Electronic Notes in Theoretical Computer Science 66 No. 1 (2004)
URL: http://www.elsevier.nl/locate/entcs/volume66.html 12 pages

Sequential computability of a function -diagonal space and limiting recursion-

Yoshiki Tsujii[2] Mariko Yasugi[1,3] Takakazu Mori[4]

Faculty of Science
Kyoto Sangyo University
Kyoto, Japan

Abstract

We consider the real sequences in $I = [0, 1)$ and real functions on I. A computability notion with respect to the uniformity $\{U_n\}$, where $U_n(x) = [\frac{k}{2^n}, \frac{k+1}{2^n})$ if $x \in [\frac{k}{2^n}, \frac{k+1}{2^n})$, will be called $\mathcal{D}$-computability. An $\mathbf{R}$-computable sequence from I will be shown to be approximated by a recursive sequence of rational numbers with a limiting recursive modulus of convergence with respect to $\{U_n\}$. Using this result, we relate two extended notions of sequential computability of a function or a function sequence, one formulated in terms of limiting recursion and one in terms of $\{U_n\}$.

Key words: Effective diagonal uniformity, Two notions of sequential computability of a function and of a function sequence, Effective uniform continuity of a function

1 Introduction

The standard notion of computability of a real number or of a sequence of real numbers as well as that of computability of a continuous or of a sequence of continuous functions is generally agreed. As for a continuous real function f defined on a compact interval, for example, f is called computable if the two conditions below are satisfied ([6]).

(i) (Sequential computability) Given a computable sequence $\{x_m\}$, $\{f(x_m)\}$ is a computable sequence of real numbers.

[1] This work has been supported in part by Kayamori Foundations of Informational Sciences Advancement K15VIII No.157 and Science Foundations of JSPS No.16340028.
[2] Email: tsujiiy@cc.kyoto-su.ac.jp
[3] Email: yasugi@cc.kyoto-su.ac.jp
[4] Email: morita@cc.kyoto-su.ac.jp

(ii) (Effective uniform continuity) There is a recursive function $\alpha : \mathbf{N} \to \mathbf{N}$ with which holds that $|x-y| < \frac{1}{2^{\alpha(p)}}$ implies $|f(x) - f(y)| < \frac{1}{2^p}$.

However, we also compute values and draw graphs of some discontinuous functions. So, we would like to attribute a certain kind of computability to some discontinuous functions. In that case, the conditions (i) and (ii) above have to be modified.

There are many theories of computation of discontinuous functions. We too have proposed some approaches to this problem. Among them, one is to express the value of a function at a jump point in terms of a "limiting recursive" modulus of convergence instead of a recursive one (Yasugi, Brattka, Washihara:[9]). Another is to change the topology of the domain of a function (Tsujii, Mori, Yasugi:[8]). In some cases, these two approaches are equivalent ([10]). As for a sequence of functions with varying jump points, we took up, as an example, the system of Rademacher functions $\{\phi_l\}$ (Yasugi, Washihara:[13]). In [13], it was claimed that $\{\phi_l\}$ admits a "weak computation" in the following sense: input a recursive information of a computable sequence of real numbers $\{x_m\}$, a recursive sequence of rational numbers $\{s_{lmn}\}$ converging to $\{\phi_l(x_m)\}$ with a "limiting recursive" modulus of convergence can be produced. Such a property of a function (function sequence) will be called $\mathcal{L}^*$-sequentially computable.

In [12] we presented an alternative way of expressing a notion of computability of the Rademacher function system by changing the topology of the real interval $I = [0, 1)$, first by decomposing it into $\{[\frac{k}{2^\nu}, \frac{k+1}{2^\nu})\}_k$ for each ν ($k \leq 2^\nu - 1$), and then taking a kind of the limit with respect to ν. In this way, we obtain an "effective uniform space" as the limit of an "effective sequence of uniform spaces." The computability of a sequence from I with respect to this limit space is called "$\mathcal{D}$-computatbility" ($\mathcal{D}$ representing "diagonal"). It is based on our theory of the effective uniform space (cf. [8], [11], [10]). In [12], it is shown that, input a $\mathcal{D}$-computable sequence of real numbers, $\{x_m\}$, the double sequence of values $\{\phi_l(x_m)\}$ is $\mathbf{R}$-computable (computable in the Euclidean topology). Such a property of a function (function sequence) will be called "$\mathcal{D}$-sequentially computable." We took up the Rademacher function system for the reason that it is the basis of Walsh analysis and fits our interest.

The two notions of sequential computability of a function (a function sequence) appear quite apart, but they can be related as in the case of [10].

The *aim* of this article is to *relate these two notions of sequentia computability*, one with respect to the diagonal space and one with respect to limiting recursive functions. The domain of discourse is restricted to I. It is first shown that an $\mathbf{R}$-computable sequence of real numbers is approximated by a recursive sequence of rational numbers with a limiting recursive modulus of convergence (Section 3: Theorem 3.3). We then show that, for an effectively $\mathcal{D}$-uniformly continuous (*educ*) function, $\mathcal{D}$-sequential computability and $\mathcal{L}^*$-sequential computability are equivalent (Section 4: Theorems 4.2 and 4.4). As an application, we show that the Rademacher function system is *educ* and is $\mathcal{L}^*$-sequentially computable (Section 5: Theorem 5.3).

There are other discontinuous functions (in the Euclidean topology) which are not *educ* but are effectively $\mathcal{D}$-continuous in some sense. This in mind, as a way of remark, we describe "locally uniformly $\mathcal{D}$-computable" functions and "$\mathcal{D}$-computable" functions, and give some examples for respective ones.

In Section 2, we list some definitions and notations from preceding references for the reader's convenience.

We only list those references which have direct applications to the present work. We have consulted [3] for the uniform space and [7] for the Rademacher functions. As for some notions of computable functions on I, we have also consulted [4] and [5]. Although our subject is closely related to the Fine metric space (cf. [1],[4],[5]), we will leave investigations into their effective relationship to a later work.

2 Preliminaries

For details of basic definitions below, see [6] and [7]. A sequence of rational numbers $\{r_n\}$ is called *recursive* if there is a recursive way to compute r_n for each n. A real number x is called *computable* with respect to the Euclidean topology (**R**-computable) if it is approximated by a recursive sequence of rational numbers $\{r_m\}$ with a recursive modulus of convergence α. We will express such a circumstance as $x \simeq \langle r_m, \alpha(p)\rangle$, or for short $x \simeq \langle r_m, \alpha\rangle$. These definitions can be extended to a *computable sequence* of real numbers.

We will henceforth confine the domain of discourse to $I = [0, 1)$.

Rademacher functions are step functions from $I = [0, 1)$ to $\{-1, 1\}$. The lth *Rademacher function* $\phi_l : I \to \mathbf{R}$ is defined as follows.

$$\phi_0(x) = 1;\quad \phi_l(x) = \begin{cases} 1, & x \in [\frac{2i}{2^l}, \frac{2i+1}{2^l}) \\ -1, & x \in [\frac{2i+1}{2^l}, \frac{2i+2}{2^l}) \end{cases}$$

$(l \geq 1, i \leq 2^l - 1)$. $\{\phi_l\}$ will be called the *Rademacher function system.*

Instead of elaborating the general theory of an effective sequence of uniform spaces and its limit space (the diagonal space) as developed in [8] and [12], we will review just what is needed here.

For $\nu = 0, 1, 2, \cdots$ and $0 \leq k \leq 2^\nu - 1$, define subintervals of I, $\{I_k^\nu\}$, as well as a sequence of maps $U_n^\nu : I \to P(I)$ as follows, where $P(I)$ represents the powerset of I. Put

$$I_k^\nu = [\frac{k}{2^\nu}, \frac{k+1}{2^\nu}),$$

and let k be, in particular, the unique k, $0 \leq k \leq 2^\nu - 1$, such that $x \in I_k^\nu$. Then define

$$U_n^\nu(x) = I_k^\nu \cap (x - \frac{1}{2^n}, x + \frac{1}{2^n}).$$

$\{U_n^\nu\}$ forms an "effective sequence of uniformities" on I (Definition 51 in [12]).

The sequence $\{U_n^n\}$ will be called the *diagonal sequence* of $\{U_n^\nu\}$, and will be denoted by $\{U_n\}$ (Definition 71 in [12]). The sequence $\{U_n\}$ forms an "effective uniformity" which is topologically effectively equivalent to the "effective limit" $\{U_k^\nu\}_{\nu k}$ (Proposition 71 in [12]). Notice that when $y \in U_n(x)$, $U_n(x) = U_n(y)$.

A sequence of real numbers $\{x_m\} \subset I$ is $\mathcal{D}$-*computable* ($\mathcal{D}$ denoting "diagonally") (Definition 73 in [12]) if there is a recursive sequence $\{q_{mp}\} \subset I$ of rational numbers which converges to $\{x_m\}$ effectively with respect to $\{U_n\}$ in a manner that, for a recursive γ and for $l \geq \gamma(m,p)$, $q_{ml} \in U_p(x_m)$. We will write this property as $x_m \simeq_{\mathcal{D}} \langle q_{ml}, \gamma \rangle$. The definition can be extended to multiple sequences.

The family of $\mathcal{D}$-computable sequences of real numbers forms a "computability structure" for $\langle I, \{U_n\} \rangle$ (Proposition 72 in [12]). In [12], it is shown that a $\mathcal{D}$-computable sequence of real numbers is **R**-computable but not conversely, while, for a single real number, two notions of computability coincide.

Let $r, s \geq 0$ be integers and let g and $\phi_1, \cdots, \phi_r$ be recursive functions, where $g : \mathbf{N}^{r+s+1} \to \mathbf{N}$ and $\phi_i : \mathbf{N} \to \mathbf{N}, i = 1, 2, \cdots r$. The partial function h defined as follows will be called *limiting recursive* (cf. Gold: [2]):

$$h(p_1, \cdots, p_s) = \lim_n g(\tilde{\phi}_1(n), \cdots, \tilde{\phi}_r(n), p_1, \cdots, p_s, n),$$

where $\tilde{\phi}(n)$ is a code for the finite sequence

$$\langle \phi(0, p_1, \cdots, p_s), \cdots, \phi(n, p_1, \cdots, p_s) \rangle .$$

Subsequently when we mention a sequence (of numbers or functions), it may be a multiple sequence. For example, a recursive sequence of rational numbers may mean a single sequence $\{r_j\}_j$ or a multiple sequence $\{r_{\mathbf{j}}\}_{\mathbf{j}}$ ($\mathbf{j} = j_1 j_2 \cdots j_n$), as the case may be.

3 R-computability and $\mathcal{D}$-computability

For a later use, we first prepare the following lemma.

Lemma 3.1 A recursive sequence of rational numbers is $\mathcal{D}$-computable.

Definition 3.2 (Weak $\mathcal{D}$-computability) Let $\{x_m\}$ be a sequence from I. If there are a recursive sequence of rational numbers $\{z_{mp}\}$ and a function $\nu : \mathbf{N}^2 \to \mathbf{N}$ which is (recursive or) limiting recursive, satisfying

$$z_{mp} \in U_n(x_m) \quad \text{for} \quad p \geq \nu(m,n),$$

then we say that $\{x_m\}$ is *weakly* $\mathcal{D}$-computable with respect to $\{z_{mp}\}$ and ν. [5]

The first theorem claims "weak $\mathcal{D}$-computability" of an **R**-computable sequence.

[5] The term "weak $\mathcal{D}$-computability" has been suggested by the referee.

Theorem 3.3 (From **R**-computability to weak $\mathcal{D}$-computability) Let $\{x_m\}$ be an **R**-computable sequence from I. Then, there are a recursive sequence of rational numbers $\{z_{mp}\}$ and a function $\nu : \mathbf{N}^2 \to \mathbf{N}$ which is (recursive or) limiting recursive such that $\{x_m\}$ is weakly $\mathcal{D}$-computable with respect to $\{z_{mp}\}$ and ν (cf. Definition 3.2).

Proof Suppose $x_m \simeq \langle r_{mq}, \alpha \rangle$ with recursive $\{r_{mq}\}$ and $\alpha(m,q)$, that is, for $l \geq \alpha(m,q)$, $|x_m - r_{ml}| < \frac{1}{2^q}$. Then define

$$z_{mq} = r_{m\alpha(m,q)} + \frac{1}{2^q}.$$

It is obvious that $\{z_{mq}\}$ is a recursive sequence of rational numbers. By virtue of Lemma 3.1, it is $\mathcal{D}$-computable. It also holds that $x_m < z_{mp} < x_m + \frac{2}{2^p}$. Put

$$k(m,q,n) = \max\{k; \frac{k}{2^n} \leq z_{mq}\}.$$

$k(m,q,n)$ is recursive. Define next $l(m,q,n)$, also recursive, as follows.

$$l(m,q,n) = \begin{cases} 0, & z_{mq} < \frac{k(m,q,n)}{2^n} + \frac{4}{2^q} \\ 1, & \frac{k(m,q,n)}{2^n} + \frac{4}{2^q} \leq z_{mq} \leq \frac{k(m,q,n)+1}{2^n} - \frac{2}{2^q} \\ 2, & \frac{k(m,q,n)+1}{2^n} - \frac{2}{2^q} < z_{mq}. \end{cases}$$

Notice that $l(m,q,n) = 1$ implies $q > n+2$.

We have at our disposal the following five facts.

Fact 1. If $l(m,q,n) = 1$, then $k(m,q,n) = k(m,p,n)$ for any $p \geq q$.

Fact 2. There exists a q_0 such that, for any $q \geq q_0$, $l(m,q,n) \neq 2$.

Fact 3. If $l(m,q,n) = 1$ for some q, then, for any $p \geq q$, $l(m,p,n) = 1$.

Fact 4. If $\lim_{q\to\infty} l(m,q,n) = 0$, then $x_m = L_n(x_m)$, where $L_n(x)$ represents the left end point of $U_n(x)$.

Fact 5. $z_{mq} \in [\frac{k(m,q,n)}{2^n}, \frac{k(m,q,n)+1}{2^n})$.

We delay the proofs of Facts 1$\sim$Fact 5, and complete the proof of the theorem using these Facts.

Define $l(m,n)$ and $\nu(m,n)$ successively as follows.

$$l(m,n) = \lim_{q\to\infty} l(m,q,n).$$

By Facts 2 and 3, $\lim_{q\to\infty} l(m,q,n)$ exists, and the value is either 1 or 0. So, by definition, $l(m,n)$ is limiting recursive.

$$\nu(m,n) = \begin{cases} \min\{q | l(m,q,n) = 1\}, & l(m,n) = 1 \\ n+1, & l(m,n) = 0. \end{cases}$$

By Facts 2 and 3 and the definition of $l(m,q,n)$, ν is recursive in $l(m,n)$, hence is limiting recursive. Let us remark that from the comment above, $\nu(m,n) \geq n+1$ in either case.

Note that by the definition of z_{mq}, $x_m < z_{mq} < x_m + \frac{2}{2^q}$, and so $z_{mq} \in U_n(x_m)$ if $q \geq \nu(m,n)$, since $\nu(m,n) \geq n+1$.

Suppose first $l(m,n) = 1$. By the definition of ν and Fact 3, it follows successively for any $p \geq \nu(m,n)$ $(= \max(\min\{q | l(m,q,n) = 1\}, n+1))$,

$$\frac{k(m,p,n)}{2^n} + \frac{4}{2^p} \leq z_{mp} \leq \frac{k(m,p,n)+1}{2^n} - \frac{2}{2^p},$$

$$\frac{k(m,p,n)}{2^n} + \frac{4}{2^p} - \frac{2}{2^p} < x_m < \frac{k(m,p,n)+1}{2^n} - \frac{2}{2^p} - \frac{1}{2^p} + \frac{1}{2^p},$$

$$\frac{k(m,p,n)}{2^n} + \frac{2}{2^p} < x_m < \frac{k(m,p,n)+1}{2^n}.$$

This tells us that

$$U_n(x_m) = [\frac{k(m,p,n)}{2^n}, \frac{k(m,p,n)+1}{2^n})$$

and hence by Fact 5 $z_{mp} \in U_n(x_m)$ for $p \geq \nu(m;n)$.

Suppose next $l(m,n) = 0$. Fact 4 shows that $x_m = L_n(x_m)$. So,

$$L_n(x_m) = x_m < z_{mp} < L_n(x_m) + \frac{2}{2^p} < L_n(x_m) + \frac{1}{2^n}$$

for $p \geq n+1$. It means that for $p \geq \nu(m,n) = n+1$,

$$z_{mp} \in U_n(x_m) = [L_n(x_m), L_n(x_m) + \frac{1}{2^n}).$$

This completes the proof of the theorem. Now the proofs of five facts above.
Fact 1: Suppose $l(m,q,n) = 1$. It is sufficient to show

$$\frac{k(m,q,n)}{2^n} \leq z_{mp} < \frac{k(m,q,n)+1}{2^n} \tag{1}$$

for $p \geq q$, since then $k(m,p,n) = k(m,q,n)$. $l(m,q,n) = 1$ implies

$$\frac{k(m,q,n)}{2^n} + \frac{4}{2^q} - \frac{1}{2^q} \leq r_{m\alpha(m,q)} \leq \frac{k(m,q,n)+1}{2^n} - \frac{2}{2^q} - \frac{1}{2^q}. \tag{2}$$

From (2) and

$$r_{m\alpha(m,q)} - \frac{1}{2^q} < x_m < r_{m\alpha(m,q)} + \frac{1}{2^q},$$

$$\frac{k(m,q,n)}{2^n} + \frac{2}{2^q} < x_m < \frac{k(m,q,n)+1}{2^n} - \frac{2}{2^q}. \tag{3}$$

Now suppose $p \geq q$. Then

$$z_{mp} - \frac{1}{2^p} = r_{m\alpha(m,p)} + \frac{1}{2^p} - \frac{1}{2^p} < x_m + \frac{1}{2^p} < r_{m\alpha(m,p)} + \frac{1}{2^p} + \frac{1}{2^p} = z_{mp} + \frac{1}{2^p}.$$

From this and (3), it holds

$$\frac{k(m,q,n)}{2^n} + \frac{2}{2^q} - \frac{1}{2^p} + \frac{1}{2^p} < z_{mp} < \frac{k(m,q,n)+1}{2^n} - \frac{2}{2^q} + \frac{2}{2^p}, \tag{4}$$

from which follows (1).
Fact 2: There exists an integer k_{mn} such that

$$\frac{k_{mn}}{2^n} \leq x_m < \frac{k_{mn}+1}{2^n}.$$

It then follows that there is a positive integer q_0 satisfying

$$\frac{k_{mn}+1}{2^n} - x_m > \frac{4}{2^{q_0}}. \tag{5}$$

With these k_{mn} and q_0, we will show that, for any $q \geq q_0$,

$$\frac{k_{mn}}{2^n} < z_{mq} < \frac{k_{mn}+1}{2^n} - \frac{2}{2^q}. \tag{6}$$

If (6) holds, then $k(m,q,n) = k_{mn}$ and $l(m,q,n) \neq 2$ for any $q \geq q_0$, proving Fact 2.

For the proof of (6), we have for any $q \geq q_0$,

$$\frac{k_{mn}}{2^n} \leq x_m < z_{mq} < x_m + \frac{2}{2^q} \leq x_m + \frac{2}{2^{q_0}} \tag{7}$$

and, by (5),

$$x_m + \frac{2}{2^{q_0}} < \frac{k_{mn}+1}{2^n} - \frac{2}{2^{q_0}} < \frac{k_{mn}+1}{2^n} - \frac{2}{2^q}.$$

This and (7) yield (6).
Fact 3: Suppose $l(m,q,n) = 1$, and put $k \equiv k(m,q,n)$. Then,

$$\frac{k}{2^n} + \frac{4}{2^q} \leq z_{mq} \leq \frac{k+1}{2^n} - \frac{2}{2^q}.$$

Suppose $p > q$. (If $p = q$, then the assertion in Fact 3 already holds.) When $l(m,q,n) = 1$, (3) holds. Furthermore, $k(m,p,n) = k(m,q,n) = k$ by Fact 1. So, we obtain $l(m,p,n) = 1$ as follows. From (4) we have hence

$$\frac{k}{2^n} + \frac{2}{2^q} < z_{mp} < \frac{k+1}{2^n} - \frac{2}{2^q} + \frac{2}{2^p},$$

$$\frac{k(m,p,n)}{2^n} + \frac{4}{2^p} < z_{mp} < \frac{k(m,p,n)+1}{2^n} - \frac{2}{2^p}.$$

Fact 4: Assume $\lim_{q\to\infty} l(m,q,n) = 0$. Suppose also that $\frac{k}{2^n} \leq x_m < \frac{k+1}{2^n}$ for an integer k. For all sufficiently large q, we have that $x_m < z_{mq} < \frac{k+1}{2^n}$, hence $k = k(m,q,n)$. $\lim_{q\to\infty} l(m,q,n) = 0$ means that

$$\frac{k}{2^n} \leq x_m < z_{mq} < \frac{k}{2^n} + \frac{4}{2^q} \tag{8}$$

for all sufficiently large q, hence $x_m = \frac{k}{2^n} = L_n(x_m)$.
Fact 5: This fact follows from the choice of $k(m,q,n)$. $\square$

4 Sequential computability of a function

We will define two notions of sequential computability of a real function on I, one with respect to the Euclidean topology and one with respect to $\{U_n\}$, and then relate them.

Definition 4.1 (Sequential computability) Let $f : I \to \mathbf{R}$ be a real function defined on I.

1) f is called *$\mathcal{D}$-sequentially computable* if, for every $\mathcal{D}$-computable sequence of real numbers $\{x_m\} \subset I$, $\{f(x_m)\}$ is $\mathbf{R}$-computable (cf. Section 2).

2) f is called *$\mathcal{L}^*$-sequentially computable* if, for any $\mathbf{R}$-computable sequence $\{x_m\} \subset I$, there are a recursive sequence of rational numbers $\{t_{mq}\}$ and a function $\delta : \mathbf{N}^2 \to \mathbf{N}$ which is recursive in ν satisfying

$$f(x_m) \simeq \langle t_{mq}, \delta(m,p) \rangle \, ,$$

where ν is the limiting recursive modulus of convergence of $\{z_{mp}\}$ to $\{x_m\}$ in Theorem 3.3.

Note $\mathcal{L}^*$-sequential computability is stated slightly differently from the corresponding notion in [10].

We will show that the two notions of sequential computability in Definition 4.1 coincide for functions satisfying a certain effective continuity. The Remark below will be useful.

Remark If $\{x_m\}$ is $\mathcal{D}$-computable with recursive $\{q_{mp}\}$ and β, that is, $x_m \simeq_{\mathcal{D}} \langle q_{mp}, \beta(m,n) \rangle$, then in Theorem 3.3, we can take $z_{mp} = q_{mp}$ and $\nu(m,n) = \beta(m,n)$, hence ν (a limiting recursive modulus of convergence of $\{z_{mp}\}$ to $\{x_m\}$) can in fact be taken to be recursive.

Theorem 4.2 (From $\mathcal{L}^*$-sequential computability to $\mathcal{D}$-sequential computability) If f is $\mathcal{L}^*$-sequentially computable, then f is $\mathcal{D}$-sequentially computable.

Proof Suppose f is $\mathcal{L}^*$-sequentially computable and let $\{x_m\} \subset I$ be $\mathcal{D}$-computable. Then it is $\mathbf{R}$-computable. By definition, there are a recursive sequence of rational numbers t_{mq} and a function δ which is recursive in ν satisfying $f(x_m) \simeq \langle t_{mq}, \delta(m,p) \rangle$. By virtue of the Remark above, ν can in fact be recursive for this $\{x_m\}$, hence δ can be recursive, and so $\{f(x_m)\}$ is $\mathbf{R}$-computable. □

We will next consider the family of functions which are "effectively uniformly continuous" with respect to $\{U_n\}$. A function in this family is generally not continuous but is piecewise continuous in the Euclidean topology.

Definition 4.3 (Effective uniform continuity) A function $f : I \to \mathbf{R}$ is called *effectively $\mathcal{D}$-uniformly continuous* (abbreviated to *educ*) if there is a recursive function γ_0 such that, for $n \geq \gamma_0(p)$,

$$y \in U_n(x) \Rightarrow |f(x) - f(y)| < \frac{1}{2^p}.$$

Note In claiming *educ* of f, it suffices to claim that $y \in U_{\gamma_0(p)}(x)$ implies $|f(x) - f(y)| < \frac{1}{2^p}$.

Theorem 4.4 (From $\mathcal{D}$-sequential computability to $\mathcal{L}^*$- sequential computability) If f is *educ* and $\mathcal{D}$-sequentially computable, then f is $\mathcal{L}^*$-sequentially computable.

Proof Suppose f is $\mathcal{D}$-sequentially computable, and suppose $\{x_m\}$ is $\mathbf{R}$-computable. Then by Theorem 3.3 there is a recursive sequence of rational numbers $\{z_{mp}\}$ and a limiting recursive function ν such that $\{x_m\}$ is weakly $\mathcal{D}$-computable, that is,

$$z_{mp} \in U_n(x_m) \quad \text{for} \quad p \geq \nu(m, n). \tag{9}$$

Since $\{z_{mq}\}$ is a recursive sequence of rational numbers, it is $\mathcal{D}$-computable (Lemma 3.1). So, by the assumption that f is $\mathcal{D}$-sequentially computable, $\{f(z_{mq})\}$ is $\mathbf{R}$-computable. Namely, there are a recursive sequence of rational numbers $\{s_{mqt}\}$ and a recursive $\beta(m, q, n)$ such that $f(z_{mq}) \simeq \langle s_{mql}, \beta \rangle$, namely,

$$|f(z_{mq}) - s_{mql}| < \frac{1}{2^n} \quad \text{if} \quad l \geq \beta(m, q, n).$$

Now define a recursive sequence of rational numbers $\{t_{mn}\}$ by

$$t_{mn} := s_{mn\beta(m,n,n)}.$$

Next define $\delta(m, p)$, using γ_0 in Definition 4.3, by

$$\delta(m, p) := \max(\nu(m; \gamma_0(p + 1)), p + 1).$$

δ is limiting recursive (recursive in ν). Suppose $q \geq \delta(m, p) \geq \nu(m, \gamma_0(p + 1)), p + 1$. Then, by (9), $z_{mq} \in U_{\gamma_0(p+1)}(x_m)$, hence by *educ* (Definition 4.3) it holds that

$$|f(x_m) - f(z_{mq})| < \frac{1}{2^{p+1}}.$$

$$|f(z_{mq}) - t_{mq}| = |f(z_{mq}) - s_{mq\beta(m,q,q)}| < \frac{1}{2^q} \leq \frac{1}{2^{\delta(m,p)}} \leq \frac{1}{2^{p+1}}.$$

We have thus

$$|f(x_m) - t_{mq}| < \frac{1}{2^p}$$

if $q \geq \delta(m, p)$, that is, $f(x_m)$ is approximated by recursive $\{t_{mq}\}$ with a modulus of convergence δ which is recursive in ν. □

By virtue of Theorems 4.2 and 4.4, it will be reasonable to define the uniform $\mathcal{D}$-computability of a function as follows.

Definition 4.5 ($\mathcal{D}$-uniformly computable function) A function $f : I \to \mathbf{R}$ is called $\mathcal{D}$-uniformly computable if it is $\mathcal{D}$-sequentially computable and *educ*. The first condition can be replaced by $\mathcal{L}^*$-sequential computability at the presence of the second condition.

Note A function which is effectively $\mathcal{D}$-uniformly continuous (Definition 4.3) and is $\mathcal{D}$-sequentially computable is uniformly computable in the sense of Definition 4.5 of [8]. In [8], we presented a weaker notion of computabillity in the effective uniform space.

For a single function, Theorem 4.4 may not be so striking because of the assumption *educ*, which is a rather strong property. With a sequence of functions, however, it exhibits more power, since then we can deal with a case where the jump points vary according to the functions in the sequence. We will present an example of such in the next section.

5 $\mathcal{L}^*$-sequential computability of the Rademacher function system

We can easily extend the definitions and propositions in the previous section to a sequence (or multiple sequence) of functions. Below we state the extended definitions and propositions corresponding to Definitions 4.1 and 4.3, and Theorems 4.2 and 4.4.

Definition 5.1 (Sequential computability notions of a function sequence) Consider a function sequence $\{f_i\}$ from I to $\mathbf{R}$.

1) $\{f_i\}$ is called $\mathcal{D}$-sequentially computable if, for any $\mathcal{D}$-computable $\{x_m\}$, $\{f_i(x_m)\}$ is an $\mathbf{R}$-computable double sequence of real numbers.

2) $\{f_i\}$ is called $\mathcal{L}^*$-sequentially computable if, for any $\mathbf{R}$-computable $\{x_m\}$, there are a recursive sequence of rational numbers $\{t_{imq}\}$ and a function δ which is recursive in ν satisfying

$$f_i(x_m) \simeq \langle t_{imq}, \delta(i, m, p)\rangle\,,$$

where ν is a limiting recursive function claimed in Theorem 3.3.

3) $\{f_i\}$ is called *educ* if there is a recursive function γ_0 satisfying

$$y \in U_n(x), n \geq \gamma_0(i, p) \Rightarrow |f_i(x) - f_i(y)| < \frac{1}{2^p}.$$

Proposition 5.2 (Sequential computability notions of a function sequence) 1) If $\{f_i\}$ is a $\mathcal{D}$-sequentially computable sequence of functions which is *educ*, then $\{f_i\}$ is $\mathcal{L}^*$-sequentially computable.

2) If $\{f_i\}$ is $\mathcal{L}^*$-sequentially computable, then $\{f_i\}$ is $\mathcal{D}$-sequentially computable.

Now we will deal with a specific system of the Rademacher functions $\{\phi_i\}$ (cf. Section 2). For any recursive sequence of rationals $\{q_{mn}\} \subset I$, the sequence of values $\{\phi_i(q_{mn})\}$ is a recursive (triple) sequence of rational numbers (integers). Furthermore $\{\phi_i\}$ is *educ* and is $\mathcal{D}$-sequentially computable. (These properties are essentially included in Theorem 2 of [12].) Then 1) of Proposition 5.2 implies that it is $\mathcal{L}^*$-sequentially computable. We thus have the following characterization of $\{\phi_i\}$.

Theorem 5.3 (Sequential computability of the Rademacher function system) The Rademacher function system $\{\phi_i\}$ is *educ* and is $\mathcal{L}^*$-sequentially computable.

Remark There are many piecewise continuous functions on I in the Euclidean topology which are not $\mathcal{D}$-uniformly continuous, that is, not uniformly continuous in the space $\langle I, \{U_n\}\rangle$. Some of them can be characterized as "locally uniformly continuous functions" and "continuous functions" in the space $\langle I, \{U_n\}\rangle$. Corresponding to these notions of continuity, we can introduce the notions of "locally uniform $\mathcal{D}$-computability" and "$\mathcal{D}$-computability" by rephrasing the definitions in Brattka and Mori ([1] [4],[5]).

Let $\{e_i\}$ be an effective enumeration of binary rational numbers in I.

We call a function $f : I \to \mathbf{R}$ *locally uniformly $\mathcal{D}$-computable* if the following conditions hold: f is $\mathcal{D}$-sequentially computable, and there are recursive functions γ and α satisfying that $|f(x) - f(y)| < \frac{1}{2^p}$ if $x, y \in U_{\gamma(i)}(e_i)$ and $y \in U_{\alpha(i,p)}(x)$; $\cup_{i=1}^{\infty} U_{\gamma(i)}(e_i) = I$.

A funtion $f : I \to \mathbf{R}$ is called *$\mathcal{D}$-computable* if it satisfies the following: f is $\mathcal{D}$-sequentially computable, and f is effectively $\mathcal{D}$-continuous, that is, there exists a recursive funtion $\gamma(i, p)$ such that $x \in U_{\gamma(i,p)}(e_i)$ implies $|f(x) - f(e_i)| < \frac{1}{2^p}$ and $\cup_{i=1}^{\infty} U_{\gamma(i,p)}(e_i) = I$ for each p.

The following are some examples of locally uniformly $\mathcal{D}$-computable functions which are not uniformly $\mathcal{D}$-continuous.

Pick up disjoint intervals $\{I^n_{2^n-2}\}$, where $I^n_{2^n-2} = [\frac{2^n-2}{2^n}, \frac{2^n-1}{2^n})$. Then it holds that $\cup_{n=1}^{\infty} I^n_{2^n-2} = [0, 1) = I$. Define a function by $\phi(x) = n$ if $x \in I^n_{2^n-2}$. ϕ is not $\mathcal{D}$-uniformly continuous, but it is locally uniformly continuous with respect to $\{U_n\}$. $\psi(x) = n + x, x \in I^n_{2^n-2}$ is also an example of this sort. Another such example is the following function: $\mu(x) = \frac{1}{1-2x}$ if $0 \leq x < \frac{1}{2}$ and $\mu(x) = 1$ if $\frac{1}{2} \leq x < 1$. These functions are in fact locally uniformly $\mathcal{D}$-computable.

An example of a $\mathcal{D}$-computable function which is not locally uniformly $\mathcal{D}$-continuous has been constructed by Brattka in Theorem 12 of [1].

A theorem corresponding to Theorem 4.4 for effectively $\mathcal{D}$-continuous functions can be proved with more elaboration, the details of which will appear elsewhere.

Note In [8], notions of "computability" and "uniform computability" have been defined for the effectively uniform space. $\mathcal{D}$-computability satisfies the condition of "computability" in [8], and $\mathcal{D}$-uniform computability satisfies the condition of "uniform computability" in [8]. Locally uniform $\mathcal{D}$-computability is between "uniform computability" and "computability" in the sense of [8].

References

[1] Brattka,V., *Some notes on Fine computability*, JUCS, 8(3) (2002),382-395.

[2] Gold, E.M., *Limiting recursion*, JSL, 30-1 (1965), 28-48

[3] Kawada, Y., Mimura, Y., *Outline of Modern Mathematics* (in Japanese), Iwanami Shoten (1965)

[4] Mori, T., *Computabilities of Fine continuous functions*, Computability and Complexity in Analysis, LNCS 2064, Springer (2001) 200-221

[5] Mori, T., *On the computability of Walsh functions*, TCS 284 (2002) 419-436

[6] Pour-El, M.B., Richards, J.I., *Computability in Analysis and Physics*, Springer-Verlag (1989)

[7] Schipp, F., Wade, W.R., Simon, P., *Walsh Series*, Adam Hilger (1990)

[8] Tsujii, Y., Yasugi, M., Mori, T., *Some properties of the effectively uniform topological space*, Computability and Complexity in Analysis, LNCS 2064, Springer (2001) 336-356

[9] Yasugi, M., Brattka, V., Washihara, M., *Computability aspects of some discontinuous functions*, Scientiae Mathematicae Japonicae (SCMJ) Online, vol.5 (2001) 405-419

[10] Yasugi, M., Tsujii, Y., *Two notions of sequential computability of a function with jumps*, ENTCS 66 No.1 (2002) 11 pages;
http://www.elsevier.nl/locate/entcs/volume66.html.
Revised version: *Computability of a function with jumps -Effective uniformity and limiting recursion-*, to appear in Topology and its Applications(Elsevier Science).

[11] Yasugi, M., Tsujii, Y., Mori, T., *Metrization of the uniform space and effective convergence*, Math. Log. Quart. 48 (2002) Suppl.1, 123-130

[12] Yasugi, M., Tsujii, Y., Mori, T., *Effective sequence of uniformities and the Rademacher function system* (Preliminary Notes);
http://www.kyoto-su.ac.jp/ yasugi/page/recent.html

[13] Yasugi, Y., Washihara, M., *A note on Rademacher functions and computability*, Words, Languages and Combinatorics III, World Scientific (2003) 466-475