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Gauged Twistor Models of Massless and Massive Particles

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Contents

1-1. Introduction

- **Twistor theory** was proposed by Penrose in 1967 and has been developed with the aim of finding a unified framework for

  space-time, **gravity**, **elementary particles**, **quantum physics**.

- Although twistor theory has provided many interesting ideas, it *cannot* be said that Penrose's ambitious plan of finding a unified framework has achieved success.

- From a mathematical viewpoint, twistor theory can be regarded as a skillful tool for solving

  anti-self-dual YM equation, Bogomolny equation, etc.

- Relatively recently, a new method for calculating gluon scattering amplitudes in QCD has been studied based on **twistor string theory** (E. Witten, CMP 252 (2004) 189; R. Boels *et al.*, PLB 648 (2007) 90).

- In 1983, Shirafuji presented a model of massless spinning particles formulated in terms of twistors.

- Since then, *various* generalizations of this model have been proposed until recently. For example,

  - $D = 6$ massless particle model (I. Bengtsson and M. Cederwall, NPB (1988) 81),

  - Tensorial space-time model (I. Bandos and J. Lukierski, MPLA (1999) 1257),

  - Massive particle models (S. Fedoruk *et al*, IJMPA 21 (2006) 4137; S. Fedoruk and J. Lukierski, PLB 733 (2014) 309).

★ In this talk, we shall consider gauging of the Shirafuji model and its generalization to massive particles.

1-2. A definition of twistor

- The bispinor notation $p_{\alpha\dot{\alpha}}$ and the 4-vector notation p_μ are related by

$$\begin{pmatrix} p_{0\dot{0}} & p_{0\dot{1}} \\ p_{1\dot{0}} & p_{1\dot{1}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix}.$$

- A twistor Z^A is defined as a pair of two *bosonic* Weyl spinors:

$$Z^A \equiv (\omega^\alpha, \pi_{\dot{\alpha}}), \quad A = 0, 1, 2, 3, \quad \alpha = 0, 1, \quad \dot{\alpha} = \dot{0}, \dot{1}.$$

$\pi_{\dot{\alpha}}$: Weyl spinor specifying a light-like vector,

$$p_{\alpha\dot{\alpha}} = \bar{\pi}_\alpha \pi_{\dot{\alpha}} \iff p_{\alpha\dot{\alpha}} p^{\alpha\dot{\alpha}} = 0 \qquad (\pi_{\dot{\alpha}} \pi^{\dot{\alpha}} = 0).$$

ω^α : Weyl spinor defined by $\omega^\alpha = i z^{\alpha\dot{\alpha}} \pi_{\dot{\alpha}}$.

Here, $z^{\alpha\dot{\alpha}}$ are coordinates of a point in the *complexified* Minkowski space:

$$z^{\alpha\dot{\alpha}} = x^{\alpha\dot{\alpha}} + iy^{\alpha\dot{\alpha}},$$

where $x^{\alpha\dot{\alpha}}$ denote a point in the *ordinary* Minkowski space.

2-1. Twistor formulation of massless particles

- In 1983, Shirafuji gave an action for a massless spinning particle in Minkowski space (T. Shirafuji, PTP 70 (1983) 18):

$$S_0 = \int d\tau \left[i\bar{Z}_A \frac{d}{d\tau} Z^A \right],$$

where $Z^A = Z^A(\tau)$ is a twistor depending on the world line parameter τ, and $\bar{Z}_A = \bar{Z}_A(\tau)$ its dual twistor.

★ S_0 remains invariant under the global $\mathrm{U}(1)$ transformation

$$Z^A \to e^{i\theta} Z^A, \quad \bar{Z}_A \to e^{-i\theta} \bar{Z}_A \quad (\,\theta : \text{constant real parameter}\,).$$

★ S_0 can be expressed as

$$S_0 = \int d\tau \Bigg[\underbrace{-\bar{\pi}_\alpha \pi_{\dot{\alpha}} \frac{dx^{\alpha\dot{\alpha}}}{d\tau}}_{\text{orbital part}} \underbrace{-iy^{\alpha\dot{\alpha}} \left(\bar{\pi}_\alpha \frac{d\pi_{\dot{\alpha}}}{d\tau} - \pi_{\dot{\alpha}} \frac{d\bar{\pi}_\alpha}{d\tau} \right)}_{\text{spin part}} \Bigg].$$

- To see that S_0 actually describes a massless spinning particle, it is convenient to consider the Pauli-Lubanski spin (pseudo)vector

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma} \implies W^\mu = sP^\mu \text{ (massless particles)}$$

P_ν : 4-momentum vector, $M_{\rho\sigma}$: angular-momentum tensor, s : helicity.

In the Shirafuji model, s is determined to be

$$s = \frac{1}{2}\bar{Z}_A Z^A = -y^{\alpha\dot{\alpha}}\bar{\pi}_\alpha \pi_{\dot{\alpha}}\,.$$

- The canonical momentum conjugate to Z^A is

$$\frac{\delta S_0}{\delta \dot{Z}^A} = i\bar{Z}_A \qquad \left(\cdot = \frac{d}{d\tau}\right).$$

Hence, $\bar{Z}_A$ is treated as a momentum variable, while Z^A is a coordinate variable.

- The canonical quantization is carried out by replacing Z^A and $\bar{Z}_A$ with the corresponding operators $\hat{Z}^A$ and $\hat{\bar{Z}}_A$. The commutation relations are found to be

$$\big[\hat{Z}^A, \hat{\bar{Z}}_B\big] = \delta^A_B\,, \quad \big[\hat{Z}^A, \hat{Z}^B\big] = \big[\hat{\bar{Z}}_A, \hat{\bar{Z}}_B\big] = 0\,.$$

Then, $\hat{Z}^A$ and $\hat{\bar{Z}}_A$ can be represented as $\hat{Z}^A = Z^A,\ \hat{\bar{Z}}_A = -\dfrac{\partial}{\partial Z^A}$.

(R. Penrose, IJTP 1 (1968) 61; S.D. and J. Note, JMP 54 (2013) 072304.)

- The Weyl-ordered helicity operator is given by $\hat{s} = \dfrac{1}{4}\Big(\hat{\bar{Z}}_A \hat{Z}^A + \hat{Z}^A \hat{\bar{Z}}_A\Big)$.

The helicity eigenvalue equation $\hat{s} f = s f$ becomes

$$-\frac{1}{2}\Big(Z^A \frac{\partial}{\partial Z^A} + 2\Big) f(Z) = s f(Z)\,, \quad f(Z) : \text{twistor wave function.}$$

This is valid for any homogeneous function f of degree $-2s-2$. This degree must be an *integer* so that f can be single-valued; hence, $s = \dfrac{n}{2}\ (n \in \mathbb{Z})$.

- We can construct massless spinor fields of higher-rank by the Penrose transform of $f(Z)$.

1. Positive helicity fields: $s = \dfrac{n}{2}$ $(n \in \mathbb{N})$ [Penrose 1969]

$$\Psi_{\dot{\alpha}_1 \dot{\alpha}_2 \cdots \dot{\alpha}_n}(z) = \frac{1}{2\pi i} \oint_{\Gamma_z} \pi_{\dot{\alpha}_1} \pi_{\dot{\alpha}_2} \cdots \pi_{\dot{\alpha}_n} f(Z) \pi_{\dot{\beta}} d\pi^{\dot{\beta}} .$$

   Here, Γ_z denotes a contour on a Riemannian surface $\mathbb{C}\mathbf{P}^1$ parametrized by a pair of $\pi_{\dot{1}}/\pi_{\dot{0}}$ and $\pi_{\dot{0}}/\pi_{\dot{1}}$.

2. Negative helicity fields: $s = -\dfrac{n}{2}$ $(n \in \mathbb{N})$ [Hughston 1973]

$$\Psi_{\alpha_1 \alpha_2 \cdots \alpha_n}(z) = \frac{1}{2\pi i} \oint_{\Gamma_z} \frac{\partial}{\partial \omega^{\alpha_1}} \frac{\partial}{\partial \omega^{\alpha_2}} \cdots \frac{\partial}{\partial \omega^{\alpha_n}} f(Z) \pi_{\dot{\beta}} d\pi^{\dot{\beta}} .$$

We can show that the spinor fields satisfy the generalized Weyl equations

$$\partial^{\beta\dot{\beta}} \Psi_{\dot{\beta}\dot{\alpha}_2 \cdots \dot{\alpha}_n} = 0 \, , \qquad \partial^{\beta\dot{\beta}} \Psi_{\beta \alpha_2 \cdots \alpha_n} = 0 \, .$$

2-2. A gauged twistor model of massless particles

- Now we carry out *gauging* of the global $\mathrm{U}(1)$ transformation of twistors, introducing a $\mathrm{U}(1)$ gauge field $a = a(\tau)$ on the world line. (I. Bars and M. Picón, PRD 73 (2006) 064002; S.D., T. Egami and J. Note, PTP 124 (2010) 969.) The local $\mathrm{U}(1)$ transformation is

$$Z^A \to e^{i\theta(\tau)} Z^A , \quad \bar{Z}_A \to e^{-i\theta(\tau)} \bar{Z}_A , \quad a \to a + \frac{d\theta(\tau)}{dt} .$$

★ Replace $\dfrac{d}{d\tau}$ in S_0 by the covariant derivative $D = \dfrac{d}{d\tau} - ia$.

★ Add the 1-dim. CS term $S_{\mathrm{CS}} = -2s \int d\tau a$ to the gauged action.

In this way, we have

$$\begin{aligned}\tilde{S}_0 &= \int d\tau \big[i\bar{Z}_A D Z^A - 2sa \big] \\ &= \int d\tau \Big[i\bar{Z}_A \frac{d}{d\tau} Z^A + a \big(\bar{Z}_A Z^A - 2s \big) \Big] .\end{aligned}$$

★ The EL equation $\dfrac{\delta\tilde{S}_0}{\delta a}=0$ leads to the helicity condition $\dfrac{1}{2}\bar{Z}_A Z^A = s$.

This fact implies that $\tilde{S}_0$ governs a massless spinning particle with *a fixed value* s *of helicity*.

★ The CS coefficient s is *quantized* in connection with the quantization of twistor.

- Remarkably, $\tilde{S}_0$ is equivalent to the action of the model so-called *massless particle with rigidity* (S.D. and T. Suzuki, PLB 731 (2014) 337):

$$S_{\mathrm{rp}} = -|s| \int d\tau \sqrt{-\dot{x}}\, \underbrace{\sqrt{\frac{\dot{x}^2\ddot{x}^2-(\dot{x}\ddot{x})^2}{(-\dot{x}^2)^3}}}_{\text{extrinsic curvature of the particle world line}}$$

where $\qquad \dot{x}^2 := \dot{x}_\mu \dot{x}^\mu, \quad \ddot{x}^2 := \ddot{x}_\mu \ddot{x}^\mu, \quad \dot{x}\ddot{x} := \dot{x}_\mu \ddot{x}^\mu.$

This model describes a classical analog of the *Zitterbewegung* of a massless spinning particle (M.S. Plyushchay, PLB 243 (1990) 383).

3-1. Twistor formulation of massive particles

- To describe a massive particle in terms of twistors, we need to introduce more than two twistors [Penrose 1975, Perjés 1975, Hughston 1979] :

$$Z_i^A \equiv \left(\omega_i^\alpha, \pi_{i\dot{\alpha}}\right), \quad i = 1, 2, \ldots N, \quad A = 0, 1, 2, 3,$$

  with the condition

$$\omega_i^\alpha = i z^{\alpha\dot{\alpha}} \pi_{i\dot{\alpha}} \qquad \left(z^{\alpha\dot{\alpha}} = x^{\alpha\dot{\alpha}} + i y^{\alpha\dot{\alpha}}\right)$$

  Hereafter, we consider the case $N = 2$. (The cases of $N \geq 3$ turn out to be *trivial* in the specific model that we consider.)

- The 4-momentum of a massive particle is expressed as

$$p_{\alpha\dot{\alpha}} := \bar{\pi}^1_\alpha \pi_{1\dot{\alpha}} + \bar{\pi}^2_\alpha \pi_{2\dot{\alpha}} = \bar{\pi}^i_\alpha \pi_{i\dot{\alpha}}, \quad i = 1, 2.$$

  Then,

$$p_{\alpha\dot{\alpha}} p^{\alpha\dot{\alpha}} = \underbrace{\left|\pi_{1\dot{\alpha}} \pi_1^{\dot{\alpha}}\right|^2}_{=0} + \underbrace{\left|\pi_{2\dot{\alpha}} \pi_2^{\dot{\alpha}}\right|^2}_{=0} + 2\left|\pi_{1\dot{\alpha}} \pi_2^{\dot{\alpha}}\right|^2 = 2\left|\pi_{1\dot{\alpha}} \pi_2^{\dot{\alpha}}\right|^2 \neq 0,$$

$$(\text{ iff } \pi_{1\dot{\alpha}} \neq c\pi_{2\dot{\alpha}}, \quad c \in \mathbb{C}).$$

- The mass-shell condition $p_{\alpha\dot{\alpha}}p^{\alpha\dot{\alpha}} = m^2$ can be written as

$$2\left|\pi_{1\dot{\alpha}}\pi_2^{\dot{\alpha}}\right|^2 - m^2 = 0\,,$$

which is equivalent to

$$\epsilon^{ij}\pi_{i\dot{\alpha}}\pi_j^{\dot{\alpha}} - \sqrt{2}me^{i\varphi} = 0\,, \quad \epsilon_{ij}\bar{\pi}^i_\alpha\bar{\pi}^{j\alpha} - \sqrt{2}me^{-i\varphi} = 0$$

(S. Fedoruk and J. Lukierski, PLB 733 (2014) 309)

Here, φ is a real parameter.

- We adopt FL's complex mass-shell condition. Then the Shirafuji action S_0 can *naively* be generalized for massive particles:

$$S_m = \int d\tau\left[i\bar{Z}^i_A\frac{d}{d\tau}Z^A_i + h\left(\epsilon^{ij}\pi_{i\dot{\alpha}}\pi_j^{\dot{\alpha}} - \sqrt{2}me^{i\varphi}\right) + \bar{h}\left(\epsilon_{ij}\bar{\pi}^i_\alpha\bar{\pi}^{j\alpha} - \sqrt{2}me^{-i\varphi}\right)\right],$$

where h is a complex Lagrange multiplier.

The action S_m remains invariant under

1. Global $\mathrm{U}(1)$ transformation:

$$Z_i^A \longrightarrow e^{i\theta} Z_i^A\,, \quad \bar{Z}_A^i \longrightarrow e^{-i\theta} \bar{Z}_A^i, \quad h \longrightarrow e^{-2i\theta} h\,, \quad \bar{h} \longrightarrow e^{2i\theta} \bar{h}\,,$$

$$\varphi \longrightarrow \varphi + 2\theta\,.$$

(θ : constant real parameter)

2. Global $\mathrm{SU}(2)$ transformation:

$$Z_i^A \longrightarrow U_i{}^j Z_j^A, \quad \bar{Z}_A^i \longrightarrow \bar{Z}_A^j U^\dagger{}_j{}^i, \quad \pi_{i\dot{\alpha}} \longrightarrow U_i{}^j \pi_{j\dot{\alpha}}, \quad \bar{\pi}_\alpha^i \longrightarrow \bar{\pi}_\alpha^j U^\dagger{}_j{}^i,$$

h, $\bar{h}$, and φ do not change.

($U \in \mathrm{SU}(2)$: constant matrix)

- Unlike the earlier twistor models of massive particles, we *systematically* derive appropriate constraints by *gauging* the global $\mathrm{U}(1)$ and $\mathrm{SU}(2)$ transformations, as will be seen later.

3-2. Gauged twistor models of massive particles

- Now we carry out gauging of the global $\mathrm{U}(1)$ and $\mathrm{SU}(2)$ transformations:

$$\theta \Rightarrow \theta(\tau)\,, \qquad U \Rightarrow U(\tau)\,.$$

★ We introduce a $\mathrm{U}(1)$ gauge field $a = a(\tau)$ and a $\mathrm{SU}(2)$ gauge field $b = b(\tau) = \sum_{r=1}^{3} b_r(\tau)\sigma_r$ [σ_r: Pauli matrices], which obey the gauge transformation rules

$$a \longrightarrow a + \frac{d\theta}{dt}\,, \qquad b \longrightarrow UbU^{\dagger} - i\,\frac{dU}{dt}U^{\dagger}\,.$$

★ Replace $\dfrac{d}{d\tau}$ in S_m by $D_i{}^j = \delta_i{}^j \dfrac{d}{d\tau} - i\delta_i{}^j a - ib_r\sigma_{ri}{}^j$.

★ Add the 1-dim. CS term for a, $S_{\mathrm{CS}} = -2s\int d\tau a$, to the gauged action.

Note that the 1-dim. CS term for b vanishes: $S'_{\mathrm{CS}} = -2t\int d\tau \underbrace{\mathrm{Tr}\, b}_{=0} = 0\,.$

In this way, we have

$$
\begin{aligned}
\tilde{S}_m &= \int d\tau \Big[i\bar{Z}^i_A D_i{}^j Z^A_j - 2sa \\
&\qquad + h\big(\epsilon^{ij}\pi_{i\dot{\alpha}}\pi^{\dot{\alpha}}_j - \sqrt{2}me^{i\varphi}\big) + \bar{h}\big(\epsilon_{ij}\bar{\pi}^i_\alpha\bar{\pi}^{j\alpha} - \sqrt{2}me^{-i\varphi}\big)\Big], \\
&= \int d\tau \Big[i\bar{Z}^i_A \frac{d}{d\tau} Z^A_i + a\big(\bar{Z}^i_A Z^A_i - 2s\big) + b_r \bar{Z}^i_A \sigma_{ri}{}^j Z^A_j \\
&\qquad + h\big(\epsilon^{ij}\pi_{i\dot{\alpha}}\pi^{\dot{\alpha}}_j - \sqrt{2}me^{i\varphi}\big) + \bar{h}\big(\epsilon_{ij}\bar{\pi}^i_\alpha\bar{\pi}^{j\alpha} - \sqrt{2}me^{-i\varphi}\big)\Big].
\end{aligned}
$$

★ The EL equation $\dfrac{\delta \tilde{S}_m}{\delta a} = 0$ gives $\dfrac{1}{2}\bar{Z}^i_A Z^A_i = s$, while $\dfrac{\delta \tilde{S}_m}{\delta b_r} = 0$ gives

$$
\bar{Z}^i_A \sigma_{ri}{}^j Z^A_j = 0 \qquad (\, r = 1, 2, 3\,)\,.
$$

It eventually turns out that this set of constraints is too strong and allows only *spinless* fields. Therefore we need to modify the model to involve *spinfull* fields.

- To construct a modified model, we consider the non-linear realization of $\mathrm{SU}(2)$ by introducing the coset space $\mathrm{SU}(2)/\mathrm{U}(1)$. Coset representative elements $V(\xi,\bar{\xi})$ ($V \in \mathrm{SU}(2)$, $\xi(\tau) \in \mathbb{C}$) are chosen from each left coset of $\mathrm{U}(1)[\subset \mathrm{SU}(2)]$. The V obeys the transformation rule

$$V(\xi,\bar{\xi}) \to V(\xi',\bar{\xi}') = U(\tau)V(\xi,\bar{\xi})e^{-i\vartheta(\tau)\sigma_3},$$

where $\vartheta(\tau)$ is a real parameter of the $\mathrm{U}(1)$ transformation generated by σ_3, which is hereafter denoted by $\widetilde{\mathrm{U}}(1)$.

- Now we define the modified action

$$S = \int d\tau \Big[i\bar{Z}^i_A D_i{}^j Z^A_j - 2sa - 2t\Big(b_r \mathcal{V}_r{}^3 - \dot{\xi} e_\xi{}^3 - \dot{\bar{\xi}} e_{\bar{\xi}}{}^3\Big) - k\sqrt{2g_{\xi\bar{\xi}} D\xi D\bar{\xi}}$$
$$+ h\Big(\epsilon^{ij}\pi_{i\dot{\alpha}}\pi^{\dot{\alpha}}_j - \sqrt{2}me^{i\varphi}\Big) + \bar{h}\Big(\epsilon_{ij}\bar{\pi}^i_\alpha \bar{\pi}^{j\alpha} - \sqrt{2}me^{-i\varphi}\Big)\Big],$$

where $\mathcal{V}_r{}^3$, $e_\xi{}^3$, $e_{\bar{\xi}}{}^3$, and $g_{\xi\bar{\xi}}$ are constructed from V, while s, t, and $k(>0)$ are constants.

The action S remains invariant under

1. the local $\mathrm{U}(1)$ transformation,
2. the local $\mathrm{SU}(2)$ transformation,
3. the reparametrization $\tau \to \tau'$.

- In the particular gauge $\xi(\tau) = \xi_0$ such that $V(\xi_0, \bar{\xi}_0) = 1$, the action S takes a simpler form

$$S = \int d\tau \Big[i\bar{Z}^i_A \frac{d}{d\tau} Z^A_i + a\big(\bar{Z}^i_A Z^A_i - 2s\big) + b_3\big(\bar{Z}^j_A {\sigma_{3j}}^k Z^A_k - 2t\big) + b_i \bar{Z}^j_A {\sigma_{ij}}^k Z^A_k - k\sqrt{b_i b_i} + h\big(\epsilon^{ij}\pi_{i\dot{\alpha}}\pi^{\dot{\alpha}}_j - \sqrt{2}me^{i\varphi}\big) + \bar{h}\big(\epsilon_{ij}\bar{\pi}^i_\alpha \bar{\pi}^{j\alpha} - \sqrt{2}me^{-i\varphi}\big)\Big].$$

In this form, the local $\mathrm{SU}(2)$ symmetry *hides*, while the local $\widetilde{\mathrm{U}}(1)$ symmetry is realized in addition to the local $\mathrm{U}(1)$ symmetry.

- The EL equations for a, b_3, b_i, h, $\bar{h}$, and φ are

$$\begin{cases} T_0 - s = 0\,, \quad T_3 - t = 0\,, \quad T_i - \dfrac{kb_i}{2\sqrt{b_j b_j}} = 0 \quad (i = 1, 2)\,, \\ \epsilon^{ij}\pi_{i\dot{\alpha}}\pi_j^{\dot{\alpha}} - \sqrt{2}me^{i\varphi} = 0\,, \quad \epsilon_{ij}\bar{\pi}^i_{\alpha}\bar{\pi}^{j\alpha} - \sqrt{2}me^{-i\varphi} = 0\,, \quad he^{i\varphi} - \bar{h}e^{-i\varphi} = 0\,, \end{cases}$$

where

$$T_0 := \frac{1}{2}\bar{Z}^i_A Z^A_i\,, \qquad T_r := \frac{1}{2}\bar{Z}^j_A \sigma_{rj}{}^k Z^A_k \quad (r = 1, 2, 3).$$

These equations are also derived as the secondary constraints in the canonical formulation based on the action S. The third constraint gives

$$T_i T_i - \frac{k^2}{4} = 0\,.$$

The Dirac brackets of the twistors are found to be

$$\left\{Z^A_i, \bar{Z}^j_B\right\}_{\mathrm{D}} = -i\delta^j_i\delta^A_B\,, \qquad \left\{Z^A_i, Z^B_j\right\}_{\mathrm{D}} = \left\{\bar{Z}^i_A, \bar{Z}^j_B\right\}_{\mathrm{D}} = 0\,.$$

3-3. Canonical quantization of the model

- Quantization of the model is performed with the commutation relations

$$\left[\hat{Z}_i^A, \hat{\bar{Z}}_B^j\right] = \delta_i^j \delta_B^A\,, \qquad \left[\hat{Z}_i^A, \hat{Z}_j^B\right] = \left[\hat{\bar{Z}}_A^i, \hat{\bar{Z}}_B^j\right] = 0\,.$$

The operators $\hat{T}_0 := \frac{1}{4}\left(\hat{\bar{Z}}_A^i \hat{Z}_i^A + \hat{Z}_i^A \hat{\bar{Z}}_A^i\right)$ and $\hat{T}_r := \frac{1}{2}\hat{\bar{Z}}_A^j \sigma_{rj}{}^k \hat{Z}_k^A$ satisfy

$$\left[\hat{T}_0, \hat{T}_r\right] = 0\,, \qquad \left[\hat{T}_p, \hat{T}_q\right] = i\epsilon_{pqr}\hat{T}_r\,.$$

- We now take the representation $\hat{Z}_i^A = Z_i^A$, $\hat{\bar{Z}}_A^i = -\dfrac{\partial}{\partial Z_i^A}$. The above mentioned (first-class) constraints are read as the eigenvalue equations

$$\begin{cases} \left(\hat{T}_0 - s\right) f(Z) = 0\,, \quad \left(\hat{T}_3 - t\right) f(Z) = 0\,, \quad \left(\hat{T}_i \hat{T}_i - \dfrac{k^2}{4}\right) f(Z) = 0\,, \\ \left(\epsilon^{ij}\pi_{i\dot{\alpha}}\pi_j^{\dot{\alpha}} - \sqrt{2}me^{i\varphi}\right) f(Z) = 0\,, \quad \left(\epsilon_{ij}\dfrac{\partial^2}{\partial\omega_i^\alpha \partial\omega_{j\alpha}} - \sqrt{2}me^{-i\varphi}\right) f(Z) = 0\,, \end{cases}$$

(The constraint $he^{i\varphi} - \bar{h}e^{-i\varphi} = 0$ is classified as a second-class constraint.)

- The single-valuedness of $f(Z)$ restricts the possible values of s and t to

$$s = \frac{n_s}{2}\,, \quad t = \frac{n_t}{2} \qquad (n_s, n_t \in \mathbb{Z}).$$

By using the eigenvalue equation for the Casimir operator of $\mathrm{SU}(2)$

$$\hat{T}_r \hat{T}_r f(Z) = j(j+1) f(Z) \qquad \left(j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \right),$$

the possible values of k are determined to be

$$k = 2\sqrt{j(j+1) - t^2} \qquad (\, t = -j,\ -j+1, \ldots, j-1,\ j\,).$$

In this way, k as well as the CS coefficients s and t is *quantized*. The twistor function $f(Z)$ is characterized by the quantum numbers (s, j, t), so that it can be denoted as $f_{s,j,t}(Z)$.

- We can construct *massive* spinor fields of higher-rank by the (generalized) Penrose transform of the homogeneous function $f(Z)$ of degree $-2s-4$:

$$\Psi^{i_1\ldots i_m}_{\alpha_1\ldots\alpha_m;\,j_1\ldots j_n,\,\dot\alpha_1\ldots\dot\alpha_n} = \frac{1}{(2\pi i)^2}\oint_{\Sigma_z}\pi_{j_1\dot\alpha_1}\cdots\pi_{j_n\dot\alpha_n}\frac{\partial}{\partial\omega^{\alpha_1}_{i_1}}\cdots\frac{\partial}{\partial\omega^{\alpha_m}_{i_m}}f(Z)$$
$$\times\,\pi_{1\dot\beta}d\pi^{\dot\beta}_1\wedge\pi_{2\dot\gamma}d\pi^{\dot\gamma}_2\,.$$

Here, $s=(n-m)/2$ $(m,n\in\mathbb{N}_0)$, and Σ_z denotes a 2-dimensional contour. Using the mass-shell conditions at the quantum level, we can show that this spinor field satisfies the generalized Dirac equations

$$i\partial^{\beta\dot\beta}\Psi^{i_1\ldots i_m}_{\alpha_1\ldots\alpha_m;\,j_1\ldots j_n,\,\dot\beta\dot\alpha_2\ldots\dot\alpha_n}+\frac{m}{\sqrt2}e^{i\varphi}\epsilon^{\beta\gamma}\epsilon_{j_1k}\Psi^{ki_1\ldots i_m}_{\gamma\alpha_1\ldots\alpha_m;\,j_2\ldots j_n,\,\dot\alpha_2\ldots\dot\alpha_n}=0\,.$$

$$i\partial^{\beta\dot\beta}\Psi^{i_1\ldots i_m}_{\beta\alpha_2\ldots\alpha_m;\,j_1\ldots j_n,\,\dot\alpha_1\ldots\dot\alpha_n}+\frac{m}{\sqrt2}e^{-i\varphi}\epsilon^{\dot\beta\dot\gamma}\epsilon^{i_1k}\Psi^{i_2\ldots i_m}_{\alpha_2\ldots\alpha_m;\,kj_1\ldots j_n,\,\dot\gamma\dot\alpha_1\ldots\dot\alpha_n}=0\,.$$

- In the simplest cases $(m,n) = (0,1)$ and $(n,m) = (1,0)$. the above equations become

$$\sqrt{2} i \partial_{\alpha\dot{\alpha}} \epsilon^{ij} \Psi_j^{\dot{\alpha}} - m \Psi_\alpha^i = 0\,, \qquad \sqrt{2} i \partial^{\alpha\dot{\alpha}} \Psi_\alpha^i - m \epsilon^{ij} \Psi_j^{\dot{\alpha}} = 0\,.$$

These equations can be arranged in the form of the ordinary Dirac equation:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0\,, \qquad (i\gamma^\mu \partial_\mu - m)\psi^{\mathrm{c}} = 0\,,$$

where

$$\psi = \underbrace{\begin{pmatrix} \Psi_\alpha^1 \\ \Psi_2^{\dot{\alpha}} \end{pmatrix}}_{t = -\frac{1}{2}} \begin{matrix} s = -\frac{1}{2} \\ s = \frac{1}{2} \end{matrix}\,, \qquad \psi^{\mathrm{c}} = \underbrace{\begin{pmatrix} \Psi_\alpha^2 \\ \Psi_1^{\dot{\alpha}} \end{pmatrix}}_{t = \frac{1}{2}} \begin{matrix} s = -\frac{1}{2} \\ s = \frac{1}{2} \end{matrix}\,.$$

We see that s is a quantum number for chirality, while t is a quantum number for characterizing particles and anti-particles.

4-1. Summary

- We have found the gauged Shirafuji model of a massless spinning particle by considering the local $\mathrm{U}(1)$ transformation of twistors.
  - The gauged model describes a massless spinning particle with a *fixed* value of helicity.
  - This model is equivalent to the model of a massless particle with rigidity.

- We have constructed a gauged twistor model of massive spinning particles by considering the local $\mathrm{U}(1)$ and $\mathrm{SU}(2)$ transformations of twistors.
  - A non-linear realization of $\mathrm{SU}(2)$ has been incorporated in the model so as to be able to treat massive *spinfull* fields.
  - Appropriate constraints have been derived in a systematic manner by virtue of gauging the $\mathrm{U}(1)$ and $\mathrm{SU}(2)$ symmetries.
  - Higher-rank massive spinor fields (with $\mathrm{SU}(2)$ indices) have been obtained by the (generalized) Penrose transform.

4-2. Future issues

- Consider coupling to external electromagnetic and gravitational fields.
- Compare to the Barut-Zanghi model, another massive spinning particle model involving coupling of an external electromagnetic field.