role in the primary breakdown along the J_5 direction (eq. 28) we locate the SU2 $_L$ $\times$ U1 $_Y$ $\rightarrow \left] (1, 3)_{-1} \right[_{-6}$ triplet therein (seesaw of type II [f2]) . The complete decomposition of all $f \times f'$ couplings is given in Appendix E , from which we display eq. 141 as eq. 30 below . The two relatively hermitian conjugate triplets and their e.m. charges are

(29)
$$[126] \subset \left(T^{0}, T^{-}, T^{--} \right) \leftrightarrow \left(\overline{T}^{0}, \overline{T}^{+}, \overline{T}^{++} \right) \subset [\overline{126}]$$

$$\overline{T} = T^{*}$$

$\rightarrow$

f3

$$
\begin{array}{lll|l}
[126] & & \{\overline{15}\}_{-6} & \rbrack(\overline{6}\,,\,1)_{+\frac{2}{3}}\lbrack_{-6} + \rbrack(\overline{3}\,,\,2)_{-\frac{1}{6}}\lbrack_{-6} + \underline{\rbrack(1\,,\,3)_{-1}\lbrack_{-6}} \\
 & & & \qquad\qquad\qquad c.c.\ \updownarrow \\
[126] & [120] & \{45\}_{-2} & \\
[120] & & \{\overline{45}\}_{+2} & \left[\begin{array}{c} \begin{pmatrix} \rbrack(6\,,\,1)_{+\frac{1}{3}}\lbrack_{2} + \\ \rbrack(\overline{3}\,,\,3)_{+\frac{1}{3}}\lbrack_{2} \end{pmatrix} + \begin{pmatrix} \rbrack(8\,,\,2)_{-\frac{1}{2}}\lbrack_{2} + \\ \rbrack(1\,,\,2)_{-\frac{1}{2}}\lbrack_{2} \end{pmatrix} + \\ \rbrack(3\,,\,2)_{+\frac{7}{6}}\lbrack_{2} + \rbrack(3\,,\,1)_{-\frac{4}{3}}\lbrack_{2} + \rbrack(\overline{3}\,,\,1)_{+\frac{1}{3}}\lbrack_{2} \end{array}\right] \\
[126] & & \{\overline{50}\}_{+2} & \left[\begin{array}{c} \begin{pmatrix} \rbrack(6\,,\,3)_{+\frac{1}{3}}\lbrack_{2} + \\ \rbrack(\overline{3}\,,\,1)_{+\frac{1}{3}}\lbrack_{2} \end{pmatrix} + \rbrack(8\,,\,2)_{-\frac{1}{2}}\lbrack_{2} + \rbrack(3\,,\,2)_{+\frac{7}{6}}\lbrack_{2} \\ + \rbrack(\overline{6}\,,\,1)_{-\frac{4}{3}}\lbrack_{2} + \rbrack(1\,,\,1)_{+2}\lbrack_{2} \end{array}\right]
\end{array}
$$

(30)

$\longrightarrow$

f4

We complete the classification of the $[\,126\,]$ **(p)scalar multiplet (eq. 140 , Appendix E)**

[10] [120]	$\{5\}_{\,-2}$	$\big]\,(3\,,\,1)_{\,-\frac{1}{3}}\big[_{\,+3} + \big]\,(1\,,\,\overline{2})_{\,+\frac{1}{2}}\big[_{\,+3}$
[10] [126] [120]	$\{\overline{5}\}_{\,+2}$	$\big]\,(\overline{3}\,,\,1)_{\,+\frac{1}{3}}\big[_{\,-3} + \big]\,(1\,,\,2)_{\,-\frac{1}{2}}\big[_{\,+3}$
[126]	$\underline{\{1\}_{\,+10}}$	$\underline{\big]\,(1\,,\,1)_{\,0}\big[_{\,+10}}$
[126] [120]	$\{10\}_{\,+6}$	$\big]\,(3\,,\,2)_{\,-\frac{1}{6}}\big[_{\,+6} + \big]\,(\overline{3}\,,\,1)_{\,-\frac{2}{3}}\big[_{\,+6} + \big]\,(1\,,\,1)_{\,+1}\big[_{\,+6}$
[120]	$\{\overline{10}\}_{\,-6}$	$\big]\,(\overline{3}\,,\,\overline{2})_{\,+\frac{1}{6}}\big[_{\,-6} + \big]\,(3\,,\,1)_{\,+\frac{2}{3}}\big[_{\,-6} + \big]\,(1\,,\,1)_{\,-1}\big[_{\,-6}$

(31)

While a direct $\nu\,\nu$ **mass term could be induced by a** ***small*** **vacuum expected value**

(32) $$\langle\,\Omega\,|\,\overline{T}^{\,0}\,|\,\Omega\,\rangle \;\rightarrow\; \mathcal{H}_{\,\nu\nu} \;=\; \frac{1}{2}\,m_{\,F\,G}\,\nu^{\,F}_{\,\dot\gamma}\,\nu^{\,\dot\gamma\,G} \;+\; h.c.$$

we do not consider this (p)scalar hierarchy of v.e.v. in the following – by hypoythesis, that primary and e.w. breaking is associated with ***one*** **(p)scalar v.e.v for** ***one*** **representation of SO (10) :**
$[\,126\,]\;,\;[\,45\,]\;,\;[\,10\,]$ **respectively .** $\longrightarrow$

2-2a 'Mass from mixing' only [9-1977] , [10-1979/80]

The mass matrix for the 6 neutrino flavors forming 3 families of $[\,16\,]$ spin (10) representations, whence considered in the left-chiral basis takes the reduced form (eq. 24)

$$\left(\nu^{\,1,2,3} \;=\; \nu^{\,F=1,2,3} \,,\; \nu^{\,4,5,6} \;=\; \mathcal{N}^{\,F=1,2,3} \right)^{\dot{\gamma}}$$

$$\mathcal{H}_{\,\mathcal{M}} \;=\; \frac{1}{2}\,\nu^{\,j}_{\,\dot{\gamma}}\,\mathcal{M}_{\,jk}\,\nu^{\,\dot{\gamma}\,k} \;+\; h.c. \quad ; \quad j\,,\,k \;=\; 1,\cdots,6$$

$$\mathcal{M} \;=\; \left(\begin{array}{cc} 0 & \mu^{\,T} \\ \mu & M \end{array} \right)$$

(33)

$$\begin{array}{lll} \mu & \leftrightarrow & y_{\,F\,G}\,\langle\,\Omega\,|\;\varphi^{\,*}_{\,0}\,[\,10\,]\;|\,\Omega\,\rangle\;\mathcal{N}^{\,F}_{\,\dot{\gamma}}\;[\,16\,]\;\nu^{\,\dot{\gamma}\,G}\;[\,16\,] \\ M & \leftrightarrow & Y_{\,F\,G}\,\langle\,\Omega\,|\;\phi^{\,*}_{\,0}\,[\,126\,]\;|\,\Omega\,\rangle\;\mathcal{N}^{\,F}_{\,\dot{\gamma}}\;[\,16\,]\;\mathcal{N}^{\,\dot{\gamma}\,G}\;[\,16\,] \end{array}$$

$$\begin{array}{c} y \\ Y \end{array} \;\leftrightarrow\; \left[\begin{array}{c} [\,10\,] \\ [\,\overline{126}\,] \end{array} \right] \left(\,[\,16\,]\;\times\;[\,16\,]\,\right)_{\,s}$$

The $0_{\,3\times\,3}$ entry in $\mathcal{M}$ is the consequence of our hypothesis $\mathcal{H}_{\,\nu\nu} \;=\; 0$ in eq. 32 . This is potentially fruitful ground for applying discrete symmetries to the (p)scalar self interactions . $\longrightarrow$

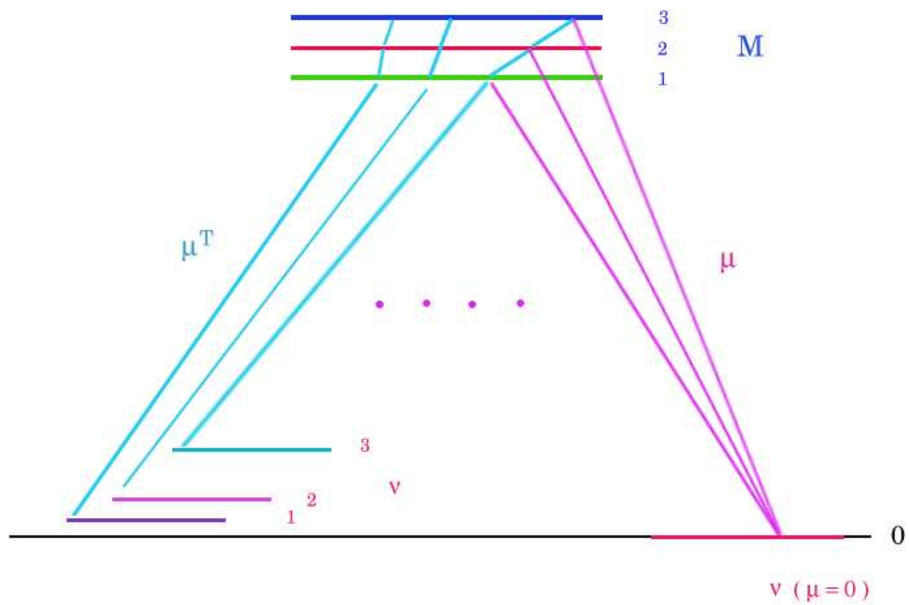

Fig F2 : Mass from mixing

Chiral fermionic structure ensures positive physical eigenvalues, for arbitrary complex μ and symmetric but otherwise arbitrary M . This would similarly guarantee positive masses for scalars, for *(p)scalar* **mass from mixing, only in a supersymmetric setting .**

I proceed *reviewing* **properties of mixing and** *the* **mass relation following from the structure of $\mathcal{M}$ as defined in eq. 33 .** $\longrightarrow$

'seesaw' $\swarrow\!\!\nearrow$

The relative 'size' of μ and M defines the 'mass from mixing' situation and segregates 3 heavy neutrino flavors from the 3 light ones :

(34)
$$\swarrow \qquad ||\,\mu\,|| \;\ll\; ||\,M\,|| \qquad \nearrow$$
$$||\,\mu\,||^{\,2} \;=\; tr\,\mu\,\mu^{\,\dagger} \quad , \quad ||\,M\,||^{\,-2} \;=\; tr\,M^{\,-1}\,\overline{M^{\,-1}}$$

Diagonalization of $\mathcal{M}$

We use the generic expansion parameter $\vartheta \;=\; ||\,\mu\,||\,/\,||\,M\,|| \;\ll\; 1$ – and determine $\longrightarrow$

a unitary 6×6 matrix U with the property [a]

(35)
$$\mathcal{M} = U \mathcal{M}_{diag} U^T \rightarrow \mathcal{M}_{diag} =$$
$$\mathcal{M}_{diag}(m_1, m_2, m_3; M_1, M_2, M_3)$$
$$0 \leq m_1 \leq \cdots \leq M_3 \;,\; m_3 \ll M_1$$
and $U = TU_0 \;;\; T^{-1} \mathcal{M} T^{-1\,T} = \mathcal{M}_{bl.diag.} \rightarrow$
$$= \begin{pmatrix} \mathcal{M}_1 & 0 \\ 0 & \mathcal{M}_2 \end{pmatrix} = U_0 \mathcal{M}_{diag} U_0^T$$

The matrix T in eq. 35 describes the mixing of light and heavy flavors, determined from a 3×3 submatrix t.

(36)
$$T = \begin{pmatrix} (1 + t\,t^\dagger)^{-1/2} & (1 + t\,t^\dagger)^{-1/2}\, t \\ -t^\dagger\,(1 + t\,t^\dagger)^{-1/2} & (1 + t^\dagger\, t)^{-1/2} \end{pmatrix}$$

$\rightarrow$

[a] **To account for inverted hirarchy , the order of the light masses can be accordingly permuted.**

The upper left 3×3 block of T (eq. 36) $\left(1 + t\, t^{\dagger} \right)^{-1/2}$ causes the (3×3) mixing matrix governing oscillations of light (anti)neutrino's to deviate from unitarity , i.e. it becomes subunitary, but by a tiny amount since as we will discuss below

(37) $$|| \, t \, ||^{\,2} = \sum_{kl=1}^{3} | \, t_{\,kl} \, |^{\,2} = O \, \left(10^{\,-21} \right)$$

The matrix t in eq. 36 is reduced to diagonal form through two unitary 3×3 matrices u and w [a]

(38) $$\begin{array}{l} t = u \, (\tan \, a_{\,diag} \,) \, w^{\,-1} \quad ; \quad a_{\,diag} = a_{\,diag} \, (\, a_{\,1} \, , \, a_{\,2} \, , \, a_{\,3} \,) \\ 0 \leq a_{\,k} \leq \pi \, / \, 2 \quad ; \quad a_{\,k} \ll \pi \, / \, 2 \text{ for } \vartheta = || \, \mu \, || \, / \, || \, M \, || \ll 1 \end{array}$$

t is determined from the quadratic equation

(39) $$t = \mu^{\,T} \, M^{\,-1} - t \, \mu \, \bar{t} \, M^{\,-1}$$

which can be solved recursively $\longrightarrow$

---

[a] In eq. 38 $a_{\,diag}$ defines the three (real) heavy-lightmixing angles $a_{\,1,2,3}$, which without loss of generality can be chosen in the first quadrant, but which are small for $\vartheta = || \, \mu \, || \, / \, || \, M \, || \ll 1$

setting

(40)
$$t_{n+1} = \mu^T M^{-1} - t_n \mu \bar{t}_n M^{-1} \;;\; t_0 = 0 \;,\; t_1 = \mu^T M^{-1},$$
$$t_2 = t_1 - \mu^T M^{-1} \mu \mu^\dagger \overline{M}^{-1} M^{-1} \;, \; \cdots$$
$$\lim_{n \to \infty} t_n = t$$

The sequence defined in eq. 40 is convergent for $\vartheta = || \mu || / || M || < 1$ **.**

u , w **in eq. 38 contain *all* 9 CP violating phases, pertaining to T .**

$t = u (\tan a_{diag}) w^{-1}$ **defined in eq. (39) and its determining equation, repeated below**

(39)
$$t = \mu^T M^{-1} - t \mu \bar{t} M^{-1}$$

lead to block diagonal form of $\mathcal{M}_{bl.diag.}$ **.**

(41)
$$\mathcal{M}_{bl.diag.} = T^{-1} \mathcal{M} T^{-1\,T} \;;\; \mathcal{M}_{bl.diag.} = \begin{pmatrix} \mathcal{M}_1 & 0 \\ 0 & \mathcal{M}_2 \end{pmatrix}$$

$(\mathcal{M}_1 , \mathcal{M}_2)$ **(eq. 41)** $\rightarrow$

become [a]

$$\mathcal{M}_1 = (1 + t t^\dagger)^{-1/2} \left[- t \mu - \mu^T t^T + t M t^T \right] (1 + t t^\dagger)^{-1/2\,T}$$

(42) $$\mathcal{M}_2 = (1 + t^\dagger t)^{-1/2} \left[\mu \bar{t} + t^\dagger \mu^T + M \right] (1 + t^\dagger t)^{-1/2\,T}$$

$$\rightarrow \mathcal{M}_1 = - t \mathcal{M}_2 t^T$$

It follows from the assumptions detailed in footnote a , that $Det\, t \neq 0$ and hence the heavy-light mixing angles $a_{1,2,3} > 0$ defined in eq. 38 are strictly bigger than 0.
The lowest approximation, $t \rightarrow t_1$ and and $\mathcal{M}_2 \rightarrow M$, yields the first nontrivial approximation of the light neutrino mass matrix in second order mixing

(43) $$\mathcal{M}_1 \sim \mathcal{M}_1^{(2)} = - \mu^T M^{-1} \mu$$

Remaining dagonalization of $\mathcal{M}_{bl.diag.}$

We go back to eq. 35 $U = T U_0$: U_0 diagonalizes the remaining 3×3 blocks . U_0 is determined modulo diagonal (orthogonal , 6×6) matrices $I = I_{diag}$ $\rightarrow$

[a] In the scenario adopted here, we further assume $Det\, M \neq 0$ and $Det\, \mu \neq 0$.

as shown in eq. 44 representing the discrete abelian group $(Z_2)^{\otimes 6}$

$$T^{-1}\,\mathcal{M}\,T^{-1\,T} = \mathcal{M}_{bl.diag.} \;\; ; \;\; \mathcal{M}_{bl.diag.} = \begin{pmatrix} \mathcal{M}_1 & 0 \\ 0 & \mathcal{M}_2 \end{pmatrix}$$

(44) $$U_0 = \begin{pmatrix} u_0 & 0 \\ 0 & v_0 \end{pmatrix} \sim U_0\,I \; ; \; I = I_{diag}(\pm 1, \cdots, \pm 1)$$

$$\begin{array}{l} \mathcal{M}_1 = u_0\, m_{diag}(m_1, m_2, m_3)\, u_0^T \\ \mathcal{M}_2 = v_0\, M_{diag}(M_1, M_2, M_3)\, v_0^T \end{array} \; ; \; \mathcal{M}_1 = -t\,\mathcal{M}_2\, t^T$$

3-1 Generic mixing and mass estimates

We introduce the arithmetic mean measure for 3×3 **matrices A , not to be confused with the norms** $||\,.\,||$ **defined in eq. 34**

(45) $$|A| = |Det\,A|^{1/3}$$

Eq. 42 then implies

$$|\mathcal{M}_1| / |\mathcal{M}_2| = |t|^2$$

(46) $$|\mathcal{M}_1| = |m_{diag}| = (m_1 m_2 m_3)^{1/3}$$

$$|\mathcal{M}_2| = |M_{diag}| = (M_1 M_2 M_3)^{1/3}$$

We consider the arithmetic mean of the light and heavy neutrino masses and the coorresponding 'would be' masses if μ and μ^T would be the only parts of the full 6×6 mass matrix $\mathcal{M}$

$$\overline{m} = (m_1 m_2 m_3)^{1/3} \quad , \quad \overline{M} = (M_1 M_2 M_3)^{1/3}$$

(47)

$$\mu = u_\mu \, \mu_{diag}(\mu_1, \mu_2, \mu_3) \, v_\mu^{-1} \quad ; \quad \overline{\mu} = (\mu_1, \mu_2, \mu_3)^{1/3}$$

Then beyond eq. 46 there is one more (exact) relation

$$\widehat{t} = (\tan a_1 \tan a_2 \tan a_3)^{1/3} = |t|$$

(48) $$|\mu|^2 = |\mathcal{M}_1||\mathcal{M}_2| \rightarrow \overline{m}/\overline{\mu} = \widehat{t} \ , \ \overline{m}/\overline{M} = \widehat{t}^{\,2}$$

or equivalently $\overline{m} = \widehat{t}\,\overline{\mu}$ $\swarrow \ \nearrow$ $\overline{M} = \widehat{t}^{\,-1}\,\overline{\mu}$

$\longrightarrow$

The estimates below are based on the assumption that the scalar doublets (2) are part of a complex (p)scalar multiplet in $[\,10\,]$ of SO10 .

It follows that at the unification scale we have

(49) $$\mu = \mu^{T} = \mu_{u}$$

We shall use the relation at a scale near 100 GeV

(50) $$\mu \sim \frac{1}{3}\,(\,\mu_{u}\,)$$

The factor $\frac{1}{3}$ accounts for the color rescaling reducing the (colored) up-quark mass matrix from the unification scale down to 100 GeV .

Using the definitions in eq. 47 and the quark masses $m_{u} \sim 5.25$ MeV , $m_{c} \sim 1.25$ GeV and $m_{t} \sim 172.5$ GeV

(51) $$\overline{\mu}_{u} = (\,m_{u}\,m_{c}\,m_{t}\,)^{1/3} \sim 1 \text{ GeV} \longrightarrow \overline{\mu} \sim \frac{1}{3} \text{ GeV}$$

Further lets approximate the mass square differences obtained from the combined neutrino oscillation measurements by

(52) $$\Delta\, m^{2}_{1\,2} \sim 10^{-4} \text{ eV}^{2} \quad , \quad \Delta\, m^{2}_{2\,3} \sim 2.5\; 10^{-2} \text{ eV}^{2}$$

$\longrightarrow$

Finally 'pour fixer les idées' I set the lowest light neutrino mass $\sim\ 1$ meV and assume hierarchical (123) light masses. This implies

$$m_1 \sim 1 \text{ meV}, \; m_2 \sim 10 \text{ meV}$$

(53)

$$m_3 \sim 50 \text{ meV} \longrightarrow \overline{m} \sim 8 \text{ meV}$$

and

$$\widehat{t} = \overline{m} / \overline{\mu} \sim 2.5\, 10^{-11} \quad , \quad \widehat{t}^{\,2} \sim 6.0\, 10^{-22}$$

(54)

$$\overline{M} = \overline{\mu} / \widehat{t} \sim 1.4\, 10^{10} \text{ GeV}$$

CQO Conclusions , questions and outlook

C5 The origin of neutrino mass can indeed be understood within the specific structure of spin (10) as charge-like gauge group . Boson fields appear to correspond to the full set of local $f(x) f'(x)$ binary products with $f, f' \subset [16] \oplus [\overline{16}]$.
This brings us to a starting point along the path of unification of gravitational and charge like gauge groups

$$\begin{array}{llccc} & G_0 & = & \text{spin}\,(1,3) & \otimes & \text{spin}\,(0,10) \\ & & & \downarrow & & \downarrow \\ (55) & & & \text{gauging} & & \text{gauging} \\ & & & \text{orientation} & & \text{charges} \\ & \text{space-time} & & \mathcal{M}_3^1 & \times & \mathcal{M}_{10}^0 \end{array}$$

Q1 Is the geometric association of spin $(0, 10)$ in eq. 55 indicating internal space-like *coordinates* extending the geometric origin of spin $(1, 3)$ from space-time ? [a]

Q2 What is the nature of coordinates in extended space-time ? $\longrightarrow$

---

[a] I cite here just one reference : Élie Cartan, 'Sur une classe remarquable d'espaces de Riemann', Bull. Soc. Math. de France, 54 (1926) 214 , and 55 (1927) 114 , [h1] .

Q2 *continued*

proposed structures for general $(\mathcal{M} \mid X)$, within superstring theories

$$
X \subset \mathcal{M} \quad \begin{matrix} \nearrow \\ \searrow \end{matrix} \left\{ \begin{array}{c|lcl} x^{\mu}, \vartheta_{\alpha}, \overline{\vartheta}_{\dot{\gamma}} & x^{\mu} & : & \textbf{c-numbers} \\ \textbf{base superspace} & \vartheta, \overline{\vartheta} & : & \textbf{Grassmann variables} \\ \widehat{x}^{\mu}, \widehat{f}^{A}, \widehat{f}^{*B} & \widehat{x}^{\mu} & : & \textbf{bosonic q-numbers} \\ \textbf{target superspace} & \widehat{f}^{A}, \widehat{f}^{*B} & : & \textbf{fermionic q-numbers} \end{array} \right.
$$

(56)

but the question addressed is more general and may not necessarily concern a space $(\mathcal{M} \mid X)$, endowed with a supersymmetric structure .

Outlook

The pathways of nature , entangled indeed

make tremble the doubtful who may not proceed

yet build on assurance acquired to feed

the hope to discover those road signs to read.

Thank you

- - - - - - -

Complementary material to

'The origin of neutrino mass'

stations along the path of cognition

Contribution to – Discrete'08 –

Symposium on the Prospects in the Physics of Discrete Symmetries
11.-16. December 2008, IFIC, Valencia, Spain

4-1 From Lorentzian (p , q) to conformal groups [1-a]

$$M_{\mu\nu} \sim i\,(x_\mu \partial_\nu - x_\nu \partial_\mu) \;\; ; \;\; \partial_\alpha x_\beta = \eta_{\alpha\beta}\,\Big|_{p\ time\ ,\ q\ space}$$

(57)
$$[\,M_{\mu\nu}\,,\,M_{\sigma\tau}\,] = i \left\{ \begin{array}{c} +\eta_{\mu\tau} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\tau} \\ -\eta_{\nu\tau} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\tau} \end{array} \right\} \rightarrow \textbf{Lie}\ (\,SO\,(\,p\,,\,q\,)\,)$$

Given an associative (p , q) Clifford algebra Γ

$$\{\,\gamma_\mu\,,\,\gamma_\nu\,\} = 2\,\eta_{\mu\nu}\,\P \;\rightarrow\; \sigma_{\mu\nu} = \frac{i}{2}\,[\,\gamma_\mu\,,\,\gamma_\nu\,]$$

$$s_{\mu\nu} = \frac{1}{2}\,\sigma_{\mu\nu} \;\rightarrow\; s_{\mu\nu} \sim M_{\mu\nu} \;\rightarrow\; \textbf{spin}\,(\,p\,,\,q\,;\,\Gamma\,)$$

(58)
$$[\,s_{\mu\nu}\,,\,s_{\sigma\tau}\,] = i \left\{ \begin{array}{c} +\eta_{\mu\tau}\, s_{\nu\sigma} - \eta_{\mu\sigma}\, s_{\nu\tau} \\ -\eta_{\nu\tau}\, s_{\mu\sigma} + \eta_{\nu\sigma}\, s_{\mu\tau} \end{array} \right\}$$

Completing the conformal Lie algebra with conformal infinitesimal boosts

(59)
$$K_\mu \sim i\,\left(2\,x_\mu\,x^\alpha\,\partial_\alpha - x^2\,\partial_\mu\right) \;\; ; \;\; D \sim i\,x^\alpha\,\partial_\alpha$$

with the commutation relations

$$[K_\mu , K_\nu] = 0 \ ; \ [P_\mu , K_\nu] = 2i\,(\eta_{\mu\nu} D - M_{\mu\nu})$$

$$[K_\mu , K_\nu] = -\left\{ \begin{array}{l} 4\,[x_\mu x^\alpha \partial_\alpha , x_\nu x^\beta \partial_\beta] - 2\,[x^2 \partial_\mu , x_\nu x^\beta \partial_\beta] \\ +\,[x^2 \partial_\mu , x^2 \partial_\nu] - 2\,[x_\mu x^\alpha \partial_\alpha , x^2 \partial_\nu] \end{array} \right\}$$

(60)
$$= -\left\{ \begin{array}{l} 4\,(x_\mu x^\alpha\,[\partial_\alpha , x_\nu x^\beta \partial_\beta] - [x_\nu x^\beta \partial_\beta , x_\mu x^\alpha]\,\partial_\alpha) \\ -2\,(x^2\,[\partial_\mu , x_\nu x^\beta \partial_\beta] - [x_\nu x^\beta \partial_\beta , x^2]\,\partial_\mu - \mu \leftrightarrow \nu) \\ +\,(x^2\,[\partial_\mu , x^2 \partial_\nu] - [x^2 \partial_\nu , x^2]\,\partial_\mu) \end{array} \right\}$$

$$= -\left\{ \begin{array}{l} -2\,(-x^2 x_\nu \partial_\mu - \mu \leftrightarrow \nu) \\ +2\,(x^2 x_\mu \partial_\nu - \mu \leftrightarrow \nu) \end{array} \right\} = 0$$

$$[P_\mu , K_\nu] = -\left[\partial_\mu , 2x_\nu x^\alpha \partial_\alpha - x^2 \partial_\nu\right]$$

(61) $$= -\left\{ \begin{array}{l} [\partial_\mu , 2x_\nu x^\alpha]\,\partial_\alpha \\ -\,[\partial_\mu , x^2]\,\partial_\nu \end{array} \right\} = -2\,(\eta_{\mu\nu} x^\alpha \partial_\alpha + x_\nu \partial_\mu - x_\mu \partial_\nu)$$

$$= 2i\,(\eta_{\mu\nu} D - M_{\mu\nu})$$

This can be completed to the Lie algebra of SO $(2_{0,5}, 4_{1\cdots 4})$ **using a dummy mass scale m**

$$\mu = 0 \cdots 3 : \left\{ \begin{array}{l} M_{\mu 4} = \frac{1}{2}\,(m^{-1} P_\mu - m K_\mu) \\ M_{\mu 5} = \frac{1}{2}\,(m^{-1} P_\mu + m K_\mu) \end{array} \right\}$$

(62) $$[M_{\mu 4}, M_{\nu 5}] = \frac{1}{4}\{[P_\mu , K_\nu] + \mu \leftrightarrow \nu\} \quad = \quad i\,\eta_{\mu\nu} D$$

$$= \quad -i\,\eta_{\mu\nu} M_{45} \quad \rightarrow$$

$$M_{45} = -D$$

For completeness we also verify the commutation rules

$$[\, M_{\mu\,4\,(5)}\,,\, M_{\nu\,4\,(5)}\,] = \mp\tfrac{1}{4}\left\{\begin{array}{c} [\,P_\mu\,,\,K_\nu\,] \\ \\ [\,K_\mu\,,\,P_\nu\,] \end{array}\right\} = \mp\tfrac{1}{4}\,(\,[\,P_\mu\,,\,K_\nu\,] - \mu \leftrightarrow \nu\,)$$

(63)

$$= \pm\, i\, M_{\mu\nu} \qquad = i \begin{pmatrix} -\,\eta_{44} \\ \\ -\,\eta_{55} \end{pmatrix} M_{\mu\nu} \quad \surd$$

as well as

$$[\, M_{\mu\,4\,(5)}\,,\, M_{4\,5}\,] \;=\; -\tfrac{1}{2}\,[\,(\,m^{-1}\,P_\mu \mp m\,K_\mu\,)\,,\,D\,]$$

(64)

$$= \tfrac{1}{2}\left\{\begin{array}{l} [\,m^{-1}\,\partial_\mu\,,\,x^\beta\,\partial_\beta\,] \\ \mp\,[\,2\,m\,x_\mu\,x^\alpha\,\partial_\alpha\,,\,x^\beta\,\partial_\beta\,] \\ \pm\,[\,m\,x^2\,\partial_\mu\,,\,x^\beta\,\partial_\beta\,] \end{array}\right\}$$

$$\left[m^{-1} \partial_\mu , x^\beta \partial_\beta \right] = 1 \, m^{-1} \partial_\mu$$

$$\begin{aligned}
\left[2 m x_\mu x^\alpha \partial_\alpha , x^\beta \partial_\beta \right] &= 2 m \left\{ \begin{array}{l} x_\mu x^\alpha \left[\partial_\alpha , x^\beta \partial_\beta \right] \\ + \left[x_\mu x^\alpha , x^\beta \partial_\beta \right] \partial_\alpha \end{array} \right\} \\
&= 2 m \left\{ x_\mu x^\alpha \partial_\alpha - x^\beta \left(\partial_\beta x_\mu x^\alpha \right) \partial_\alpha \right\} \\
&= (- 1) \, 2 m x_\mu x^\alpha \partial_\alpha \\
\left[m x^2 \partial_\mu , x^\beta \partial_\beta \right] &= m \left\{ \begin{array}{l} x^2 \left[\partial_\mu , x^\beta \partial_\beta \right] \\ + \left[x^2 , x^\beta \partial_\beta \right] \partial_\mu \end{array} \right\} \\
&= (- 1) \, m x^2 \partial_\mu
\end{aligned} \tag{65}$$

Commutation with $x^\beta \partial_\beta$ returns the *mass* dimension . Thus eq. 64 becomes $\longrightarrow$

$$\left[M_{\mu\,4\,(5)} \, , \, M_{4\,5} \right] \;=\; \frac{1}{2} \left\{ \begin{array}{l} m^{-1} \, \partial_{\mu} \\ \\ \pm \left(2\, m\, x_{\mu}\, x^{\alpha}\, \partial_{\alpha} \;-\; m\, x^{2}\, \partial_{\mu} \right) \end{array} \right\}$$

(66)
$$= \; - i \, \tfrac{1}{2} \, \left\{ m^{-1} \, P_{\mu} \; \pm \; m \, K_{\mu} \right\}$$

$$= \; - i \, M_{\mu\,5\,(4)} \; = \; i \left(\begin{array}{c} \eta_{44} \, M_{\mu\,5} \\ \\ - \, \eta_{55} \, M_{\mu\,4} \end{array} \right) \qquad \surd$$

4-1a Details of Lorentzian (p , q) $\times \; P_{\mu} \; \longrightarrow$ conformal (p + 1 , q + 1) - extension

for the Majorana setting : p = 1 , q = 3

It becomes clear from the derivations in section 4-1 that the extension from the motion group in $d \; = \; p \; + \; q$ dimensions with p time- and q space-signatures follows the same rules for all p , q .
The extended group structure becomes simple and forms the conformal group SO (p + 1 , q + 1) .
For the corresponding $\Gamma -$ algebra extension we discuss here just the characteristic Majorana setting inherent to p = 1 , q = 3 , illustrating the induced extension : spin (p , q) $\longrightarrow$ spin (p + 1 , q + 1) .
It will become clear only after the above comparison of extensions , that they are *not* in any way related.

$\longrightarrow$

To study the Majorana representations of signatured (and associative) Clifford algebras (eq. 58) it is necessary to adopt a real form of the Dirac equation , i.e. to pass from the matrices *and conventions* $\gamma_{\mu} \rightarrow \Gamma_{\mu} = i \gamma_{\mu}$ **and** $\eta_{\mu\nu} \rightarrow - \eta_{\mu\nu}$ **, which satisfy the relations**

$$\begin{aligned} \eta_{\mu\nu} &= \textbf{diag} \; (1_{p-times} \, 1 \, ; \, -1_{q-times} \, -1) \\ \{ \gamma_{\mu} , \gamma_{\nu} \} &= 2 \, \eta_{\mu\nu} \, ¶ \\ \{ \Gamma_{\mu} , \Gamma_{\nu} \} &= 2 \, (- \eta_{\mu\nu}) \, ¶ \;\; ; \;\; \Gamma_{\mu} = i \gamma_{\mu} \\ - \eta_{\mu\nu} &= \textbf{diag} \; (-1_{p-times} \, -1 \, ; \, 1_{q-times} \, 1) \end{aligned} \tag{67}$$

The full Γ **algebra over the complex numbers is the same for all space-time signatures. It shall be denoted** $\{ {}_d \Gamma \, ; \, \mathbb{C} \}$ **and identified with its unique – modulo (inner) automorphisms – irreducible matrix representation.**

$$\textbf{dim} \; \{ {}_d \Gamma ; \mathbb{C} \} = 2^{\left[\frac{d}{2} \right]} \;\; ; \;\; d = p + q \;\; ; \;\; \left[\frac{d}{2} \right] = \begin{cases} \frac{d}{2} \; \textbf{for } d \textbf{ even} \\ \frac{d-1}{2} \; \textbf{for } d \textbf{ odd} \end{cases} \tag{68}$$

If within $\{ {}_d \Gamma \, ; \, \mathbb{C} \}$ **and given p , q signatures the** Γ_{μ} **matrices (eq. 67) can be chosen real, we deal with a Majorana representation, discussed for p = 1 , q = 3 in appendix A.** $\longrightarrow$

Equipped with the Majorana representation $\{\,\Gamma_{\,p=1\,,\,q=3}\,;\,\mathbb{R}\,\}$ we extend it to $\{\,\Gamma_{\,p=2\,,\,q=4}\,;\,\mathbb{R}\,\}$ below , keeping in mind that there is no Majorana representation for general p , q values. Also care must be taken in the numbering of coordinates beyond the four pertaining to $d\,=\,1\,+\,3$. We continue the enumeration of space-time dimensions, always starting with extended time followed by extended 3-space, using red color for the time-like dimension numbers and signatures , as follows

$$x^{\,0}\;=\;t\;,\;x^{\,k},k\;=\;1,2,3\cdots q\;;\;x^{\,q+r}\;,\;r\;=\;1,\cdots,\;p\;-\;1\quad;\quad\textbf{for p=2 , q=4}\;\longrightarrow$$

(69)
$$\begin{array}{cccccccc} x^{\,0} & x^{\,1} & x^{\,2} & x^{\,3} & x^{\,4} & ; & x^{\,5} \\ + & - & - & - & - & & + \end{array}$$

In order to distinguish the space-time dimensionality and its associated $\Gamma\,-$ algebra we shall use (or substitute) the notation

(70)
$$\Gamma_{\,\mu}\;\longrightarrow\;{}_{d}\,\Gamma_{\,\mu}\quad;\quad d\;=\;p\;+\;q\quad,\quad\mu\;=\;0,1\cdots d-1$$

Thus $\Gamma_{\,5},\;\gamma_{\,5\;R(L)}$ (eq. 90) for general d , p , q ***and general basis*** **become**

(71)
$$_{d}\,\Gamma_{\,d+1}\;=\;{}_{d}\,\left(\,\Gamma_{\,0}\,\Gamma_{\,1}\,\cdots\,\Gamma_{\,d-1}\,\right)$$

In the product in eq. 71 the prefix $_{d}$ is not repeated for brevity .

4-1b Product representation for the Clifford algebra $\{ {}_{d_1+d_2}\Gamma \,;\, \mathbb{C} \}$ [2-a]

We consider two Clifford algebras corresponding to *even* dimensions d_1 , d_2 respectively

(72)
$$\left.\begin{array}{lll} d_1 = 2\nu_1 & : & {}_{d_1}\left(\Gamma_0 , \cdots \Gamma_{d_1 - 1} \right) \\ d_2 = 2\nu_2 & : & {}_{d_2}\left(\Sigma_0 , \cdots \Sigma_{d_2 - 1} \right) \end{array}\right\} D = d_1 + d_2$$

$$\textbf{dim} \left\{ {}_{d_j}\Gamma \,;\, \mathbb{C} \right\} = 2^{\nu_j} = N_j \quad ; \quad j = 1, 2$$

Then we construct the direct product representation of $\{ {}_D\Gamma \,;\, \mathbb{C} \}$ in the following way (two ways)

(73)
$$\begin{array}{lll} {}_D\Gamma_\alpha = {}_{d_1}\Gamma_\alpha \otimes \P_{N_2 \times N_2} & ; & \alpha = 0, \cdots d_1 - 1 \\ {}_D\Gamma_{d_1+\beta} = {}_{d_1}\Gamma_{d_1+1} \otimes {}_{d_2}\Sigma_\beta & ; & \beta = 0, \cdots d_2 - 1 \end{array}$$

$${}_D\Gamma_{D+1} = \left({}_{d_1}\Gamma_{d_1+1} \right)^{1 + d_2} \otimes \left({}_{d_2}\Gamma_{d_2+1} \right)$$

As long as we work over the field $\mathbb{C}$, signatures i.e. the value of the squares $\longrightarrow$

$$({}_D\Gamma_x)^2 = -\eta_{xx}|_{\textbf{no sum}}\, ¶ = ({}_D\sigma_x)\, ¶ \;;\; x = 0, \cdots, D-1$$

(74) $$({}_D\Gamma_{D+1})^2 = ({}_D\sigma_{D+1})\, ¶ \;;\; ({}_D\Gamma_{D+1}) = ({}_D\sigma_{D+1})|_{\Pi}$$

$$\{({}_D\sigma_x) = \pm 1 \,|\, x = 0, \cdots, D-1, D+1\}$$

are immaterial . Nevertheless the 'straight' product definition of ${}_d\Gamma_{d+1}$ **for** d **even in eq. 71 implies , always within even** $D = d_1 + d_2$, d_1, d_2 **, by eqs. 72 , 73 ,** *recursively*

(75) $$({}_D\sigma_{D+1})|_{\Pi} = \left({}_{d_1}\sigma_{d_1+1}\right)\left({}_{d_2}\sigma_{d_2+1}\right)$$

The suffix $_\Pi$ **of the signature** $({}_D\sigma_{D+1})|_{\Pi}$ **in eqs. 74 and 75 shall indicate that this quantity** *depends* **on the chosen form of the product representation .**

It becomes obvious that if the ${}_d\sigma_x$ **parities are assigned the direct product composition , as defined in eq. 73 , this may not be compatible from** $d_1 \,\&\, d_2$ **to** $D \leftrightarrow d_1 \otimes d_2$ **.**

To this end we compute , for generic even d , the quantities ${}_d\sigma_{d+1}$ **for any one of assigned parities** ${}_d\sigma_x$ **for** $x = 0 \cdots d-1$ **, maintaining the 'strict' product representation of** ${}_d\Gamma_{d+1}$ **as defined in eq. 71** $\longrightarrow$

(76)
$$\left(\prod_{x=0}^{d-1} ({}_{d}\Gamma_{x}) \right)^{2} = \left(\prod_{x} ({}_{d}\Gamma_{x})^{2} \right) \sigma_{rev} (d)$$
$$\sigma_{rev} (d) = (-1)^{1+2+\cdots+d-1} = (-1)^{\nu} \quad ; \quad d = 2\nu$$

Hence , continuing to work over $\mathbb{C}$, we can upon multiplication and/or rearrangement in ordering of individual ${}_{d}\Gamma_{x}$; $x = 0 , \cdots d-1$ elements with appropriate powers of i , assign arbitrary signatures and ${}_{d}\sigma_{x}$ parities , yielding a signature $(p , q) ; p + q = d$ $\rightarrow$

(77)
$${}_{d}\sigma_{d+1} (p , q) = (-1)^{\frac{1}{2} (p - q)} = (-1)^{\frac{1}{2} (q - p)} = {}_{d}\sigma_{d+1} (q , p)$$

It is the symmetry with respect to exchange of $p \leftrightarrow q$ signatures, which renders the direct product , defined in eq. 73 ***nonsymmetric*** **yet consistent with eq. 75. However the assigned ${}_{d_{1,2}}\sigma_{x}$ parities or signatures are not directly transferred to the ordered product $\{ {}_{D}\Gamma_{y} ; y = 0 , \cdots D-1 \}$**

(78)
$${}_{D}\sigma_{\alpha} = \left({}_{d_1}\sigma_{\alpha} \right) \qquad \textbf{for} \quad \alpha = 0 , \cdots d_1 - 1$$
$${}_{D}\sigma_{\alpha} = \left({}_{d_1}\sigma_{d_1+1} \right) (p_1 , q_1) \left({}_{d_2}\sigma_{\alpha} \right) \qquad \textbf{for} \quad \alpha = d_1 , \cdots D$$
$$\rightarrow \quad (p , q) \; (D) |_{\Pi} = \begin{cases} (p_1 + p_2 , q_1 + q_2) & \textbf{for} \quad q_1 - p_1 = 0 \textbf{ mod } 4 \\ (p_1 + q_2 , q_1 + p_2) & \textbf{for} \quad q_1 - p_1 = 2 \textbf{ mod } 4 \end{cases}$$

4-1c Reduction of a Majorana representation $\{\, \Gamma_{p\,,\,q}\,;\,\mathbb{R}\,\} \;\longrightarrow\; \{\, \Gamma_{p-1\,,\,q-1}\,;\,\mathbb{R}\,\}$

Given a Majorana representation $\{\, \Gamma_{p\,,\,q}\,;\,\mathbb{R}\,\}$ with $p\,,q \,\geq\, 1$ we single out the last two real matrices for each signature respectively , using here the reordered numbering

(79)
$$\left\{ \Gamma_1^{\,-}\,,\,\cdots\,,\,\Gamma_p^{\,-}\,;\,\Gamma_1^{\,+}\,,\,\cdots\,,\,\Gamma_q^{\,+} \right\}$$

$$\left\{ \Gamma_p^{\,-}\,,\,\Gamma_q^{\,+} \right\}\;;\;\left\{ \left(\Gamma_p^{\,-}\right)^2\,,\,\left(\Gamma_q^{\,+}\right)^2 \right\} \;=\; (\,¶\,)_{2^{\nu}\times 2^{\nu}}\;\{\,-1\,,\,+1\,\}\;;\;2\,\nu \;=\; p\,+\,q$$
(80)

Next we consider the product

$$\Pi \;=\; \Gamma_p^{\,-}\,\Gamma_q^{\,+} \quad \textbf{with} \quad (\,\Pi\,)^2 \;=\; -\,\left(\Gamma_p^{\,-}\right)^2\,\left(\Gamma_q^{\,+}\right)^2 \;=\; +\,(\,¶\,)_{2^{\nu}\times 2^{\nu}}$$

(81)
$$\longrightarrow\; Pr_{\pm} \;=\; \tfrac{1}{2}\,\left(\,¶_{2^{\nu}\times 2^{\nu}}\,+\,\Pi\,\right)\quad;\quad \left\{ \begin{array}{c} Pr_{\pm}^{\,2} \;=\; Pr_{\pm}\quad,\quad Pr_{\pm}\;\mathrm{Pr}_{\mp} \;=\; 0 \\[2ex] Pr_{+}\,+\,Pr_{-} \;=\; ¶_{2^{\nu}\times 2^{\nu}} \end{array} \right\}$$

As a consequence of the real nature of the matrices $\Gamma_{p\,,\,q}$ the projectors $Pr_{\pm}$ defined in eq. 81 are real symmetric and thus hermitian matrices , projecting on two orthogonal subspaces $\mathcal{S}_{\pm}$ of dimension $2^{\,\nu\,-\,1}$ respectively . Furthermore these projectors commute with the remaining Γ matrices $\longrightarrow$

InvPI2

$$\left\{ \Gamma_1^-, \cdots, \Gamma_{p-1}^-; \Gamma_1^+, \cdots, \Gamma_{q-1}^+ \right\}$$

(82)
$$\left[\mathrm{Pr}_{\pm}, \Gamma_k^- \right] = 0 \quad , \quad k = 1, \cdots, p-1$$
$$\left[\mathrm{Pr}_{\pm}, \Gamma_j^+ \right] = 0 \quad , \quad j = 1, \cdots, q-1$$

It follows that the projected matrices

(83)
$$\widehat{\Gamma}_k^- = \Gamma_k^- Pr_{\pm} \quad , \quad k = 1, \cdots, p-1$$
$$\widehat{\Gamma}_j^+ = \Gamma_j^+ Pr_{\pm} \quad , \quad j = 1, \cdots, q-1$$

form – for either sign of $Pr_{\pm}$ **separately – irreducible representations over** $\mathbb{R}$
$\longrightarrow \{ \Gamma_{p-1\,,\,q-1} ; \mathbb{R} \}$ **with** $\widehat{p} = p-1, \widehat{q} = q-1; \widehat{p} + \widehat{q} = 2(\nu - 1)$ **.**

4-2 The two base sets of Majorana representations $\left\{ \Gamma_{p=2\nu^{-},\, q=0}\, ;\, \mathbb{R}\,(-) \right\}$ and $\left\{ \Gamma_{p=0,\, q=2\nu^{+}}\, ;\, \mathbb{R}\,(+) \right\}$

We give the case $p = 2\,\nu^{-} = d^{-}\,;\; q = 0$ a label $(-)$ and conversely $q = 0\,;\; q = 2\,\nu^{+}$ the label $(+)$. It now follows from eq. 109

(84)
$$\begin{array}{ll|l} p = d^{-} & q = 0 & M\,(-) = {}_4M^{\,d^{-}}_{(1)} + {}_4M^{\,d^{-}}_{(2)} \\ p = 0 & q = d^{+} & M\,(+) = {}_4M^{\,d^{+}}_{(3)} + {}_4M^{\,d^{+}}_{(2)} \end{array}$$

$d^{\pm} = 2\,\nu^{\pm}$ **even**

From eq. 126 we obtain

(85)
$${}_4M^{\,2\nu}_{(l)} = 2^{\,\nu-1}\left(2^{\,\nu-1} + F\,[\,2\,\nu - 2\,(l)\,]\right)$$

Combining eqs. 84 and 85 we obtain $\longrightarrow$

$$M(\mp) = \frac{1}{2}\, 2^{\nu} \left[2^{\nu} + \left\{ \begin{array}{lll} F(2\nu - 2) + F(2\nu - 4) & \textbf{for} & (-) \\ F(2\nu - 6) + F(2\nu - 4) & \textbf{for} & (+) \end{array} \right\} \right]$$

(86)

$$= \frac{1}{2}\, 2^{\nu} \left[2^{\nu} - 1 \right]$$

$\nu \to \nu^{\mp}$ **respectively**

The quantities $M(\mp)$ represent the number of antisymmetric $2^{\nu} \times 2^{\nu}$ matrices forming the full Clifford algebras $\left\{ \Gamma_{p=2\nu^{-},\, q=0} ; \mathbb{R}(-) \right\}$ and $\left\{ \Gamma_{p=0,\, q=2\nu^{+}} ; \mathbb{R}(+) \right\}$.

Comparing the two relations for $M(\mp)$ in eq. 86 it follows using the labels $p\,(= 2\nu^{-})$ for $\left\{ \Gamma_{p=2\nu^{-},\, q=0} ; \mathbb{R}(-) \right\}$ and $q\,(= 2\nu^{+})$ for $\left\{ \Gamma_{p=0,\, q=2\nu^{+}} ; \mathbb{R}(+) \right\}$ $\rightarrow$

(87)
$$F\,(p-2)+F\,(p-4)=-1 \quad \textbf{for} \quad (-) \quad \longrightarrow \quad p=6\,,\,8$$
$$F\,(q-6)+F\,(q-4)=-1 \quad \textbf{for} \quad (+) \quad \longrightarrow \quad q=2\,,\,8$$

In order to illustrate the solutions to eq. 87 I display the function $F\,(j)\,;\,j\,=$ **even from eq. 125 below**

$$F\,(j)=F\,(-j)=F\,(j+8)$$

(88)
$$\begin{array}{c|cccccccc} j & -6 & -4 & -2 & 0 & 2 & 4 & 6 & 8 \\ F & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \end{array}$$

$\longrightarrow$

d4

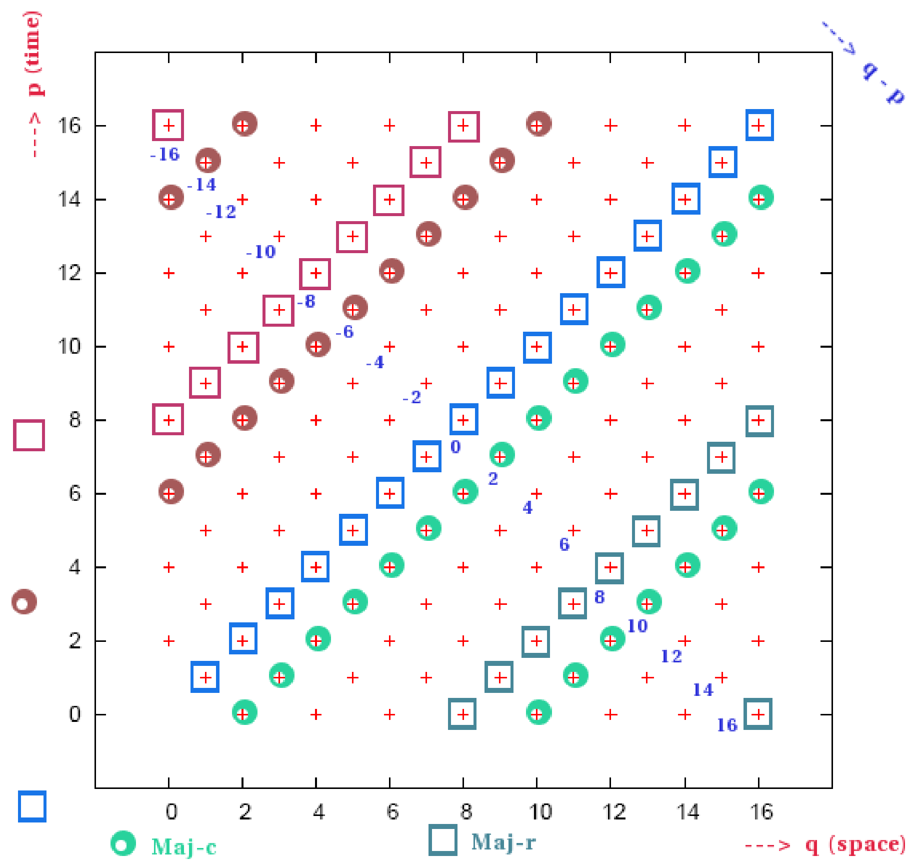

Fig 2 : The complex and real Majorana representations

MajCR $(\,p\,,\;q\,) \quad \longleftrightarrow$

Appendix A : The Majorana representation of $\{ \Gamma_{p\,,\,q}\,;\,\mathbb{R} \}$ for p = 1 , q = 3

If for a given signature p , q a Majorana representation over the real numbers $\{ {}_d\Gamma\,;\,\mathbb{C} \}$ exists, this representation shall be denoted $\{ \Gamma_{p\,,\,q}\,;\,\mathbb{R} \}$.
For p = 1 , q = 3 the (left- and right-) chiral basis over the field $\mathbb{C}$ corresponds to the Γ_μ matrices

$$\Gamma^{(\chi)}_\mu = \begin{pmatrix} 0 & i\,\sigma_\mu \\ i\,\widetilde{\sigma}_\mu & 0 \end{pmatrix} \quad ; \quad \begin{array}{l} \sigma_\mu = (\sigma_0\,;\,\sigma_k) \\ \widetilde{\sigma}_\mu = (\sigma_0\,;\,-\sigma_k) \end{array} \quad ; \quad k = 1,2,3$$

$$\sigma_0 = \P_{2\times 2} \;\; ; \;\; \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;\; ; \;\; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \;\; ; \;\; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Gamma_\mu \equiv \eta_{\mu\nu}\,\Gamma^\nu \;\; , \;\; \Gamma_5 = \Gamma_0\,\Gamma_1\,\Gamma_2\,\Gamma_3 \;\; , \;\; \Gamma_5^2 = -\,\P_{4\times 4}\,\big|_{p=1 q=3} \;\; ; \tag{89}$$ **in any basis**

In the chiral basis we have

$$\Gamma^{(\chi)}_5 = i \begin{pmatrix} \P & 0 \\ 0 & -\P \end{pmatrix} = i\,\gamma_{5\,R} = -\,i\,\gamma_{5\,L} \tag{90}$$

$\longrightarrow$

In the Majorana basis $\Gamma_5^{(Maj)}$ is real , antisymmetric.

In the chiral basis the substrate of the spinor (a 4-dimensional column 'vector') is of the form

(91) $$\begin{pmatrix} \varphi_\alpha \\ \widetilde{\varepsilon}^{\,\dot\gamma\dot\delta} \, (\psi_\delta)^* \end{pmatrix}^{(\chi)} \; ; \quad \alpha , \delta , \, \dot\gamma , \dot\delta \, = \, 1 , 2 \; ; \; \widetilde{\varepsilon}^{\,\dot\gamma\dot\delta} \, = \, - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with the (Majorana-) reality condition $\phi_\alpha = \psi_\alpha$

We introduce two component and 2 by 2 matrix notation

(92) $$(\varphi_\delta , \psi_\delta) \rightarrow \varphi , \psi \; ; \; \widetilde{\varepsilon}^{\,\dot\gamma\dot\delta} \rightarrow \widetilde{\varepsilon} \equiv \varepsilon^{-1} = - \varepsilon$$

$$\varepsilon = i \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$\longrightarrow$

The representation for a spinor (eq. 91) in the chiral basis becomes

(93) $$\begin{pmatrix} \varphi \\ \varepsilon^{-1}\,\psi^{*} \end{pmatrix}^{(\chi)} \quad ; \quad \text{with the Majorana condition } \varphi = \psi$$

The Majorana basis obtains whence φ , ψ in eqs. 91 , 93 are identified and then decomposed into real and imaginary parts (component by component)

(94) $$\varphi = \psi = x + i\,y \;\;;\;\; x = \tfrac{1}{2}\,(\varphi + \varphi^{*}) \;\;;\;\; y = \tfrac{1}{2i}\,(\varphi - \varphi^{*})$$

The action of $\Gamma_{\mu}^{(\chi)}$ (eq. 89) then becomes

$$\Gamma_{\mu}^{(\chi)} \begin{pmatrix} x + i\,y \\ \varepsilon^{-1}\,(x - i\,y) \end{pmatrix} = \begin{pmatrix} i\,\sigma_{\mu}\,\varepsilon^{-1}\,(x - i\,y) \\ i\,\widetilde{\sigma}_{\mu}\,(x + i\,y) \end{pmatrix} = \begin{pmatrix} x' + i\,y' \\ \varepsilon^{-1}\left(x' - i\,y'\right) \end{pmatrix}_{\mu}$$

(95)

$\longrightarrow$

Thus we obtain , working out the action of $\Gamma_{\mu}^{(\chi)}$ (eq. 95) separately for each μ

$$\begin{pmatrix} x' + i\,y' \\ \varepsilon^{-1}\left(x' - i\,y'\right) \end{pmatrix}_0 = \begin{pmatrix} \varepsilon^{-1}(\,y + i\,x\,) \\ (\,-y + i\,x\,) \end{pmatrix}$$

(96)

$$\begin{aligned} x'_0 &= -\varepsilon\,y \\ y'_0 &= -\varepsilon\,x \end{aligned} \quad \longrightarrow \quad \Gamma_0^{(Maj)} = \begin{pmatrix} 0 & -\varepsilon \\ -\varepsilon & 0 \end{pmatrix}$$

$$\begin{aligned} x'_1 &= \sigma_3\,y \\ y'_1 &= \sigma_3\,x \end{aligned} \quad \longrightarrow \quad \Gamma_1^{(Maj)} = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}$$

$$\begin{aligned} x'_2 &= x \\ y'_2 &= -y \end{aligned} \quad \longrightarrow \quad \Gamma_2^{(Maj)} = \begin{pmatrix} ¶ & 0 \\ 0 & -¶ \end{pmatrix}$$

$$\begin{aligned} x'_3 &= -\sigma_1\,y \\ y'_3 &= -\sigma_1\,x \end{aligned} \quad \longrightarrow \quad \Gamma_3^{(Maj)} = \begin{pmatrix} 0 & -\sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}$$

$\longrightarrow$

The Majorana representation of $\{ \Gamma_{p=1\,,\,q=3}\,;\,\mathbb{R} \}$ constructed in eq. 96 allows inner automorphisms

$$\Gamma'_{\mu} = R\,\Gamma_{\mu}^{(Maj)}\,R^{-1} \;\;;\;\; R : \textbf{real}_{\,4\times 4} \;\;,\;\; \{ R \,|\, Det\,R = 1 \} \simeq SL(4, \mathbb{R})$$

(97)

$$dim\,(SL(4, \mathbb{R})) = 15$$

forming the special linear (real, simple, noncompact) group in 4 dimensions .

We include $\Gamma_5^{(\chi)}$ (eq. 90) and transform it to the Majorana representation, using eq. 95

$$\Gamma_5^{(\chi)} \begin{pmatrix} x + iy \\ \varepsilon^{-1}(x - iy) \end{pmatrix} = \begin{pmatrix} i\,(x + i\,y) \\ -i\,\varepsilon^{-1}\,(x - iy) \end{pmatrix} = \begin{pmatrix} x' + i\,y' \\ \varepsilon^{-1}\left(x' - iy'\right) \end{pmatrix}_5$$

$$= \begin{pmatrix} -y + i\,x \\ \varepsilon^{-1}\,(-y - ix) \end{pmatrix}$$

$$\begin{array}{l} x'_5 = -\,y \\ y'_5 = x \end{array} \;\;\color{red}{\longrightarrow}\;\; \Gamma_5^{(Maj)} = \begin{pmatrix} 0 & -\,¶ \\ ¶ & 0 \end{pmatrix}$$

(98)

Appendix B : The Majorana representation $\{ \Gamma_{p,q} ; \mathbb{R} \}$
counting of (p) antisymmetric real Γ_x matrices

Lets first reorder the usual numbering of Γ matrices such that for signature (p , q) we have

$$0 \to 1 , \cdots , p-1 \to p \; ; \; p \to 1 , \cdots , p+q-1 \to q$$

(99)
$$\left(\Gamma_r^- \right) = (-1) \P_{2^\nu \times 2^\nu} \; ; \; r = 1 , \cdots , p \; ; \; d = 2\nu = p + q$$
$$\left(\Gamma_s^+ \right) = (+1) \P_{2^\nu \times 2^\nu} \; ; \; s = 1 , \cdots , q$$

Given a Majorana representation the first level Γ^+ matrices can be brought to *symmetric* , the Γ^- matrices to *antisymmetric* form .
The level λ product of Γ matrices is thus of the form

$$\Pi^\lambda_{\varrho\sigma} = \left(\Gamma^-_{r_1} \cdots \Gamma^-_{r_\varrho} \right) \left(\Gamma^+_{s_1} \cdots \Gamma^+_{r_\sigma} \right) \; ; \quad 1 \le r_1 \cdots \le r_\varrho \le p$$

(100)
$$1 \le s_1 \cdots \le s_\sigma \le q$$

$$\lambda = \varrho + \sigma \; ; \; 0 \le \lambda \le d$$

$\longrightarrow$

The signature of any member of $\Pi^{\lambda}_{\varrho\sigma}$ is

(101)
$$sig\ \left(\Pi^{\lambda}_{\varrho\sigma}\right) = s\,(\,\lambda\,;\,\varrho\,,\,\sigma\,)\ =\ (\,-\,1\,)^{\,\varrho}\ (\,-\,1\,)^{\,\frac{1}{2}\,\lambda\,(\lambda-1)}$$
$$\lambda\ =\ \varrho\ +\ \sigma\quad;\quad 0\ \leq\ \lambda\ \leq\ d\quad;\quad 0\ \leq\ \varrho\ \leq\ p\quad;\quad 0\ \leq\ \sigma\ \leq\ q$$

The factor $(\,-\,1\,)^{\,\frac{1}{2}\,\lambda\,(\lambda-1)}$ segregates λ into the four classes mod 4

(102)
$$(\,-\,1\,)^{\,\frac{1}{2}\,\lambda\,(\lambda-1)}\ =\ \begin{cases} +1 \quad \textbf{for} \quad \lambda\ =\ \begin{cases} 4\,m\ +\ 0\ \rightarrow\ (\lambda)_{\,0} \\ 4\,m\ +1\ \rightarrow\ (\lambda)_{\,1} \end{cases} \\ -\,1 \quad \textbf{for} \quad \lambda\ =\ \begin{cases} 4\,m\ +2\ \rightarrow\ (\lambda)_{\,2} \\ 4\,m\ +3\ \rightarrow\ (\lambda)_{\,3} \end{cases} \end{cases}$$

$$m\ =\ 0\,,\,1\ \cdots\quad;\quad 0\ \leq\ \lambda\ \leq\ 2\,\nu$$

The signature $s\,(\,\lambda\,;\,\varrho\,,\,\sigma\,)$ defined in eq. 101 then separates the (integer) indices further $\longrightarrow$

(103)

$$s(\lambda;\varrho,\sigma) = \begin{cases} +1 \quad \textbf{for} \quad \varrho = \begin{cases} \textbf{even} \ \&(\lambda)_{0\ \&\ 1} \\ \textbf{odd} \ \&(\lambda)_{2\ \&\ 3} \end{cases} \\ -1 \quad \textbf{for} \quad \varrho = \begin{cases} \textbf{even} \ \&(\lambda)_{2\ \&\ 3} \\ \textbf{odd} \ \&(\lambda)_{0\ \&\ 1} \end{cases} \end{cases}$$

Hence the power M of the set $\mathcal{S}_- = \{\varrho, \sigma \mid s(\lambda;\varrho,\sigma) = -1\}$ **is the number of antisymmetric** $2^\nu \times 2^\nu$ **matrices**

$$M = \sum_{\mathcal{S}_-} \begin{pmatrix} p \\ \varrho \end{pmatrix} \begin{pmatrix} q \\ \sigma \end{pmatrix} = 2^{\nu-1}(2^\nu - 1) \quad ; \quad p + q = 2\nu$$

(104)

$$\mathcal{S}_- = \left\{ \begin{array}{c} \varrho \ \textbf{even} \ \&(\lambda)_{2\ \&\ 3} \\ \cup \\ \varrho \ \textbf{odd} \ \&(\lambda)_{0\ \&\ 1} \end{array} \right\} \quad ; \quad \lambda = \varrho + \sigma \quad ; \quad 0 \leq \varrho \leq p \quad ; \quad 0 \leq \sigma \leq q$$

→

Eq. 104 only holds ***provided*** **a Majorana representation** $\{\, \Gamma_{\,p\,,\,q} \,;\, \mathbb{R} \,\}$ **exists , thereby yielding a nontrivial condition . We illustrate this for p = 2 , q = 0**

$$q \;=\; 0 \;\rightarrow\; \lambda \;=\; \varrho \;\;\rightarrow$$

$$\varrho\;\textbf{even}\;\&(\lambda)_{\,2\,\&\,3} \;\rightarrow\; \varrho \;=\; 2$$

(105)

$$\varrho\;\textbf{odd}\;\&(\lambda)_{\,0\,\&\,1} \;\rightarrow\; \varrho \;=\; 1 \;\;\rightarrow\;\; M \;=\; 3$$

$$\nu \;=\; 1 \;\rightarrow\; 2^{\,\nu-1}\;(\,2^{\,\nu} \;-\; 1\,) \;=\; 1 \;\;\neq\;\; M$$

The set $\mathcal{S}_{\,-}$ **defined in eq. 104 can also be classified according to the mod 4 classes of** $\varrho\,,\;\sigma\,;\;(\varrho)\,,\;(\sigma)$ **separately** $\longrightarrow$

(106)

$$\mathcal{S}_{-} = \left\{ \begin{array}{cc|cc} (\varrho) & (\sigma) & (\varrho) & (\sigma) \\ \hline 0 & 2 & 0 & 3 \\ 1 & 0 & 1 & 3 \\ 2 & 0 & 2 & 1 \\ 3 & 1 & 3 & 2 \end{array} \right.$$

Hence the calculation of M involves a selected sum over the pair of mod 4 class sums

$$\varrho = 4\,r + (\varrho) \quad , \quad \sigma = 4\,s + (\sigma) \quad ; \quad r\,,\,s = 0\,,\,1 \cdots$$

(107)

$$M\,{}^{p}_{(\varrho)}\,{}^{q}_{(\sigma)} = \sum{}_{r,s} \begin{pmatrix} p \\ 4\,r + (\varrho) \end{pmatrix} \begin{pmatrix} q \\ 4\,s + (\sigma) \end{pmatrix}$$

$$\text{with} \quad \left\{ \begin{array}{l} 4\,r + (\varrho) \le p \\ 4\,s + (\sigma) \le q \end{array} \right.$$

The double sums for $M\,{}^{p}_{(\varrho)}\,{}^{q}_{(\sigma)}$ **in eq. 107 factorize** $\longrightarrow$

B6

$$M^{\,p\;\,q}_{\,(\varrho)\,(\sigma)} = M^{\,p}_{\,(\varrho)}\, M^{\,q}_{\,(\sigma)}$$

(108) $$M^{\,n}_{\,(\tau)} = \sum_{u=0}^{4\,u\,+\,(\tau)\,\leq\,n} \begin{pmatrix} n \\ 4\,u\,+\,(\tau) \end{pmatrix}$$

$$M^{\,n}_{\,(\tau)} = 0 \;\; \textbf{for} \;\; n < (\tau)$$

If the condition(s) $4\,u\,+\,(\tau) \leq n$ cannot be satisfied, i.e. for $n < (\tau)$ the mod 4 sum $M^{\,n}_{\,(\tau)}$ has to be set to zero , as indicated in eq. 108 .

The factorized forms thus yield for M (eq. 104)

(109) $$M = \begin{bmatrix} M^{\,p}_{\,(0)} \left(M^{\,q}_{\,(2)} + M^{\,q}_{\,(3)} \right) \\ + M^{\,p}_{\,(1)} \left(M^{\,q}_{\,(0)} + M^{\,q}_{\,(3)} \right) \\ + M^{\,p}_{\,(2)} \left(M^{\,q}_{\,(0)} + M^{\,q}_{\,(1)} \right) \\ + M^{\,p}_{\,(3)} \left(M^{\,q}_{\,(1)} + M^{\,q}_{\,(2)} \right) \end{bmatrix}$$

Appendix C: mod 4 sums of binomials and powers of 2

Pascal's triangle [11-1982]

The mod 4 sums of binomial coefficients $M_{(\tau)}^{\,n}$ defined in eq. 108 shall be endowed with the prefix 4 for clarity of notation

(110) $$M_{(\tau)}^{\,n} \;\rightarrow\; {}_4\,M_{(\tau)}^{\,n} \;=\; \sum_{u=0}^{4\,u\,+\,(\tau)\,\leq\,n} \left(\begin{array}{c} n \\ \\ 4\,u\,+\,(\tau) \end{array} \right) \quad ; \quad (\tau) \;=\; 0,1,2,3$$

The periodicity structure $\{$ mod $4\,\}$ can be mapped on the powers of the fourth roots of 1 (over $\mathbb{C}$)

$$r_{\,(\tau)} \;=\; i^{\,(\tau)} \;=\; \exp\,\left(\,\frac{2\pi}{4}\,(\tau)\,\right) \quad ; \quad (\tau) \;=\; 0,1,2,3 \;\rightarrow$$

(111) $$k \,:\, r_{\,(\tau)} \;\rightarrow\; \left(\,r_{\,(\tau)}\,\right)^{\,k} \;=\; \left(\,r_{\,(\tau)}\,\right)^{\,(k)} \;=\; x_{\,(\tau)\,(k)} \quad ; \quad (\tau)\,,\,(k) \;=\; 0,1,2,3$$

$$x_{\,(\tau)\,(k)} \;=\; (\,i\,)^{\,(\tau)\,(k)} \;=\; (\,i\,)^{\,(\,(\tau)\,(k)\,)}$$

I display the mod 4 multiplication table for the quantity $(\,(\tau)\,(k)\,)$ below $\longrightarrow$

C2

(112)

$$\begin{array}{r|cccc} (\tau)\backslash(k) & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 0 & 2 & 0 & 2 \\ 3 & 0 & 3 & 2 & 1 \end{array} \longrightarrow \quad x_{(\tau)(k)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

It follows that the inverse matrix to $x_{(\tau)(k)}$ denoted $y_{(k)(\tau)}$ filters out the mod 4 sums in conjunction with any generating function given by a power series $G(z) = \sum_{k=0}^{\infty} G_k z^k$

$$y_{(k')(\tau)} x_{(\tau)(k)} = \delta_{(k')(k)}$$

(113)

$$y_{(k)(\tau)} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} = \frac{1}{4} (i)^{-(k)(\tau)}$$

→

mod 4 filtering

Let me define the following set of generating functions associated with a given generating function $G(z)$

(114)

$$G(z) = \sum_{k=0}^{\infty} a_k z^k \rightarrow$$

$$G_{(\tau)}(z) = G\left(i^{(\tau)} z\right) \quad ; \quad (\tau) = 0, 1, 2, 3$$

$$G_{(\tau)}(z) = \sum_{k=0}^{\infty} i^{(\tau)(k)} a_k z^k$$

In eq. 114 the quantities $x_{(\tau)(k)} = (i)^{(\tau)(k)}$ defined in eq. 111 appear , multiplying the k-th not the (k)-th term in the power series for $G_{(\tau)}(z)$. It follows using y , the inverse of x defined in eqs. 111 , 113 , setting

$$\widetilde{G}^{(l)}(z) = \sum_{(\tau)} y_{(l)(\tau)} G_{(\tau)}(z) \quad ; \quad k = 4u + (k) ; \ u = 0, 1, \cdots$$

(115)

$$\begin{aligned} \widetilde{G}^{(l)}(z) &= \sum_u \sum_{(k)} a_{4u+(k)} z^{4u+(k)} \left(\sum_{(\tau)} y_{(l)(\tau)} x_{(\tau)(k)} \right) \\ &= \sum_u \sum_{(k)} a_{4u+(k)} z^{4u+(k)} \delta_{(l)(k)} \\ &= \sum_u a_{4u+(l)} z^{4u+(l)} \quad ; \quad (l) = 0, 1, 2, 3 \end{aligned}$$

$\rightarrow$

Before generating the mod 4 sums of binomial coefficient we have to settle a subtle case in the definition of ${}_4M^{0}_{(0)}$ which occurs through the properties of the set $\mathcal{S}_-$ defined in eq. 104 arising when either p or q is 0 (but not both)

(116) $${}_4M^{0}_{(0)} = 1\,,\; {}_4M^{0}_{(k)} = 0 \text{ for } (k) > 0$$

With the case n = 0 given in eq. 116 we can use for $n \geq 1$ as generating function for the mod 4 sums of binomial coefficients the generating polynomial

(117) $$\begin{aligned} G(n;z) &= (1+z)^n \qquad ; \quad n \geq 1 \\ &= \sum_{k=0}^{n} a^n_k z^k \;;\; a^n_k = \binom{n}{k} \end{aligned}$$

The base functions $G_{(\tau)}$ in eq. 114 thus become →

(118)
$$G_{(0)}(n;z) = (1+z)^n$$
$$G_{(1)}(n;z) = (1+iz)^n$$
$$G_{(2)}(n;z) = (1-z)^n$$
$$G_{(3)}(n;z) = (1-iz)^n$$

$$_4M^n_{(l)} = \sum_{(\tau)} y_{(l)(\tau)} Y^n_{(\tau)} \quad ; \quad Y^n_{(\tau)} = G_{(\tau)}(n; z=1)$$

With y determined in eq. 111 it remains to calculate the constants Y defined in eq. 118

(119)
$$Y^n_{(0)} = 2^n \quad , \quad Y^n_{(1)} = (1+i)^n$$
$$Y^n_{(2)} = 0 \quad , \quad Y^n_{(3)} = (1-i)^n = \left(Y^n_{(1)}\right)^*$$

It is the the powers $(1 \pm i)^n$ for odd (τ) within the mod 4 logic , which bring about the mod 8 dependence inherent to Majorana representations , for even $d = p + q = 2\nu$. $\rightarrow$

C6

$$n \;=\; 8\,u \;+ \{n\} \quad ; \quad \{n\} \;=\; 0, 1 \;\cdots\; 7 \quad , \quad u \;=\; 0, 1 \;\cdots$$

$$(\,1 \;+\; i\,)^{\,n} \;=\; 2^{\,4u}\;(\,1 \;+\; i\,)^{\,\{n\}}$$

(120)

$\{n\}$	0	2	4	6
$\left(\frac{1}{\sqrt{2}}\;(1+i)\right)^{\{n\}}$	1	i	-1	$-i$

$\{n\}$	1	3	5	7
$\left(\frac{1}{\sqrt{2}}\;(1+i)\right)^{\{n\}}$	$\frac{1}{\sqrt{2}}\,(1+i)$	$\frac{1}{\sqrt{2}}(-1+i)$	$\frac{1}{\sqrt{2}}\,(-1-i)$	$\frac{1}{\sqrt{2}}\,(1-i)$

$\longrightarrow$

Pascals triangle $\begin{pmatrix} n \\ k \end{pmatrix}$ **for** $n = 1, 2, \cdots, 16$

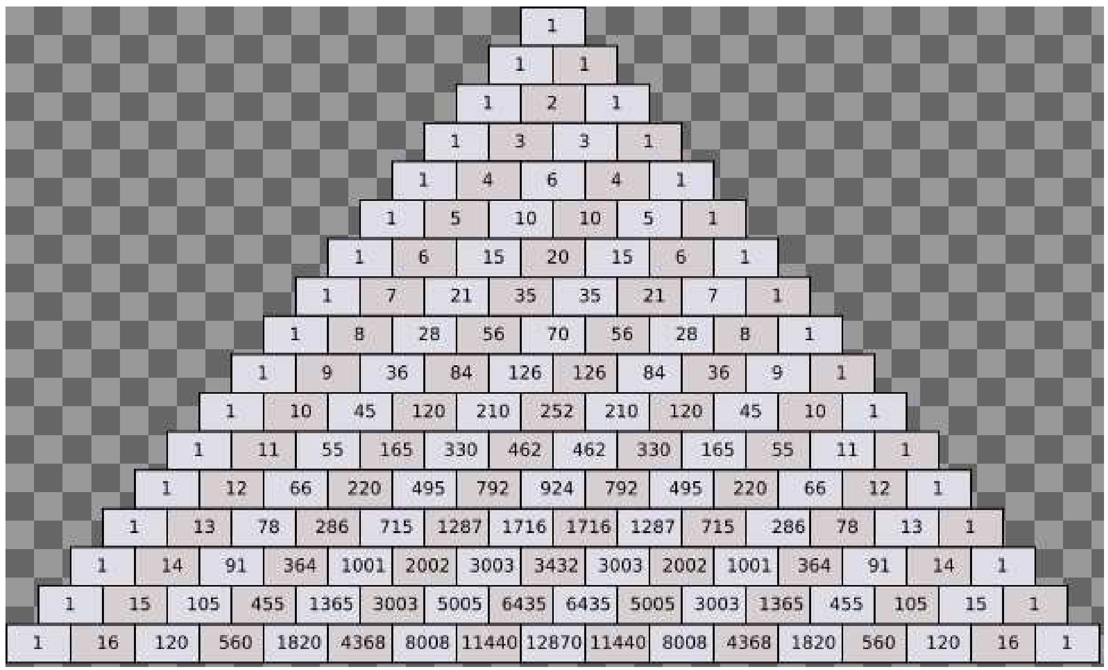

Fig 3 : Pascal's triangle

Appendix D: mod 4 sums of binomials and powers of $2^{\frac{1}{2}}$ and $i^{\frac{1}{2}} \equiv \exp\left(\frac{2\pi}{8} i\right)$

It is through the generating polynomials that half-integer powers of 2 (and i) enter . We rewrite eq. 119 and use y in the form given in eq. 113

(121)
$$Y^{\,n}_{(0)} = 2^{\,n} \quad , \quad Y^{\,n}_{(1)} = \left(2^{\,\frac{n}{2}}\right)\left(i^{\,\frac{n}{2}}\right)$$
$$Y^{\,n}_{(2)} = 0 \quad , \quad Y^{\,n}_{(3)} = \left(2^{\,\frac{n}{2}}\right)\left(i^{\,-\frac{n}{2}}\right) = \left(Y^{\,n}_{(1)}\right)^{*}$$
$$i^{\,\frac{n}{2}} = i^{\,\frac{\{n\}}{2}} \;;\; \{n\} = n \text{ mod } 8 \;;\; y_{\,(l)\,(\tau)} = \frac{1}{4}\,(\,i\,)^{\,-\,(l)\,(\tau)}$$

This yields the characteristic sums

(122)
$$Y^{\,n\,(l)} = \frac{1}{4}\sum_{(\tau)} (\,i\,)^{\,-\,(l)\,(\tau)}\, Y^{\,n}_{(\tau)} = 2^{\,n-2} + \frac{1}{4}\,\Delta^{\,n\,(l)}$$
$$\Delta^{\,n\,(l)} = 2^{\,\frac{n}{2}}\left((\,i\,)^{\,-\,(l)}\,(\,i\,)^{\,\frac{n}{2}} + (\,i\,)^{\,-\,3\,(l)}\,(\,i\,)^{\,-\frac{n}{2}}\right)$$
$$= 2^{\,\frac{n}{2}}\left((\,i\,)^{\,-\,(l)}\,(\,i\,)^{\,\frac{n}{2}} + (\,i\,)^{\,(l)}\,(\,i\,)^{\,-\frac{n}{2}}\right)$$
$$(\,i\,)^{\,-\,3\,(l)} = (\,i\,)^{\,(l)}$$

→

Eq. 122 yields

$$\Delta^{\,n\,(l)} \;=\; (\,2\,)^{\,\frac{n}{2}+1}\;\Re\,\Big(\,(\,i\,)^{\,\frac{n}{2}\,-\,(l)}\,\Big)$$

(123) $$(\,i\,)^{\,\frac{n}{2}\,-\,(l)} \;=\; \Big(\,(\,i\,)^{\,[\frac{n}{2}]\,-\,(l)}\,\Big)\begin{cases} 1 & \textbf{for} \quad n \textbf{ even} \\ i^{\,\frac{1}{2}} & \textbf{for} \quad n \textbf{ odd} \end{cases}$$

$$\left[\frac{n}{2}\right]$$

We thus rewrite the quantities $Y^{\,n\,(l)}$ **in eq. 122**

$$_4\,M^{\,n}_{\,(l)} \;\equiv\; Y^{\,n\,(l)} \;=\; 2^{\,n-2} \;+\; 2^{\,[\frac{n}{2}]\,-1}\,F\,(\,n\;-\;2\,(l)\,) \;\;;\;\; n\;\geq\;1$$

(124) $$F\,(\,n\;-\;2\,(l)\,) \;=\; \Re\,\Big(\exp\;\Big(\,i\,\frac{2\pi}{8}\,(\,n\;-\;2\,(l)\,)\Big)\,\Big)\;\times\;\begin{cases} 1 & \textbf{for} \quad n \textbf{ even} \\ \sqrt{2} & \textbf{for} \quad n \textbf{ odd} \end{cases}$$

$$F\;\rightarrow\;F\,(\,j\,)\;\;;\;\;j\;=\;0\,,\,\pm\,1\,,\,\pm\,2\,\cdots \quad \textbf{with} \quad j\;\rightarrow\;n\;-\;2\,(l)$$

The function $F\,(\,j\,)$ **defined in eq. 124 over the signed integers j takes only integer values**
$\{\,F\,\}\;=\;\{\,0\,,\,\pm\,1\,\}$

$\longrightarrow$

$$F(j) = F(-j) = F(j+8)$$

j	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
F	1	0	-1	-1	-1	0	1	1	1	0	-1	-1	-1	0	1	1

(125)

F can be visualized as projection on the real axis of a side-centered quadrangle in the complex plane , with the side centers forming an inscribed quadrangle rotated by 45 degrees , as shown in figure 1 below. We collect the formulae determining the mod 4 sums of binomial coefficients (eqs. 116 , 124)

$${}_4M^{\,0}_{(0)} = 1 \,,\; {}_4M^{\,0}_{(l)} = 0 \text{ for } (l) > 0$$

(126) $${}_4M^{\,n}_{(l)} \equiv Y^{\,n\,(l)} = 2^{\,n-2} + 2^{\left[\frac{n}{2}\right]-1} F(n - 2(l)) \;\; ; \;\; n \geq 1$$

with $F(j)$ **defined in eq. 125**

Care must be taken if using eq. 126 for n = 1 and 2 whenever $(l) > n$ **.** $\longrightarrow$

D4

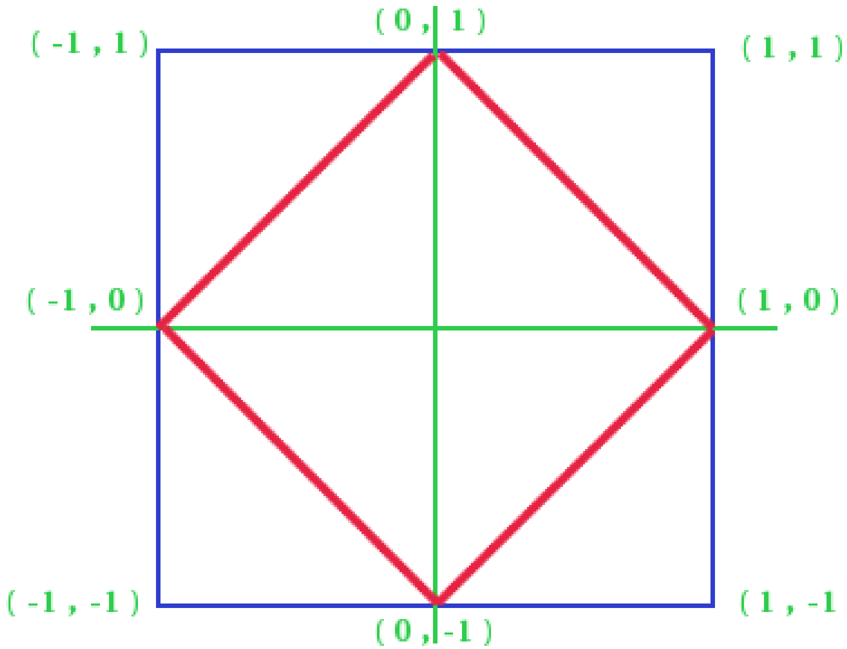

Fig 1 : The side-centered quadrangle(s) associated with the function $F\ (\ j\ )\ \longleftrightarrow$

Appendix E: The spin (10) product representations

$$\left(16 \oplus \overline{16} \right) \otimes \left(16 \oplus \overline{16} \right)$$

We follow the spin (10) decomposition discussed in section 2-1 (eq. 17 repeated below)

(17) $$\textbf{spin}\ (10) \longrightarrow \textbf{SU5} \times \textbf{U1}_{\,J_{\,5}}$$

Further let us denote representations of spin (10) as opposed to those pertaining to SU5 and associated $J_{\,5}$ quantum number by

(127) $$\textbf{spin (10) :}\quad [dim] \quad ; \quad \textbf{SU5} \times \textbf{U1}_{\,J_{\,5}} \ : \ \{dim\}_{\,J_{\,5}}$$

Thus eq. 21 translates to

(128)
$$\begin{aligned} [16] &= \{1\}_{+5} + \{10\}_{+1} + \{\overline{5}\}_{-3} \\ [\overline{16}] &= \{1\}_{-5} + \{\overline{10}\}_{-1} + \{5\}_{+3} \end{aligned}$$

In turn SU5 representations shall be decomposed along the standard model gauge group SU3 $_{c}$ $\otimes$ SU2 $_{L}$ $\otimes$ U1 $_{\mathcal{Y}}$, where $\mathcal{Y}$ denotes the electroweak hypercharge $\left(\text{with a factor}\frac{1}{2}\ \text{included}\right)$

(129) $$\mathcal{Y} = Q_{\,e.m.} / e - I_{\,3\,L}$$

$\longrightarrow$

(130) $$\{dim\} \rightarrow \sum \,](dim\textsf{SU3}_{\,c} \, , \, dim\textsf{SU2}_{\,L})_{\,\mathcal{Y}}[$$

The brackets on the right hand side of eq. 130 are reversed in order not to confuse spin (10) - and standard model representations.

Then the base $16 \; (\, \overline{16} \,)$ **decompose to**

(131) $$[16] \rightarrow \left[\begin{array}{lcl} \{1\}_{\,+5} & \rightarrow & \left\{ \quad](1\,,\,1)_{\,0}[\quad \right\}_{\,+5} \\ \{10\}_{\,+1} & \rightarrow & \left\{ \begin{array}{l}](3\,,\,2)_{\,+\frac{1}{6}}[\; + \\ ](\overline{3}\,,\,1)_{\,-\frac{2}{3}}[\; + \\ ](1\,,\,1)_{\,+1}[\end{array} \right\}_{\,+1} \\ \{\overline{5}\}_{\,-3} & \rightarrow & \left\{ \begin{array}{l}](\overline{3}\,,\,1)_{\,+\frac{1}{3}}[\; + \\ ](1\,,\,2)_{\,-\frac{1}{2}}[\end{array} \right\}_{\,-3} \end{array} \right.$$

The product representations $(\, 16 \, \oplus \, \overline{16} \,) \, \otimes \, (\, 16 \, \oplus \, \overline{16} \,)$ **generate *all* SO (10) antysymmetric tensor ones, of which we encountered the fivefold antisymmetric in section 2-1 (eq. 26).** $\longrightarrow$

To elaborate we specify the n-fold antisymmetric tensors obtained from the 10-representation of SO (10)

$$
\begin{aligned}
& [\, t_{\,0}\,] \sim 1 \\
& [\, t_{\,1}\,]^{\,A} \sim z^{\,A} \;\; ; \;\; A = 1, 2, \cdots, 10 \;\; \leftrightarrow \;\; [\, t_{\,1}\,] = \{\, \overline{5}\,\}_{\,2} \oplus \{\, 5\,\}_{\,-2} \\
& [\, t_{\,2}\,]^{\,[A_{\,1}\, A_{\,2}]} \sim \tfrac{1}{2} \left(z_{\,1}^{\,A_{\,1}}\, z_{\,2}^{\,A_{\,2}} - z_{\,1}^{\,A_{\,2}}\, z_{\,2}^{\,A_{\,1}} \right) \\
& \qquad \cdots \\
& [\, t_{\,n}\,]^{\,[A_{\,1}\, A_{\,2} \cdots A_{\,n}]} \sim \tfrac{1}{n!} \sum sgn \begin{pmatrix} 1 & \cdots & n \\ \pi_{\,1} & \cdots & \pi_{\,n} \end{pmatrix} z_{\,1}^{\,A_{\,\pi 1}}\, z_{\,2}^{\,A_{\,\pi 2}} \, \cdots \, z_{\,n}^{\,A_{\,\pi n}} \\
& n \le 10
\end{aligned}
\tag{132}
$$

The quantities $[\, t_{\,n}\,]$ defined in eq. 132 form irreducible real representations of SO (10) except for n = 5 , which is composed of the relatively complex *irreducible* representations 126 and $\overline{126}$ (eq. 26) .
The tenfold antisymmetric invariant corresponds to $[\, t_{\,n=10}\,]$. The product of two full Clifford algebras pertaining to spin (10) contains all $[\, t_{\,n}\,] \;\; ; \;\; n = 0 \cdots 10$ representations exactly once . $\longrightarrow$

E4

Treating the n = 5 tensor as one representation – it is reducible only over $\mathbb{C}$ – the dimensions of the $[\,t_n\,]$ representations follow Pascal's triangle (Fig. 3 page C7) of binomial coefficients for N = 10, whereby n even and odd shall be distinguished

$$\begin{array}{ccccccccccc} [\,t_0\,] & & [\,t_2\,] & & [\,t_4\,] & & [\,t_6\,] & & [\,t_8\,] & & [\,t_{10}\,] \\ & [\,t_1\,] & & [\,t_3\,] & & [\,t_5\,] & & [\,t_7\,] & & [\,t_9\,] & \\ 1 & & 45 & & 210 & & 210 & & 45 & & 1 \\ & 10 & & 120 & & 252 & & 120 & & 10 & \end{array} \tag{133}$$

This corresponds to the following products of $16 + \overline{16}$

(134)

	$[16]$	$[\overline{16}]$
$[16]$	$s\ :\ \begin{array}{c}[10]\ +\\ [126]\end{array}\ ,\ a\ :\ [120]$	$\begin{array}{c}[1]\ +\ [45]\ +\\ [210\,]\end{array}$
$[\overline{16}]$	$\begin{array}{c}[1]\ +\ [45]\ +\\ [210\,]\end{array}$	$s\ :\ \begin{array}{c}[10]\ +\\ [\overline{126}]\end{array}\ ,\ a\ :\ [120]$

$\longrightarrow$

The correspondence of product representations of the $16 + \overline{16} = 32$ associative Clifford algebra with the sum of antisymmetric tensor ones follows from the completeness of all products of γ matrices forming the spin (10) algebra i.e. are of dimension

(135) $$(32)^{2} = (2^{5})^{2} = 2^{10}$$

We proceed to reduce the $[16] \otimes [16]$ product with respect to J_5 , SU5 and SU3 $_c$ $\times$ SU2 $_L$ $\times$ U1 $_{\mathcal{Y}}$.
The individual products are $\left(\; _{s\,(a)} \right.$: (a)symmetric $\left. \right)$

	$\{1\}_5$	$\{10\}_1$	$\{\overline{5}\}_{-3}$
$\{1\}_5$	$\{1\}_{10\,s}$	$\{10\}_6$	$\{\overline{5}\}_2$
$\{10\}_1$	$\{10\}_6$	$\begin{pmatrix} \{\overline{5}\}_2 + \\ \{\overline{50}\}_2 \end{pmatrix}_s \left(\{\overline{45}\}_2 \right)_a$	$\begin{pmatrix} \{5\}_{-2} + \\ \{45\}_{-2} \end{pmatrix}$
$\{\overline{5}\}_{-3}$	$\{\overline{5}\}_2$	$\begin{pmatrix} \{5\}_{-2} + \\ \{45\}_{-2} \end{pmatrix}$	$\left(\{\overline{15}\}_{-6} \right)_s \left(\{\overline{10}\}_{-6} \right)_a$

(136)

→

We proceed to decompose the diagonal $\{\text{SU5}\}_{J_5}$ representations (eq. 131)

$$\left(\{10\}_1 \otimes \{10\}_1 \right)_s = \{\overline{5}\}_2 + \{\overline{50}\}_2 \qquad \downarrow$$

(137)

s	$\left](3\,,\,2)_{+\frac{1}{6}}\right[_{+1}$	$\left]\left(\overline{3}\,,\,1\right)_{-\frac{2}{3}}\right[_{+1}$	$\left](1\,,\,1)_{+1}\right[_{+1}$
$\left](3\,,\,2)_{+\frac{1}{6}}\right[_{+1}$	$\begin{pmatrix} \left](6\,,\,3)_{+\frac{1}{3}}\right[_2 + \\ \left]\left(\overline{3}\,,\,1\right)_{+\frac{1}{3}}\right[_2 \end{pmatrix}$	$\begin{pmatrix} \left](8\,,\,2)_{-\frac{1}{2}}\right[_2 + \\ \left](1\,,\,2)_{-\frac{1}{2}}\right[_2 \end{pmatrix}$	$\left](3\,,\,2)_{+\frac{7}{6}}\right[_2$
$\left]\left(\overline{3}\,,\,1\right)_{-\frac{2}{3}}\right[_{+1}$		$\left]\left(\overline{6}\,,\,1\right)_{-\frac{4}{3}}\right[_2$	$\left]\left(\overline{3}\,,\,1\right)_{+\frac{1}{3}}\right[_2$
$\left](1\,,\,1)_{+1}\right[_{+1}$			$\left](1\,,\,1)_{+2}\right[_2$

$\longrightarrow$

$$\left(\{\overline{5}\}_{-3} \otimes \{\overline{5}\}_{-3} \right)_s = \{\overline{15}\}_{-6} \qquad \downarrow$$

(138)
$$\begin{array}{c|cc} s & \left] (\overline{3}\,,\,1)_{+\frac{1}{3}} \right[_{-3} & \left] (1\,,\,2)_{-\frac{1}{2}} \right[_{-3} \\ \hline \left] (\overline{3}\,,\,1)_{+\frac{1}{3}} \right[_{-3} & \left] (\overline{6}\,,\,1)_{+\frac{2}{3}} \right[_{-6} & \left] (\overline{3}\,,\,2)_{-\frac{1}{6}} \right[_{-6} \\ \left] (1\,,\,2)_{-\frac{1}{2}} \right[_{-3} & & \left] (1\,,\,3)_{-1} \right[_{-6} \end{array}$$

$$\downarrow$$

complex e.w. triplet coupling to

$$\frac{1}{2} \left(\nu^{\,*}_{\,\dot{F}} \right)^{\alpha} \left(\nu^{\,*}_{\,\dot{G}} \right)_{\alpha}$$

Next we assemble the (anti)symmetric products $([16] \otimes [16])_s = [10] \oplus [126]$ **and** $([16] \otimes [16])_a = [120]$ **with respect to SU5** $\otimes$ **U1** $_{J_5}$ $\longrightarrow$

using eq. 136

$$([16] \otimes [16])_s = [10] \oplus [126] \qquad \downarrow$$

$$= \left\{ \begin{array}{c} \left[\begin{array}{l} \{5\}_{-2} + \\ \{\overline{5}_I\}_2 \end{array} \right] \\ \oplus \left[\begin{array}{c} \{1\}_{10} + \{\overline{5}_{II}\}_2 + \{10\}_6 + \{\overline{15}\}_{-6} \\ + \{45\}_{-2} + \{\overline{50}\}_2 \end{array} \right] \end{array} \right\} \tag{139}$$

$$([16] \otimes [16])_a = [120] \qquad \downarrow$$

$$= \left[\begin{array}{c} \{5\}_{-2} + \{\overline{5}\}_2 \\ + \{10\}_6 + \{\overline{10}\}_{-6} \\ + \{45\}_{-2} + \{\overline{45}\}_2 \end{array} \right]$$

The roman indices I, II in eq. 139 indicate that appropriate linear combinations of the *two* $\{\overline{5}\}_2$ representations form parts of $[10]$ and $[126]$ respectively . $\longrightarrow$

It remains to decompose the SU5 $\otimes$ U1 $_{J_5}$ representations in eq. 139 with respect to SU3 $_c$ $\times$ SU2 $_L$ $\times$ U1 $_{\mathcal{Y}}$. We do this associating according to the product representations as they appear in eq. 139

$$
\begin{array}{lc|c}
[10]\ [120] & \{5\}_{-2} & \left]\,(3\,,\,1)_{-\frac{1}{3}}\right[_{+3} + \left]\,(1\,,\,\overline{2})_{+\frac{1}{2}}\right[_{+3} \\
[10]\ [126]\ [120] & \{\overline{5}\}_{+2} & \left]\,(\overline{3}\,,\,1)_{+\frac{1}{3}}\right[_{-3} + \left]\,(1\,,\,2)_{-\frac{1}{2}}\right[_{+3} \\
\hline
[126] & \{1\}_{+10} & \left]\,(1\,,\,1)_{0}\right[_{+10} \\
[126]\ [120] & \{10\}_{+6} & \left]\,(3\,,\,2)_{-\frac{1}{6}}\right[_{+6} + \left]\,(\overline{3}\,,\,1)_{-\frac{2}{3}}\right[_{+6} + \left]\,(1\,,\,1)_{+1}\right[_{+6} \\
[120] & \{\overline{10}\}_{-6} & \left]\,(\overline{3}\,,\,\overline{2})_{+\frac{1}{6}}\right[_{-6} + \left]\,(3\,,\,1)_{+\frac{2}{3}}\right[_{-6} + \left]\,(1\,,\,1)_{-1}\right[_{-6}
\end{array}
$$

(140)

$\longrightarrow$

E10

[126]	$\{\overline{15}\}_{-6}$	$]\,(\overline{6}\,,\,1)_{+\frac{2}{3}}\,[_{-6} +]\,(\overline{3}\,,\,2)_{-\frac{1}{6}}\,[_{-6} +]\,(1\,,\,3)_{-1}\,[_{-6}$
[126] [120]	$\{45\}_{-2}$	$c.c.\ \updownarrow$
[120]	$\{\overline{45}\}_{+2}$	$\left[\begin{pmatrix}]\,(6\,,\,1)_{+\frac{1}{3}}\,[_{2} + \\ ]\,(\overline{3}\,,\,3)_{+\frac{1}{3}}\,[_{2} \end{pmatrix} + \begin{pmatrix}]\,(8\,,\,2)_{-\frac{1}{2}}\,[_{2} + \\ ]\,(1\,,\,2)_{-\frac{1}{2}}\,[_{2} \end{pmatrix} + \\ ]\,(3\,,\,2)_{+\frac{7}{6}}\,[_{2} +]\,(3\,,\,1)_{-\frac{4}{3}}\,[_{2} +]\,(\overline{3}\,,\,1)_{+\frac{1}{3}}\,[_{2} \right]$
[126]	$\{\overline{50}\}_{+2}$	$\left[\begin{pmatrix}]\,(6\,,\,3)_{+\frac{1}{3}}\,[_{2} + \\ ]\,(\overline{3}\,,\,1)_{+\frac{1}{3}}\,[_{2} \end{pmatrix} +]\,(8\,,\,2)_{-\frac{1}{2}}\,[_{2} +]\,(3\,,\,2)_{+\frac{7}{6}}\,[_{2} \\ +]\,(\overline{6}\,,\,1)_{-\frac{4}{3}}\,[_{2} +]\,(1\,,\,1)_{+2}\,[_{2} \right]$

(141)

$\longrightarrow$

E11

$$\left(\{10\}_{1} \otimes \{10\}_{1} \right)_{s} = \{\overline{45}\}_{2} \qquad \downarrow$$

(142)

a	$\left]\,(3\,,\,2)_{+\frac{1}{6}}\,\right[_{+1}$	$\left]\,(\overline{3}\,,\,1)_{-\frac{2}{3}}\,\right[_{+1}$	$\left]\,(1\,,\,1)_{+1}\,\right[_{+1}$
$\left]\,(3\,,\,2)_{+\frac{1}{6}}\,\right[_{+1}$	$\begin{pmatrix} \left]\,(6\,,\,1)_{+\frac{1}{3}}\,\right[_{2} + \\ \left]\,(\overline{3}\,,\,3)_{+\frac{1}{3}}\,\right[_{2} \end{pmatrix}$	$\begin{pmatrix} \left]\,(8\,,\,2)_{-\frac{1}{2}}\,\right[_{2} + \\ \left]\,(1\,,\,2)_{-\frac{1}{2}}\,\right[_{2} \end{pmatrix}$	$\left]\,(3\,,\,2)_{+\frac{7}{6}}\,\right[_{2}$
$\left]\,(\overline{3}\,,\,1)_{-\frac{2}{3}}\,\right[_{+1}$		$\left]\,(3\,,\,1)_{-\frac{4}{3}}\,\right[_{2}$	$\left]\,(\overline{3}\,,\,1)_{+\frac{1}{3}}\,\right[_{2}$
$\left]\,(1\,,\,1)_{+1}\,\right[_{+1}$			$--$

I end the collection of representation decompositions with the adjoint $[45]$ representation of SO (10) $\longrightarrow$

$$([10] \otimes [10])_a = [45] \qquad \downarrow$$

(143)

$$\begin{array}{c|cc}
a & \{5\}_{-2} & \{\overline{5}\}_{+2} \\
\hline
\{5\}_{-2} & \{10\}_{-4} & \left\{ \begin{array}{c} \{1\}_0 \leftrightarrow J_5 \\ \{24\}_0 \leftrightarrow \textbf{adjoint SU5} \end{array} \right\} \\
\{\overline{5}\}_{+2} & & \{\overline{10}\}_{+4}
\end{array}$$

It should be noted that despite coinciding dimensions the following entities are most distinct

(144)

$$[10] \neq \{10\}_{-4} \; , \; \{10\}_{6}$$

$$[45] \neq \{45\}_{-2} \; ; \; \cdots$$

References

[1-2001] P. Minkowski, 'On amplified and tiny signatures of neutrino oscillations', NO-VE International Workshop on Neutrino Oscillations, Venice, Italy 24.-26. July 2001, published in "Venice 2001, Neutrino Oscillations", p. 291 .

There is an error in the definition of the $\widehat{A}$ genus in ref. [1-2001] , which is corrected in eq. 3 here .

[2-1966] F. Hirzebruch, 'Topological methods in algebraic geometry' in "Die Grundlehren der mathematischen Wissenschaften" Band 131 , Springer Verlag, Berlin 1966 .

[3-2007] for a more complete account of left- *and* right-chiral bases see also P. Minkowski, 'Neutrino flavors light and heavy – how heavy ?', in nus-albufeira2007.pdf , URL : http://www.mink.itp.unibe.ch/ , and references cited therein .

[4-1975] H. Fritzsch and P. Minkowski , 'Unified interactions of leptons and hadrons' , Annals Phys.93 (1975) 193 and H. Georgi , 'The state of the art - gauge theories' , AIP Conf.Proc.23 (1975) 575 .

[5-1975] H. Fritzsch, M. Gell-Mann and P. Minkowski , 'Vector - like weak currents and new elementary fermions' , Phys.Lett.B59 (1975) 256 .

[6-1976] H. Fritzsch and P. Minkowski , 'Vector - like weak currents, massive neutrinos, and neutrino beam oscillations' , Phys.Lett.B62 (1976) 72 .

References

[7-1994] A. Fukugita and T. Yanagida , ’Physics of neutrinos’ in ’Physics and Astrophysics of Neutrinos’ , A. Fukugita and A. Suzuki eds. (Springer-Verlag; 1994) .

[8-1974] J. Pati and A. Salam, ’Lepton number as the fourth color’, Phys.Rev.D10 (1974) 275, Erratum-ibid. D11 (1975) 703 .

[9-1977] P. Minkowski, ’$\mu \rightarrow e\gamma$ at a rate of one out of 1-billion muon decays ?’ , Phys.Lett.B67 (1977) 421 .

[10-1979/80] Murray Gell-Mann, Pierre Ramond and Richard Slansky, ’Complex spinors and unified theories’ , published in Supergravity, P. van Nieuwenhuizen and D.Z. Freedman eds., North Holland Publ. Co., 1979 and in Stony Brook Wkshp.1979:0315 (QC178:S8:1979) .
Tsutomu Yanagida, ’Horizontal symmetry and masses of neutrinos’ , published in the Proceedings of the Workshop on the Baryon Number of the Universe and Unified Theories, O. Sawada and A. Sugamoto eds., Tsukuba, Japan, 13-14 Feb. 1979, and in (QCD161:W69:1979) .
Shelley Glashow, ’Quarks and leptons’ , published in Proceedings of the Cargèse Lectures, M. Lévy ed., Plenum Press, New York, 1980 .
Rabindra Mohapatra and Goran Senjanovič, ’Neutrino mass and spontaneous parity violation’ , Phys.Rev.Lett.44 (1980) 912 .

References

[11-1982] Robert Coquereaux, ’Modulo 8 Periodicity Of Real Clifford Algebras And Particle Physics’, Phys.Lett.B115 (1982) 389 .

[12-1974] H. Georgi and S. L. Glashow, ’Unity of all elementary particle forces’, Phys.Rev.Lett.32 (1974) 438 .

[13-1980] P. Minkowski, ’Fermionic oscillators and the embedding SO (2n) $\supset$ SUn ’, seminar at the Enrico Fermi Summer School of Physics, Varenna, Italy, Jul 21 - Aug 2, 1980 , (unpublished) .

[14-1975] F. Gursey, Pierre Ramond and P. Sikivie, ’A universal gauge theory model based on E6’, YALE-3075-118, Sep 1975, Phys.Lett.B60 (1976) 177.

[15-2005] P. Minkowski, ’Neutrino oscillations: A Historical overview and its projection’, Contributed to 11th International Workshop on Neutrino Telescopes, Venice, Italy, 22-25 Feb 2005, in *Venice 2005, Neutrino telescopes* 7-27, partially in hep-ph/0505049 .

[16-1937] E. Majorana, ’Teoria simmetrica dell’ elettrone e positrone’, Nuovo Cimento 14 (1937) 171 .

[17-2007] P. Minkowski, ’... natures way $\cdots$ our way’, in file naturesway2007.pdf, URL : http://www.mink.itp.unibe.ch , unpublished .

[18-1996] F. Gürsey and C.H. Tze , ”On the role of division- , jordan- and related algebras in particle physics”, Singapore, World Scientific (1996) 461 .

References

[f1] '#Q-1082 Lizard Carving within Polished Quartz Crystal from Russia. Detail in carving is excellent. 2.5" long' , from http://www.bestcrystals.com/ .

[f2] W. Konetschny and W. Kummer, 'Nonconservation of total Lepton number with scalar bosons', Phys. Lett. B 70 (1977) 433,

[h1] É. Cartan, 'Sur une classe remarquable d'espaces de Riemann', Bull. Soc. Math. de France, 54 (1926) 214 , and 55 (1927) 114 .

[1-a] M. F. Sohnius, 'The Conformal Group in Superspace', MPI-PAE/PTh 32/76, Oct 1976, 11p., to be publ. in Proc. of 2nd Conf. on Structure of Space and Time, Tutzing, West Germany, 1976, also in *Feldafing 1976, Proceedings, Quantum Theory and the Structure Of Time and Space, Vol.2 (In Memoriam Werner Heisenberg)*, Muenchen 1977, 241-252 and Muenchen Max-planck-Inst Phys - MPI-PAE-PTH 32-76 (76,REC.DEC) 11p.
M. F. Sohnius and P. C. West, 'Conformal invariance in N=4 supersymmetric Yang-Mills theory', Phys. Lett. 100B (1981) 245 .

[2-a] R. Brauer and H. Weyl, 'Spinors in n dimensions', Amer. J. Math., 57 no. 2 (1935), 425-449 .