場の理論、弦理論に現れるLie亜群

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§1. Introduction

Lie groupoids （**Lie 亜群**） a generalization of Lie groups
Lie algebroids （**Lie 亜代数**） a generalization of Lie algebras

History

• **Lie groups** became very important for physical theories in this 100 years, for example, a gauge symmetry, isospin, the Poincaré group, etc. Weyl, Wigner, Gell-Mann, Yang-Mills, etc.

• Many physical theories have generalized structures, Baez, Lauda '09
• understood as a **Lie groupoid** and a **Lie algebroid**. Ehresmann '58, Pradines '67

§2. Lie groupoids and Lie algebroids

Groupoids

A groupoid $\mathcal{G}$ is a set of **arrows** on a manifold M. It is denoted by $\mathcal{G} \rightrightarrows M$.

For each arrow $g \in \mathcal{G}$, $s \xrightarrow{g} t$

a **source map** $s : \mathcal{G} \to M$ a **target map** $t : \mathcal{G} \to M$

We consider a product gh of two elements of a set $\mathcal{G}$ only in a special case. A **multiplication** gh is defined only on elements such that $s(g) = t(h)$.

$$s(h) \xrightarrow{h} t(h) = s(g) \xrightarrow{g} t(g)$$

1, Require **associativity** $(gh)k = g(hk)$.

2, The **identity** 1_x satisfies $s(1_x) = t(1_x) = x \in M$.

$$s = t \circlearrowleft$$

3, An **inverse** g^{-1} is the reversed arrow.

$$s(g) = t(g^{-1}) \xleftarrow{g^{-1}} t(g) = s(g^{-1})$$

If $M = \{\text{pt.}\}$, $\mathcal{G}$ is a group.

Lie groupoid

If all operations are smooth, a groupoid is called a **Lie groupoid**.

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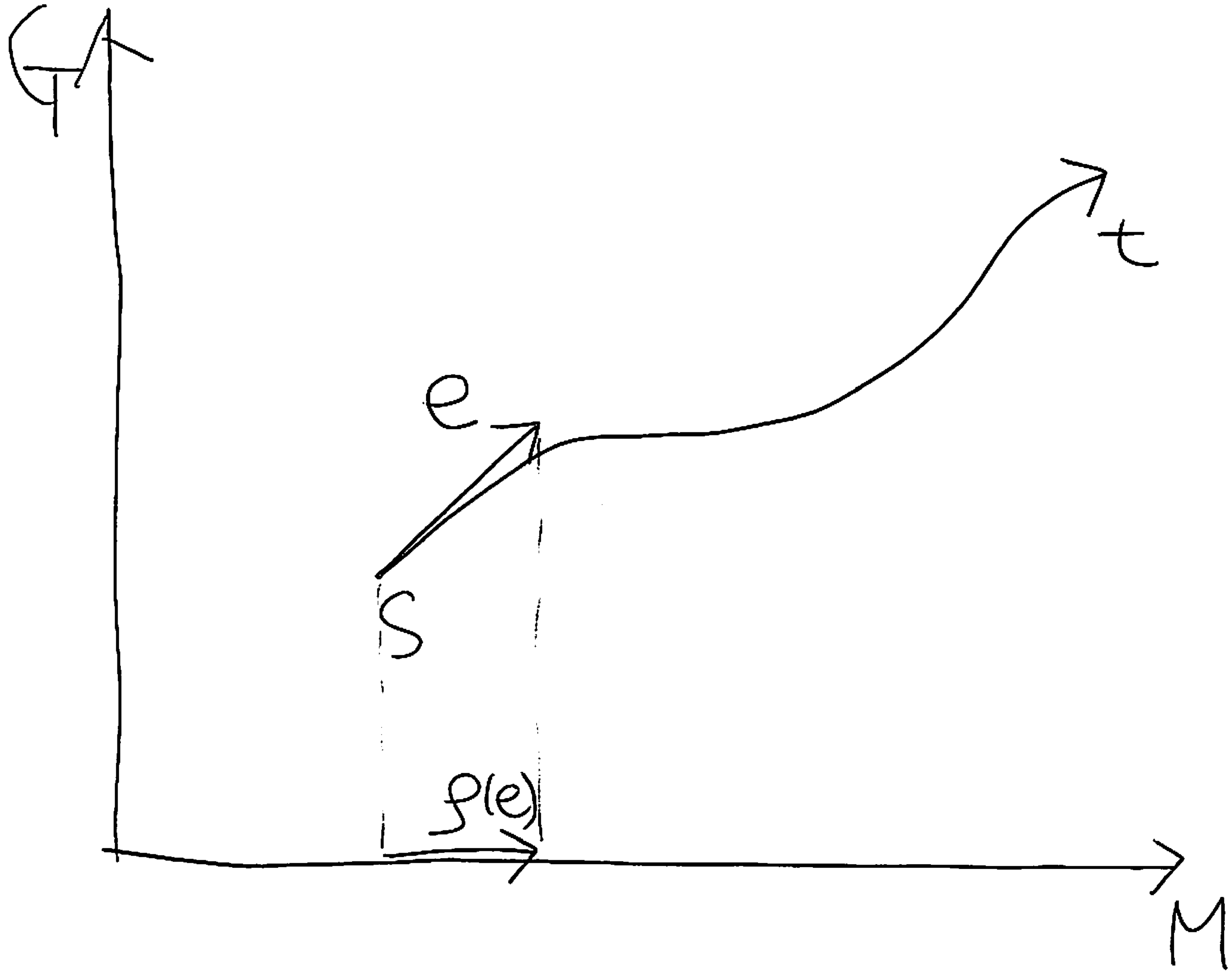
Lie algebroid

is an infinitesimal object of a Lie groupoid.



G
e
t
S
ρ(e)
M

Two operations for $e_i \in \Gamma(E), F \in C^\infty(M)$:

A **Lie bracket**: $[e_1, e_2]$, $\qquad [e_a, e_b] = f^c{}_{ab}(x) e_c$,

and

A bundle map (the **anchor map**): $\rho : E \to TM$,
$\rho(e_a)F(x) = \rho^i{}_a(x)\frac{\partial}{\partial x^i}F(x)$,

satisfying the following properties:

$$1. \qquad [\rho(e_1), \rho(e_2)] = \rho([e_1, e_2]),$$

$$2. \qquad [e_1, Fe_2] = F[e_1, e_2] + (\rho(e_1)F)e_2,$$

$$3. \qquad [[e_1, e_2], e_3] + (1, 2, 3 \text{ cylcic}) = 0.$$

Local coordinate expression

$$\rho^m{}_a \frac{\partial \rho^i{}_b}{\partial x^m} - \rho^m{}_b \frac{\partial \rho^i{}_a}{\partial x^m} + \rho^i{}_c f^c{}_{ab} = 0,$$

$$\rho^m{}_{[a} \frac{\partial f^d{}_{bc]}}{\partial x^m} + f^d{}_{e[a} f^e{}_{bc]} = 0.$$

§3. (Quasi-)Lie algebroids in physical theories

Example 1. Wilson Lines

Wilson loops on a fixed point x_0:

$$W(C_{x_0}) = \mathrm{Tr}\left(\mathcal{P}\exp i \oint_{x_0} A_\mu d\sigma^\mu\right),$$

consist of a group, but Wilson lines

$$W(L) = \mathrm{Tr}\left(\mathcal{P}\exp i \int_a^b A_\mu d\sigma^\mu\right),$$

consist of a groupoid, called a fundamental groupoid.

Example 2. Poisson Bracket

$$\{f(x), g(x)\}_{P.B.} = \frac{1}{2}\pi^{ij}(x)\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}$$

has a structure of a Lie algebroid on T^*M.

A Lie bracket is a Koszul bracket:

$$[\beta^{(1)}, \beta^{(2)}]_\pi = L_{\pi^\sharp \beta^{(1)}}\beta^{(2)} - L_{\pi^\sharp \beta^{(2)}}\beta^{(1)} - d_x\pi(\beta^{(1)}, \beta^{(2)}).$$

for example, $[df, dg]_\pi = d\pi(df, dg)$ for $\beta^{(1)} = df$, $\beta^{(2)} = dg$.

The anchor map

$$\rho(\beta) = \pi^{\sharp}\beta = \pi^{ij}(x)\beta_i \frac{\partial}{\partial x^j}$$

cf.) A Nambu bracket Nambu '73

$$\{f_1, f_2, \cdots, f_n\}$$

has an algebroid structure.

Example 3. Gravitational Theory

A simplest example is a two dimensional gravity theory with a dilaton:

$$S = \frac{1}{2}\int_{\Sigma} d^2\sigma \sqrt{-g}(\varphi R - V(\varphi)),$$

where φ is a scalar field. We rewrite the action to the Einstein-Cartan form by introducing a zweibein $e_\mu{}^a$, a spin connection $\omega_\mu{}^{ab} = \omega_\mu \epsilon^{ab}$ and an auxiliary scalar field ϕ_a.

$$S = \int_{\Sigma} d^2\sigma \left(-\frac{1}{2}\epsilon^{\mu\nu}\varphi R_{\mu\nu} - \frac{1}{2}\epsilon^{\mu\nu}\phi_a T_{\mu\nu}{}^a - eV(\varphi) \right),$$

where $R_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu$, and $T_{\mu\nu}{}^a = \partial_\mu e_\nu{}^a + \omega_\mu \epsilon^{ab} e_{\nu b} - (\mu \leftrightarrow \nu)$.

This action is invariant under the following transformation:

$$
\begin{aligned}
\delta \omega_\mu &= \partial_\mu t + \epsilon_{bc} c^b e_\mu{}^c \frac{\partial V(\varphi)}{\partial \varphi}, \\
\delta e_\mu{}^a &= -t \epsilon^{ab} e_{\mu b} + \partial_\mu c^a + \omega_\mu \epsilon^{ab} c_b, \\
\delta \varphi &= -\epsilon^{ab} c_a \phi_b, \\
\delta \phi_a &= -t \epsilon_{ab} \phi^b + \epsilon_{ab} c^b V(\varphi).
\end{aligned}
$$

This is a Lie groupoid (local gauge) symmetry. NI, Izawa '93

Example 4. T-duality

An open string with a NS B-field

$$S = \frac{1}{2}\int_\Sigma d^2\sigma (G_{IJ}(\phi)\partial^\mu\phi^I\partial_\mu\phi^J + B_{IJ}(\phi)\epsilon^{\mu\nu}\partial_\mu\phi^I\partial_\nu\phi^J),$$

where $H \equiv dB$ is an NS 3-form on a target space.

The Buscher rule in the T-duality $\phi^0 \to \tilde{\phi}^0$:

$$\tilde{G}_{00} = \frac{1}{G_{00}}, \quad \tilde{G}_{0a} = \frac{B_{0a}}{G_{00}}, \quad \tilde{G}_{ab} = G_{ab} - \frac{1}{G_{00}}(G_{0a}G_{0b} - B_{0a}B_{0b}),$$
$$\tilde{B}_{0a} = \frac{G_{0a}}{G_{00}}, \quad \tilde{B}_{ab} = B_{ab} - \frac{1}{G_{00}}(G_{0a}B_{0b} - G_{0b}B_{0a}),$$

There exists a local algebroid symmetry under the T-duality, which is the Courant algebroid on $TM \oplus T^*M$. Courant '90, Liu, Weinstein, Xu '96, Roytenberg '01

Dorfman bracket

$$[X+\xi, Y+\eta]_D = [X,Y] + L_X\eta - i_Y d\xi + i_X i_Y H,$$

for $X, Y \in TM$ and $\xi, \eta \in T^*M$.

the anchor map $\rho : X+\xi \to X$

metric $\langle X+\xi\, , Y+\eta\rangle = i_X\eta + i_Y\xi$

• The T-duality is an isomorphism of the Courant algebroid structure. Cavalcanti, Gualtieri '11 cf.) Double field theory

Hull, Zwiebach '09

Example 5. 2D $N = (2,2)$ supersymmetric sigma model

$N = (1,1)$ superfield: $\Phi^I \equiv \phi^I + \theta^+ \psi_+^I + \theta^- \psi_-^I + \theta^- \theta^+ F^I$

2D manifest $N = (1,1)$ SUSY sigma model

$$S = \frac{1}{2} \int_\Sigma d^2\sigma d^2\theta (G_{IJ}(\Phi) + B_{IJ}(\Phi)) D_+ \Phi^I D_- \Phi^J,$$

where $H \equiv dB$. This has a manifest $N = (1,1)$ supersymmetry

$$\delta_1 \Phi^I = \epsilon_1^+ D_+ \Phi^I + \epsilon_1^- D_- \Phi^I.$$

Find another supersymmetry such that

$$\delta_2 \Phi^I \;\; = \;\; \epsilon_2^+ J_+{}^I{}_J(\Phi) D_+ \Phi^J + \epsilon_1^- J_-{}^I{}_J(\Phi) D_- \Phi^J.$$

δ_1 and δ_2 are $N = (2,2)$ SUSY if and only if (G, H, J) consist of a bi-Hermitian structure. Gates, Hull, Roček '84

A bi-Hermitian structure is equivalent to a generalized Kähler structure, which is a **Dirac structure of the Courant algebroid**.

Gualtieri '04

A Dirac structure D is a subbundle of $TM \oplus T^*M$ such that

$$\langle D\,, D \rangle = 0, \quad [\Gamma D, \Gamma D]_D \subset \Gamma D, \quad \mathrm{rank}\, D = \frac{1}{2} \mathrm{rank}(TM \oplus T^*M).$$

Example 6. Topological Sigma Models

3D Chern-Simons theory

on a three dimensional manifold X with the Lie group G.

$$S = \frac{k}{4\pi}\int_X \mathrm{tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A).$$

is generalized to a gauge theory with a groupoid structure.

Courant sigma model

NI '02, Roytenberg '06

$$S = \int_X -b_i \wedge d\phi^i + \frac{1}{2}\langle A \wedge dA\rangle + \rho^i{}_a(\phi) b_i \wedge A^a + \frac{1}{3!}H_{abc}(\phi)A^a \wedge A^b \wedge A^c.$$

where ϕ^i is a scalar field and $b_i = \frac{1}{2} b_{\mu\nu i} d\sigma^\mu \wedge d\sigma^\nu$. This action is gauge invaraint under the following gauge transformation:

$$\begin{aligned}
\delta\phi^i &= -\rho^i{}_a(\phi)\epsilon^{(1)a}, \\
\delta A_\mu{}^a &= \partial_\mu \epsilon^{(2)a} + k^{ab}\rho^i{}_b(\phi)\epsilon^{(3)}_{\mu i} + k^{ab} H_{bcd}(\phi) A_\mu{}^c \epsilon^{(1)d}, \\
\delta b_{\mu\nu,i} &= \partial_\mu \epsilon^{(3)}_{\nu i} - \partial_\nu \epsilon^{(3)}_{\mu i} + \frac{\partial \rho^j{}_a}{\partial \phi^i}(\phi)(b_{\mu\nu,j}\epsilon^{(2)a} - A_\mu{}^a \epsilon^{(3)}_{\nu j} + A_\nu{}^a \epsilon^{(3)}_{\mu j}) \\
&\quad + \frac{\partial H_{abv}}{\partial \phi^i}(\phi) A_\mu{}^a A_\nu{}^b \epsilon^{(1)c},
\end{aligned}$$

if and only if a target space is the Courant algebroid.

There exists a Wilson surface observable:

$$W(\Sigma) = \mathrm{Tr}\left(\mathcal{P}\exp\frac{i}{2}\int_{\Sigma} b_{\mu\nu} d\sigma^{\mu} \wedge d\sigma^{\nu}\right).$$

This shows that the corresponding groupoid is a 2-**groupoid** from the physical argument! Mehta, Tang '10, Li-Bland, Ševera '11, Sheng, Zhu '11

We find that the Courant algebroid is a Lie 2-algebroid.

Example 7. Current Algebras

In 2D, $S^1 \times \boldsymbol{R}$, $\quad \{x^I(\sigma), p_J(\sigma')\} = \delta^I{}_J \delta(\sigma - \sigma')$.
We consider currents

$$J_{0(f)}(\sigma) = f(x(\sigma)), \quad J_{1(u,\alpha)}(\sigma) = \alpha_I(x(\sigma))\partial_\sigma x^I(\sigma) + u^I(x(\sigma))p_I(\sigma),$$

$$\begin{aligned}
\{J_{0(f)}(\sigma), J_{0(f')}(\sigma')\}_{P.B.} &= 0, \\
\{J_{1(u,\alpha)}(\sigma), J_{0(f')}(\sigma')\}_{P.B.} &= -u^I \frac{\partial f'}{\partial x^I}(x(\sigma))\delta(\sigma - \sigma'), \\
\{J_{1(u,\alpha)}(\sigma), J_{1(u',\alpha')}(\sigma')\}_{P.B.} &= -J_{1([(u,\alpha),(u',\alpha')]_D)}(\sigma)\delta(\sigma - \sigma') \\
&\quad + \langle (u,\alpha), (u',\alpha') \rangle (\sigma') \partial_\sigma \delta(\sigma - \sigma'),
\end{aligned}$$

where $f(x(\sigma))$ is a function, $\alpha(x) = \alpha_I(x)dx^I$ is a 1-form and $u(x) = u^I(x)\partial_I$ is a vector field.

$[(u,\alpha),(u',\alpha')]_D$: **Dorfman bracket** on $TM \oplus T^*M$.
$\langle(u,\alpha),(u',\alpha')\rangle = i_{u'}\alpha + i_u\alpha'$: scalar product on $TM \oplus T^*M$.

This has the structure of the **Courant algebroid**.

Alekseev, Strobl '05,

• Higher dimensional generalized current algebras have structures of Lie n-algebroids. NI, Koizumi '13, NI, Xu '13

• Analysis of flux vacua, T-duality, p-branes from algebroid current algebras Halmagyi '09, Jurco, Schupp, Vysoky '14, et.al.

Example 8. String Field Theory

Gauge symmetry in the BV master equation Hata, Zwiebach '93

The quantum master equation:

$$2i\hbar\Delta W - \{W, W\} = 0.$$

has a gauge symmetry under $W' = e^{\delta_{\Xi}^{(q)}} W$. W' satisfies the quantum master equation,

$$2i\hbar\Delta W' - \{W', W'\} = 0,$$

$$W' = e^{\mathsf{ad}_{\Xi}^{(q)}} W = -i\hbar\Delta\Xi + \cdots + W + \{W, \Xi\} + \frac{1}{2}\{\{W, \Xi\}, \Xi\} + \cdots.$$

Example: String Theory

D-brane DBI actions Asakawa, Muraki, Sasa, Watamura '14

Example: Higher gauge theory, 6D

A higher gauge theory, a gerbe, Lie 2-algebra, semi-strict Lie algebras, crossed modules are (quasi-)Lie algebroids.

Example: Integrable systems, Solvable models

Quantum groups, Hopf algebras

Example: Supergravity

Symmetry and compactefied spaces on flux vacua

§8. Future Problems

Mathematics

(A compact connected semi-simple Lie group corresponds to a compact semi-simple Lie algebra.)

• There is no 'one to one' correspondence of Lie groupoids and Lie algebroids. The Lie's 3rd theorem does not hold for a Lie algebroid. Crainic, Fernandes '01

• **Classification theory** like A to G series by the Dynkin diagram is unknown.

Physics can propose proper frameworks and ideas for the above problems.

Physics

• **Quantization** of (quasi-)lie groupoid and nonperturbative analysis of gauge theories, string and M-theories.

Thank you!