An MR-complete extension of **TRDB** and its functional interpretation

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Abstract

In this paper, we extend a functional interpretation to transfinite types. To be more precise, we define a modified realizability interpretation of a constructive arithmetic with a certain inductive definition and bar-induction. The system is called **TRDB**. We also give an extension of **TRDB**, which is called **S**, and show that **S** is a complete system with respect to the modified realizability interpretation.

In the second half, we define a truth-value of **S**-formulas under certain functionals, and show validity of **S** with respect to the truth-value.

1 Introduction

In a previous work [5], Yasugi and Hayashi formulated a system of *constructive arithmetic* with *transfinite recursion* and *bar induction*. This system was called **TRDB**, which is a streamlined version of the system used by Yasugi in [4] to prove the accessibility of an order system.

TRDB is, however, a mathematically interesting system on its own right. For this reason, it has been studied from various aspects. [5] re-formulates the arithmetic system used in [4], extracting the essence, as well as the corresponding term-system **TRM**. **TRM** is a functional system in which the algorithms inherent in a proof of **TRDB** can be interpreted. Yasugi and Hayashi in [6] also presents a set-theoretic model for **TRM** as well as an interpretation of **TRM** in **TRDB**, which is a converse of the result obtained in [5].

Takaki in [1] has established the strong normalization property of natural deductions of **TRDB**.

The present article is a sequel to these preceding works. In this paper, we deal with the *modified realizability* (abbreviated to *MR*) interpretation of a constructive arithmetic corresponding to the interpretation of **TRDB** in **TRM** obtained in [5].

A system **S** is said to be complete with respect to an interpretation R if **S** interprets itself with respect to R. As for the first order constructive arithmetic, its extensions which are complete with regards to some interpretations are already known (see, for example, Theorem 3.4.8. of [3]). We questioned if there be an extension of **TRDB** which is complete with respect to *MR*-interpretation, and have reached a conclusion.

Here an extension of **TRDB**, called **S**, is presented, and it is shown that **S** is the desired system. That is, **S** is complete with respect to MR-interpretation.

It is also shown that **TRM** interprets **S**, that is, an algorithm inherent in a proof of **S** is realized as a functional in **TRM**. This implies that **S** is of the same algorithmic strength as **TRDB**.

Some preliminaries are presented in Section 2 for the reader's convenience. It summarizes some definitions and results in [6], [1] and [2], but in much revised presentation.

Section 3 introduces the system **S** and the MR-interpretation for formulas of **S**. Then, the completeness of **S** with respect to MR-interpretation is established (Theorem 3.11). A relationship of MR-interpretation for **TRDB** and that for **S** restricted to **TRDB**-formulas is also worked out.

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In [2], we have reported a candidate for **S** and its syntactic treatment. Section 3 contains also a conclusive and comprehensive study of the system **S**.

In Section 4, a term-form system **TRM** is reviewed.

In Section 5, we investigate the semantics of **S** in terms of **TRM**, stating that the algorithm which is claimed by *MR*-interpretation of a theorem of **S** is realized by a functional in **TRM** (Theorem 5.8: validity of **S** theorem). As corollaries, we can claim that all provably total functions in **S** can be realized in **TRM** (Corollary 5.10) and that **S** is syntactically consistent (Corollary 5.11).

Incidentally, let us explain some of the notions involved in our work.

First, let us explain a little more about our system **S**. The real meaning of *completeness* of **S** is the following. The existence statement of **TRDB**, for example, $\forall x \exists y A(x, y)$ can be regarded as claiming that there is a *method* M to compute y for any x, so that $A(x, y)$ is true. **S** is capable of expressing that $\forall x \exists y A(x, y)$ is equivalent to $\exists M \forall x A(x, Mx)$.

This has been explained at length in [4] and [2].

As for the computational meaning of **TRDB**, let us recall that **HA** can be interpreted in terms of Gödel's computable functions of finite type. This means that primitive recursion of finite type is inherent in the mathematical induction applied to an arbitrary formula.

In the same line, Bar induction can be interpreted in terms of bar recursion. For details of bar induction and bar recursion, see Section 2 of this article and [5].

Definition by transfinite recursion asserts that, given a well-ordered structure $(I, <)$ and a local construction of a set, say $G[i, \{S_j\}_{j<i}]$, from the existing ones $\{S_j\}_{j<i}$ $(i \in I)$, one can define a sequence of sets $\{S_j\}_{j \in I}$ by transfinite induction along I. The union of $\{S_i\}_{i \in I}$ is itself a set.

The computational process inherent in this is explained as follows. Given a well-ordered structure $(I, <)$ and a local construction of a function, say $g[i, \{f_j\}_{j<i}]$, from the existing ones $\{f_j\}_{j<i}$ $(j \in I)$, one can define a sequence of functions $\{f_j\}_{j \in I}$ by transfinite recursion along I. The *MR*-interpretation of *definition by transfinite recursion* signifies that $\{f_j\}_{j<i}$ $(j \in I)$ is itself a legitimate function.

The term system **TRM** represents these computational methods.

Let us close this section by commenting that nested applications of *definition by transfinite recursion* along different order structures would cause no theoretical difficulty. We have not elaborated on it for the simplicity of discussion.

2 Preliminaries

In order to facilitate the reader to understand this article, we first give a concise presentation of the system **TRDB**, as well as of the notion called *type-form*. The definitions are quoted mostly from the article [5] written by Yasugi and Hayashi.

TRDB is a system of constructive arithmetic formalized in the natural deductions. As for the terminologies concerning natural deductions and constructive arithmetic, we mostly follow [3]. There are, however, some terminologies which are used in different context from those in [3]. They will be explicitly defined in this paper.

Our theory depends on a pre-supplied, primitive recursive well-ordered structure $\mathcal{I} \equiv (I, <_I)$ on natural numbers. Namely, as a set we identify I with the set of natural numbers, and we let $<_I$ be a primitive recursive well-ordering on it. We assume $\mathcal{I}$ is accessible, that is, there is a functional μ (modulus of finiteness functional for $\mathcal{I}$) such that, given any $\{i_n\}_n$, a decreasing sequence from I, for any $m \geq \mu(\{i_n\})$, i_m is in fact empty.

For the sake of simplicity, we assume $\mid I \mid < \varepsilon_0$, where $\mid I \mid$ represents the order type of I and ε_0 is the least ordinal α satisfying $\omega^\alpha = \alpha$.

For a formula A, we often write $A[t]$ in order to locate a term t in A.

Definition 2.1 (**Basic language $\mathcal{BL}$**)
(1) The language $\mathcal{BL}$ consists of the following.
(1. 1) Propositional connectives $\wedge$ (*and*), $\supset$ (*imply*).
(1. 2) There are variables, say x_p^n, where p and n are natural numbers.

A variable with the superscript n is called an n-ary variable, and represents n-ary number-theoretic functions. For $n = 0$, x_p^n is a variable which ranges over natural numbers. It will be called a number variable.
(1. 3) Constant symbols for all the number-theoretic functions which are primitive recursive in function parameters. (We regard natural numbers as 0-ary number-theoretic functions.)
(1. 4) Predicate constant symbol $=$.
(2) $\mathcal{BL}$-terms are defined by (2.1) $\sim$ (2.4) below.
(2. 1) A constant or a variable of $\mathcal{BL}$ is a term of its arity.
(2. 2) If f is an n-ary term, and if $t_1, \cdots, t_n$ are number terms, then $f(t_1 \cdots t_n)$ is a number term.
(3) Every atomic formula of $\mathcal{BL}$-language is of the form $s = t$, where s and t are number terms. The $\mathcal{BL}$-formulas are defined from atomic formulas by applications of the propositional connectives.

Remark 2.2 We identify the binary primitive recursive predicate $<_I$ with its characteristic function f_I defined in Definition 2.1 (1. 3), that is, we denote $f_I(x, y) = 0$ by $x <_I y$.

Let c be a new, unary function constant symbol. We can extend $\mathcal{BL}$ to $\mathcal{BL}(c)$ adding c to it.

Using $\mathcal{BL}(c)$-language, one can define the system **TRDB**, as in Takaki [1]. **TRDB** may be considered to be **HA** (Heyting arithmetic) with definition by *transfinite recursion* and *bar induction*. The reason why a constant c is added to the language is to justify the bar induction. See, for example, Chapter 5 in [3].

Definition 2.3 (**System TRDB**)
Symbols and Terms
(1) All symbols and terms of $\mathcal{BL}(c)$-language serve as those of **TRDB**.
(2) Special predicate constants H and Σ_H.
(3) Logical symbols $\wedge$, $\supset$, $\forall$ and $\exists$.
Formulas
(1) $s = t$ is an atomic formula of **TRDB**, where s and t are number terms.
(2) $H(s,\ t)$ is an atomic formula of **TRDB**, where s and t are number terms.
(3) $\Sigma_H(s,\ s',\ t)$ is an atomic formula of **TRDB**, where s, s' and t are number terms. The intended meaning of $\Sigma_H(s,\ s',\ t)$ is $H(s,\ t) \wedge s <_I s'$. (Cf. Remark 2.2 for $<_I$.)
(4) If A and B are formulas, then $A \wedge B$, $A \supset B$ and $\forall x A$ are formulas, where x is a variable.
(5) If A is a formula, then $\exists x A$ is a formula, where x is a number variable and x does not occur in any subformula of A which is of the form either $H(i,\ t)$ or $\Sigma_H(j,\ i,\ t)$. This restriction on the application of $\exists$ is called the admissibility.
Note. $\vee$ (*or*) can be expressed in terms of other logical symbols in **TRDB** as follows: $A \vee B$ is represented by $\exists z((z = 0 \supset A) \wedge ((z = 0 \supset 0 = 1) \supset B))$. We also abbreviate $(A \supset B) \wedge (B \supset A)$ to $A \Leftrightarrow B$. Similarly, $\neg A$ (*not* A) is expressed as $A \supset 0 = 1$.

We call a formula defined as above an *admissible formula*.
Axioms and inference rules
(1) **TRDB** contains inference rules of constructive logic formulated in natural deductions as usual, that is, introduction rules $\wedge$-I, $\supset$ -I, $\forall$-I, $\exists$-I and elimination rules $\wedge$-E, $\supset$ -E, $\forall$-E, $\exists$-E, $\perp$-E ("$\perp$" means $0 = 1$). See Section 1 of [3] for detail.
(2) **TRDB** contains axioms and inference rules on constants of $\mathbf{PRA^2}$ (primitive recursive arithmetic with function variables) (cf. Section 1 of [3]). We give these axioms and inference rules as follows.

(2. 1) For any number terms t and t',

$$t = t, \qquad \frac{\begin{array}{cc}\vdots & \vdots\\ t = t' & P[t]\end{array}}{P[t']},$$

where $P[t]$ denotes an atomic formula.
(2. 2) For any number terms t, t' and t_i $(i = 1, \cdots, n)$,

$$\frac{\begin{array}{c}\vdots\\ 0 = St\end{array}}{0 = 1}, \qquad \frac{\begin{array}{c}\vdots\\ St = St'\end{array}}{t = t'}, \qquad I_i^n(t_1, \cdots, t_i, \cdots, t_n) = t_i,$$

where S denotes a function constant expressing the successor function, and I_i^n $(i \leq n)$ denotes a function constant expressing the n-place projection function.
(2. 3) Let PRF_n denote the set of all n-ary primitive recursive functions in function parameters. Let F be the element of PRF_n obtained from a $G \in PRF_m$ and a $K_i \in PRF_n$ $(i = 1, \cdots, m)$ by functional composition:

$$F(x_1, \cdots, x_n) = G(K_1(x_1, \cdots, x_n), \cdots, K_m(x_1, \cdots, x_n)).$$

If f, g and k_i denote function constants expressing F, G and K_i respectively, then, for any number terms t_i $(i = 1, \cdots, n)$, u_j $(j = 1, \cdots, m)$ and v,

$$\frac{\begin{array}{cccc}\vdots & \vdots & & \vdots\\ g(u_1, \cdots, u_m) = v & k_1(t_1, \cdots, t_n) = u_1 & \cdots & k_m(t_1, \cdots, t_n) = u_m\end{array}}{f(t_1, \cdots, t_n) = v}.$$

(2. 4) Let F be the element of PRF_{n+1} obtained from a $K \in PRF_n$ and a $G \in PRF_{n+2}$ by primitive recursion:

$$F(x_1, \cdots, x_n, 0) = K(x_1, \cdots, x_n),$$
$$F(x_1, \cdots, x_n, Sx) = G(x_1, \cdots, x_n, x, F(x_1, \cdots, x_n, x)).$$

If f, k and g denote function constants expressing F, K and G respectively, then, for any number terms t_i $(i = 1, \cdots, n)$, t, u and v,

$$\frac{\begin{array}{c}\vdots\\ k(t_1, \cdots, t_n) = u\end{array}}{f(t_1, \cdots, t_n, 0) = u}, \qquad \frac{\begin{array}{cc}\vdots & \vdots\\ f(t_1, \cdots, t_n, t) = u & g(t_1, \cdots, t_n, t, u) = v\end{array}}{f(t_1, \cdots, t_n, St) = v}.$$

(3) **TRDB** contains *monotone bar induction* as explained below. Let $R[a]$ be a formula with a number variable a free, and suppose $R[a]$ contains neither any quantifier, H, Σ_H, nor any variable except a. Such a formula will be called *elementary*.

We consider a bijective function **F** from finite sequences of natural numbers onto natural numbers. It is well known that **F** can be defined primitive recursively. We fix such a bijection **F**.

Let $R[a]$ be an elementary formula. Then $R[a]$ is said to be *monotone* if $R[a]$ satisfies the following conditions (i) and (ii).
(i) For any unary function f, there exists a natural number n such that $R[f\lceil n]$ holds, where $f\lceil n$ denotes a number term expressing the natural number assigned to the finite sequence $\langle f(0),\ f(1), \cdots, f(n-1)\rangle$ by **F**.
(ii) For any unary function f and for any natural number n, $R[f\lceil n]$ implies $R[f\lceil m]$ if $m > n$.

Let $*$ denote a binary primitive recursive function constant satisfying the following: $a * b$ expresses the natural number assigned to $\langle a_1, \cdots, a_n, b\rangle$ by **F**, where $\langle a_1, \cdots, a_n\rangle$ is the finite sequence to which **F** assigns a. Now, our *bar induction* can be expressed as follows:

$$\frac{\begin{array}{cc}\vdots & \vdots\\ \forall z(R[z] \supset A[z]) & \forall z(\forall x A[z * x] \supset A[z])\end{array}}{A[t]} BI,$$

where $A[a]$ is an arbitrary formula, $R[a]$ is an arbitrary monotone formula and t is an arbitrary number term.
(4) **TRDB** contains *definition by transfinite recursion* $TRD(G, \mathcal{I})$. We fix a formula $G[a,\ b]$, where a and b are free number variables and G satisfies the following conditions.
(i) No free variable occurs of G except a or b.
(ii) Predicate constant H does not occur in G, and every Σ_H in G occurs in the form $\Sigma_H(j,\ a,\ s)$, where j and s are some terms.

The axiom $TRD(G, \mathcal{I})$ stands as follows:

$$\forall x \forall y\ (H(x,\ y) \Leftrightarrow G(x,\ y)).$$

(5) **TRDB** contains Σ_H-rules as follows.

$$\frac{\begin{array}{cc}\vdots & \vdots \\ H(j,\ t) & j <_I i\end{array}}{\Sigma_H(j,\ i,\ t)}\Sigma_H\text{-}I, \qquad \frac{\vdots}{\dfrac{\Sigma_H(j,\ i,\ t)}{H(j,\ t) \wedge (j <_I i)}}\Sigma_H\text{-}E$$

where i, j and t are arbitrary number terms and $j <_I i$ is a formula denoting the order of $\mathcal{I}$.

Remark 2.4 **TRDB** is specified with a certain formula G and a certain primitive recursive ordered set $\mathcal{I}$. So, **TRDB** should be written as **TRDB**$(G, \mathcal{I})$ to be precise.

Type-forms are types with parametric variables. They are briefly described below.

Definition 2.5 (**Type-form**; See Section 2 and Section 4 of [5] for detail.)
Type-forms are defined below, based on the language $\mathcal{BL}$.
(1) Symbols N and **1** are (atomic) type-forms.
(2) If α and β are type-forms, then so are $\alpha \to \beta$ (*function space*), $\{x\}\alpha$ (*parametric abstraction*), $\pi(\alpha;\ t)$ (*projection*) and $cond[A;\ \alpha,\ \beta]$ (*case definition*), where x is a variable of $\mathcal{BL}$, t is a $\mathcal{BL}$-term and A is a $\mathcal{BL}$-formula.
(3) $\mathcal{R}[i,\ t]$ (*transfinite recursion*) is a type-form, where $\mathcal{R}$ is a special letter denoting recursion operator, and i and t are number $\mathcal{BL}$-terms. ($\mathcal{R}[i,\ t]$ is characterized by a fixed type-form η. See (1. 4) of Definition 2.6 below.)
(4) $\rho[j <_I i;\ \mathcal{R}[j,\ s]]$ (*restriction*) is a type-form, where i, j and s are number $\mathcal{BL}$-terms.

Definition 2.6 (**Conversion of type-form**)
(1) Conversion rules of type-forms, say α to β, denoted by $\alpha \Rightarrow \beta$, are the following.
(1. 1) $\pi(\{x\}\alpha; t) \Rightarrow \alpha[t/x]$, where $\alpha[t/x]$ represents the substitution of t for x in α.
(1. 2) $cond[A; \alpha_1, \alpha_2] \Rightarrow \beta$, where β is α_1 if A is a true sentence in the standard interpretation of the symbols, and β is α_2 if A is a false sentence.
(1. 3) $\rho[j <_I i; \mathcal{R}[j, s]] \Rightarrow \beta$, where β is $\mathcal{R}[j, s]$ if $j <_I i$ is a true sentence, and it is **1** if $j <_I i$ is a false sentence.
(1. 4) Let Ξ be a designated symbol which is temporarily regarded as an atomic type. Let η be a type-form with Ξ but without $\mathcal{R}$, whose free variables are k and w, where k and w are of arity 0 and Xi contains only k as its free variable. $\eta[i/k, t/w]$ will denote simultaneous substitution of i for k and t for w, and $\mathcal{R}(i)$ will abbreviate $\{j\}\{x\}\rho[j <_I i; \mathcal{R}[j, x]]$.

$$\mathcal{R}[i, t] \Rightarrow \eta[i/k, t/w][\mathcal{R}(i)/\Xi],$$

where $X[Y/\Xi]$ represents the substitution of Y for Ξ in X.
(2) α is said to be 1-reduced to β if β is obtained from α by one conversion applied to a subtype of α (called a reduction of α to β), with the strategy that the conversions (1. 3) enjoys a following property: one does not go inside $\rho[j <_I i;\ \mathcal{R}[j,\ t]]$. We call this *$\rho$-strategy*. α is said to be reducible to β if there is a sequence of reductions (possibly empty) from α to β. In such a case, β is called a *reduct* of α.
(3) A type-form is said to be normal if no conversion rule applies to it.

Remark 2.7 (1) The conversion rule of $\mathcal{R}[i,\ t]$ is determined by a type-form η in Definition 2.6 (1. 4). We call such a type-form η a *central* type-form.
(2) It is obvious that the reducts of a type-form are again type-forms, and that free variables do not increase by reductions. (Cf. Section 2 of [5].)

Theorem 2.8 (**Strong normalizability of type-forms**: Yasugi and Hayashi, Theorem 1 in [5])
Every type-form is strongly normalizable to a unique normal form (under the ρ-strategy), that is, any process of reductions ends up with a normal type-form. Furthermore, there is a unique such normal type-form for every type-form.

Notation. (1) In what follows, we let $\alpha \sim \beta$ mean that α and β have the same normal form.
(2) We abbreviate $app(\alpha;\ t)$ to αt.
(3) We abbreviate $\{l\}cond[l = 0;\ \alpha,\ cond[l = 1;\ \beta,\ 1]]$ to $\alpha \times \beta$ and $\pi(\gamma;\ i)$ to $\pi_{i+1}\gamma$ for $i = 0$ or 1. $\alpha \times \beta$ represents the product types.

We define a mapping $[\![\ ]\!]$ of admissible formulas (of **TRDB**) into type-forms as well as specify the central type-form. This mapping is introduced by Yasugi and Hayashi in Definition 8.1 of [5].

Definition 2.9 (**Interpretation** $[\![\ ]\!]$)
To each (admissible) formula A (cf. Definition 2.3), we associate a type-form $[\![A]\!]$, as follows.
(1) If A is free of $\exists$ and H, then $[\![A]\!] = \mathbf{1}$.
(2) Suppose A is free of H but contains $\exists$.
(2. 1) $[\![\exists xB]\!] = N$ if $[\![B]\!] = \mathbf{1}$; $[\![\exists xB]\!] = N \times [\![B]\!]$ otherwise.
(2. 2) $[\![\forall xB]\!] = \mathbf{1}$ if $[\![B]\!] = \mathbf{1}$; $[\![\forall xB]\!] = \{x\}[\![B]\!]$ otherwise.
(2. 3) $[\![B \supset C]\!] = [\![C]\!]$ if $[\![B]\!] = \mathbf{1}$ or $[\![C]\!] = \mathbf{1}$; $[\![B \supset C]\!] = [\![B]\!] \to [\![C]\!]$ otherwise.
(2. 4) $[\![B \wedge C]\!] = [\![C]\!]$ if $[\![B]\!] = \mathbf{1}$; $[\![B \wedge C]\!] = [\![B]\!]$ if $[\![C]\!] = \mathbf{1}$; $[\![B \wedge C]\!] = [\![B]\!] \times [\![C]\!]$ otherwise.
(3) Let A be any admissible formula. We define $[\![A]\!]$ as follows.
(3. 1) $[\![H(i,\ a)]\!] = \mathcal{R}[i,\ a]$. $[\![\Sigma_H(j,\ i,\ b)]\!] = \rho[j <_I i;\ \mathcal{R}[j,\ b]]$.
(3. 2) For $\forall$, $\supset$ and $\wedge$, follow (2) above.
(4) When reductions of type-forms are concerned, we need to specify the central type-form η in Definition 2.6 (1. 4). Let G be the formula used in $TRD(G, \mathcal{I})$, and let G' be a formula-like expression obtained from G by replacing H and Σ_H with new symbols Ξ and Σ_Ξ respectively. Define $[\![\Xi(i,\ a)]\!] = \Xi(i,\ a)$ and $[\![\Sigma_\Xi(j,\ i,\ a)]\!] = \rho[j <_I i;\ \Xi(j,\ b)]$. Now apply (2) above to G', and put $\eta = [\![G']\!]$.

3 An extended system S

In this section, we introduce a system which is an extension of **TRDB**. This system, called **S**, was first considered so that **S** be complete with respect to an extended modified realizability interpretation.

Preceding the definition of **S**, we define a notation of certain type-forms.

Definition 3.1 (**Number-theoretic type-form** $N(n)$)
For a natural number n, we define an expression $N(n)$ of a type-form, which is defined by: (i) $N(0) = N$; (ii) $N(n+1) = N \to N(n)$.

The system **S** is defined by the following Definitions 3.2 and 3.7.

Definition 3.2 (**Language of S**$(G, \mathcal{I})$)
Given a system **TRDB**$(G, \mathcal{I})$, we define a language of an extended system **S**$(G, \mathcal{I})$ (often abbreviated to **S**) as follows.

(1) **Symbols**
(1. 1) All symbols of $\mathcal{BL}(c)$-language are those of **S**.
(1. 2) For each type-form α, we prepare variables (called variable-forms) X_n^α $(n = 0, 1, \cdots)$.
(1. 3) id_n $(n = 0, 1, \cdots)$ and *app* are operational symbols, which are used in order to construct terms of **S**.
(1. 4) $\mathcal{H}$ and $\Sigma_{\mathcal{H}}$ are special predicate constants of **S**.
(2) **Terms**
We induce terms of **S** in the following (2. 1)$\sim$(2. 5). We fix a central type-form η as in Definition 2.9 (4). (See also Definition 2.6 and Remark 2.7.)

By $\phi : \alpha$, we express the fact that ϕ is a term of a type-form α.
(2. 1) For any variable-form X_n^α, $X_n^\alpha : \alpha$.
Note. We distinguish "$\mathcal{BL}(c)$-variables" and "variable-forms." In **S**, Variable-forms rule on usual variables, and $\mathcal{BL}(c)$-variables rule on parameters in **S**-terms.
(2. 2) For every natural number n, $id_n : \{x^n\}N(n)$, where x^n is an n-ary $\mathcal{BL}(c)$-variable. We use this term in order to construct **S**-terms corresponding to $\mathcal{BL}(c)$-terms. See also Remark 3.3 (1) below.
(2. 3) $\phi : \beta$, whenever $\phi : \alpha$ and β is a type-form with $\alpha \sim \beta$ (α and β have the same normal form).
(2. 4) $app(\phi;\ t) : \alpha[t/x]$, whenever $\phi : \{x\}\alpha$ and t is a $\mathcal{BL}(c)$-term whose arity is the same as that of the $\mathcal{BL}(c)$-variable x.
(2. 5) $app(\phi;\ \psi) : \beta$, whenever $\phi : \alpha \to \beta$ and $\psi : \alpha$.
(3) **Formulas**
(3. 1) $\phi = \psi$ is an atomic formula of **S**, where ϕ and ψ are number terms.
(3. 2) $\mathcal{H}(i,\ t,\ X^\alpha)$ is a formula of **S**, where i and t are number $\mathcal{BL}(c)$-terms and $\alpha \sim \mathcal{R}[i,\ t]$.
(3. 3) $\Sigma_{\mathcal{H}}(j,\ i,\ t,\ X^\alpha)$ is a formula of **S**, where i, j and t are number $\mathcal{BL}(c)$-terms and $\alpha \sim \rho[j <_I i;\ \mathcal{R}[i,\ t]]$.
(3. 4) $A \wedge B$ and $A \supset B$ are formulas if A and B are formulas of **S**.
(3. 5) Let A be a formula of **S** and x is a $\mathcal{BL}(c)$-variable. Then, $\forall x A$ is a formula of **S**.
(3. 6) Let A be a formula of **S**, and let X^α be a variable-form. Then, $\forall X^\alpha A$ and $\exists X^\alpha A$ are formulas of **S** if X^α in A satisfies the following condition $\Gamma(X^\alpha, A)$:

$\Gamma(X^\alpha, A)$**:** If there is a free occurrence of X^α in A, then all $\mathcal{BL}(c)$-variables occurring in α freely are free in A. (That is, if x is a free $\mathcal{BL}(c)$-variable occurring in α, then such an occurrence of X^α is not in the scope of any $\forall x$ in A.)

Example. The following is not admitted as an **S**-formula:

$$\forall X^{\mathcal{R}[i,x]} \forall x \mathcal{H}(i,\ x,\ X^{\mathcal{R}[i,x]}).$$

Note. 1) An $\mathcal{H}$-*formula* is defined to be a formula which contains $\mathcal{H}(i,\ t,\ X^\alpha)$ or $\Sigma_{\mathcal{H}}(j,\ i,\ t,\ X^\alpha)$.
2) $\exists x$ is *not* admitted in **S**-formulas. This is because, we do not need to bound $\mathcal{BL}$-variables by $\exists$-quantifier due to the admissibility of **S**- (**TRDB**-) formulas (cf. Definition 2.3 and Remark 4.2 of [2]).

Remark 3.3 (1) For an n-ary $\mathcal{BL}(c)$-term t, the intended meaning of the **S**-term $app(id_n;\ t)$ is t. Note that $app(id_n;\ t)$ has the type-form $N(n)$ defined in Definition 3.1. In what follows, we abbreviate $app(id_n;\ t)$ to $\mathbf{t}$.
(2) In what follows, we assume that variables in a formula or a deduction are denoted by mutually distinct alphabets when they behave distinctively. For example, we do not accept formula of the form $x = x \wedge \forall x(x = x)$, but do accept $x = x \wedge \forall y(y = y)$. This does not narrow the class of formulas.

Definition 3.4 (**Morphism** $*$)

We define a morphism $*$ which sends a formula of **TRDB** to a certain formula of **S** in the following (1) and (2).
(1) Let A be a formula of **TRDB**, and let t be a $\mathcal{BL}(c)$-term in A. We define an **S**-term $S(t;\ A)$ by induction on the construction of t as follows.
(1. 1) Let t be an n-ary $\mathcal{BL}(c)$-variable x_i^n.
(i) If x_i^n is bound by an $\exists$-quantifier in A, then $S(x_i^n;\ A) = X_i^N$. Note that, in this case, the arity n of x_i^n is 0 by the admissibility of **TRDB**-formula.
(ii) Otherwise, $S(x_i^n;\ A) = app(id_n;\ x_i^n)$.
(1. 2) If t is an n-ary function constant f, then $S(f;\ A) = app(id_n;\ f)$.
(1. 3) If t is a number $\mathcal{BL}(c)$-term $f(t_1, \cdots, t_n)$ defined in Definition 2.1 (2. 2), then
$S(f(t_1, \cdots, t_n);\ A) = app(app(\cdots app(S(f;\ A);\ S(t_1;\ A)); \cdots);\ S(t_n;\ A))$.
(2) Let A be a formula of **TRDB**, and let B be a subformula of A. We define an **S**-formula $S(B;\ A)$ by induction on the construction of B, as follows.
(2. 1) If B is an atomic formula which is not an H-formula, then $S(B;\ A)$ is the formula obtained from B by replacing all terms $t_1, \cdots, t_n$ of B by the terms $S(t_1;\ A), \cdots, S(t_n;\ A)$.
(2. 2) $S(H(i,\ a);\ A) = \exists X^{\mathcal{R}[i,\ a]}\mathcal{H}(i,\ a,\ X^{\mathcal{R}[i,\ a]})$.
(2. 3) $S(\Sigma_H(j,\ i,\ a);\ A) = \exists X^{\rho[j<_I i;\ \mathcal{R}[j,\ a]]}\Sigma_{\mathcal{H}}(j,\ i,\ a,\ X^{\rho[j<_I i;\ \mathcal{R}[j,\ a]]})$.
(2. 4) $S(B \wedge C;\ A) = S(B;\ A) \wedge S(C;\ A)$.
(2. 5) $S(B \supset C;\ A) = S(B;\ A) \supset S(B;\ A)$.
(2. 6) $S(\forall x B[x];\ A) = \forall x S(B[x];\ A)$.
(2. 7) $S(\exists x B[x];\ A) = \exists X^N S(B[x];\ A)$, where $X^N = S(x;\ A)$.
Now, for every **TRDB**-formula A, we define A^* by: $A^* = S(A;\ A)$.

Example. Let A be a **TRDB**-formulas of the form

$$\forall x^1 \exists x^0 (x^2(x^1(x^0),\ 2) = 0 \supset H(1,\ x^1(0))),$$

where x^i is a $\mathcal{BL}(c)$-variable with the arity i. Then, A^* is the **S**-formula of the following form

$$\forall x^1 \exists X^N\ (t = \mathbf{0}\ \supset\ \exists X^{\mathcal{R}[1,\ x^1(0)]}\mathcal{H}(1,\ x^1(0),\ X^{\mathcal{R}[1,\ x^1(0)]})),$$

where $t = app(app(app(id_2;\ x^2);\ app(app(id_1;\ x^1);\ X^N));\ \mathbf{2})$.

Proposition 3.5 (**Well-definedness of $*$-morphism**)
The above definition of $*$-morphism is well-defined.

Proof. In order to prove this proposition, it suffices to show that, for every **TRDB**-formula A and for every subformula B, $S(B;\ A)$ is a formula of **S**. It is easily shown by induction on the construction of B. □

Remark 3.6 By virtue of the above definition, we can explain the intended meaning of $\exists X^{\rho[x<_I y;\ \mathcal{R}[x,z]]}\Sigma_{\mathcal{H}}(x,\ y,\ z,\ X^{\rho[x<_I y;\ \mathcal{R}[x,z]]})$ by the following formula $\exists X^{\mathcal{R}[x,z]}\mathcal{H}(x, z, X^{\mathcal{R}[x,z]}) \wedge (x <_I^* y)$ where $x <_I^* y$ is the **S**-formula of the form $app(app(app(id_2;\ f_I);\ app(id_0;\ x));\ app(id_0;\ y)) = 0$ and f_I is a $\mathcal{BL}(c)$-function constant defined in the definition of the formula $\Sigma_H(j,\ i,\ t)$ of Definition 2.3.

Notation. In what follows, we abbreviate an **S**-term $app(\phi;\ \psi)$ to $\phi\psi$ and $app(\phi;\ t)$ to ϕt. For $i = 0$ or 1, $app(\phi, i)$ is often denoted by $\pi_{i+1}(\phi)$.

Definition 3.7 (**Axioms and inference rules of S**)
(1) Axioms and inference rules of **S** as constructive logic are defined similarly to those of 2.3. Example. Rules on quantifiers consist of: (i) rules on $\forall$ over $\mathcal{BL}(c)$-variables, which are the same rules as in **TRDB** and (ii) rules on $\forall$ and $\exists$ over variable-forms, which are defined as follows:

$$\frac{\begin{array}{c}\vdots\\ A[X_n^\alpha]\end{array}}{\forall X_n^\alpha A[X_n^\alpha]}\forall\text{-}I, \qquad \frac{\begin{array}{c}\vdots\\ \forall X_n^\alpha A[X_n^\alpha]\end{array}}{A[\phi]}\forall\text{-}E,$$

$$\frac{\begin{array}{c}\vdots\\ A[\phi]\end{array}}{\exists X_n^\alpha A[X_n^\alpha]}\exists\text{-}I, \qquad \frac{\exists X_n^\alpha A[X_n^\alpha] \qquad \begin{array}{c}A[X_n^\alpha]\\ \vdots\\ B\end{array}}{B}\exists\text{-}E.$$

Here ϕ is a term-form whose type-form is α, and B does not contain X_n^α. $\forall$-I and $\exists$-E are assumed to satisfy the eigenvariable condition.
(2) Axioms and inference rules with respect to elementary arithmetic are obtained from those of **TRDB** by $*$-mapping. For example, axioms in (2. 2) of Definition 2.3 are of the following (see also the notation above Remark 3.3)

$$\frac{\begin{array}{c}\vdots\\ \mathbf{0 = St}\end{array}}{\mathbf{0 = 1}}, \qquad \frac{\begin{array}{c}\vdots\\ \mathbf{St = St'}\end{array}}{\mathbf{t = t'}}, \qquad \mathbf{I}_i^n(\mathbf{t}_1, \cdots, \mathbf{t}_i, \cdots, \mathbf{t}_n) = \mathbf{t}_i.$$

Moreover, we assume an axiom of the following: for an n-ary $\mathcal{BL}(c)$-term f and number terms $t_1, \cdots, t_n$,

$$app(id_0;\ f(t_1, \cdots, t_n)) = app(app(\cdots app(app(\mathbf{f};\ \mathbf{t}_1);\ \mathbf{t}_2); \cdots;\ \mathbf{t}_{n-1});\ \mathbf{t}_n).$$

(3) Bar induction of **S** is the same as that of **TRDB**.
(4) $TRD^*(G, \mathcal{I})$:

$$\forall x \forall y\ (\ \exists X^{\mathcal{R}[x,\ y]} \mathcal{H}(x,\ y,\ X^{\mathcal{R}[x,\ y]}) \Leftrightarrow G^*[x,\ y]\ ).$$

Here, $G[a,\ b]$ is the formula defined in $TRD(G, \mathcal{I})$ of **TRDB** (cf. Definition 2.3), and $G^*[a,\ b]$ is the formula obtained from $G[a,\ b]$ by the morphism $*$.
(5) $\Sigma_{\mathcal{H}}$-rule:

$$\frac{\begin{array}{c}\vdots\\ \exists X^{\mathcal{R}[j,t]} \mathcal{H}(j,\ t,\ X^{\mathcal{R}[j,t]})\end{array} \qquad \begin{array}{c}\vdots\\ j <_I i\end{array}}{\exists X^{\rho[j<_I i;\ \mathcal{R}[j,t]]} \Sigma_{\mathcal{H}}(j,\ i,\ t,\ X^{\rho[j<_I i;\ \mathcal{R}[j,t]]})}\Sigma_{\mathcal{H}}\text{-}I,$$

$$\frac{\begin{array}{c}\vdots\\ \exists X^{\rho[j<_I i;\ \mathcal{R}[j,t]]} \Sigma_{\mathcal{H}}(j,\ i,\ t,\ X^{\rho[j<_I i;\ \mathcal{R}[j,t]]})\end{array}}{\exists X^{\mathcal{R}[j,t]} \mathcal{H}(j,\ t,\ X^{\mathcal{R}[j,t]}) \wedge (j <_I i)}\Sigma_{\mathcal{H}}\text{-}E,$$

where i, j and t are arbitrary $\mathcal{BL}(c)$-terms.
(6) Implication axiom:

$$(A \supset \exists Y^\beta B[Y^\beta]) \Rightarrow \exists Y^\beta (A \supset B[Y^\beta]),$$

where A is $\exists$-free and B is any formula.
(7) Axiom of choice:

$$\forall \mathbf{X} \exists Y^\beta A[\mathbf{X},\ Y^\beta] \Rightarrow \exists Z^\gamma \forall \mathbf{X} A[\mathbf{X},\ Z^\gamma \mathbf{X}],$$

where A is an $\exists$-free formula, $\mathbf{X}$ is either a $\mathcal{BL}(c)$-variable x (in which case γ is $\{x\}\beta$) or a variable-form X^α (in which case γ is $\alpha \to \beta$).
(8) Product axiom:

$$\exists X^\alpha \exists Y^\beta A[X^\alpha,\ Y^\beta] \Rightarrow \exists Z^{\alpha \times \beta} A[\pi_1(Z^{\alpha \times \beta})/X^\alpha,\ \pi_2(Z^{\alpha \times \beta})/Y^\beta],$$

where A is $\exists$-free.

Remark 3.8 (1) Note that we restrict the axioms for equality of term-forms with type-form N. For example, **S** contains the axiom of the form $\mathbf{t = t}$, but does not contain $X^N = X^N$.
(2) In what follows, the bold-script upper case alphabet $\mathbf{X}$ expresses a variable-form X^α with the type-form α or a $\mathcal{BL}(c)$-variable x. Similarly with $\mathbf{Y}$ and $\mathbf{Z}$.

Proposition 3.9 (**Embedding of TRDB into S**)
TRDB $\vdash A$ implies **S** $\vdash A^*$ for any formula A. (See Definition 3.4 for $*$.)

Proof. Assume that there is a deduction Π of **TRDB** whose consequence is A. Let Π^* be the figure obtained form Π by replacing all formulas F of Π by F^*. By the axioms and the inference rules in Definition 3.7 (1), it is only a routine work to demonstrate that Π^* is a deduction of **S**. □

Next we define an interpretation MR on formulas of **S**.

Definition 3.10 (**Modified realizability interpretation** MR)
We define the modified realizability interpretation MR, which translates **S**-formulas to certain **S**-formulas. Given a formula A, we define $MR(A)$ by induction on the construction of A. $MR(A)$ will be of the form $\exists W^{\delta}\mathcal{A}[W^{\delta}]$, where $\mathcal{A}[\mathbf{W}]$ is an $\exists$-free formula. (In the case (1. 1) below, $\exists X$ is vacuous.)
(1) Suppose that A does not contain $\mathcal{H}$.
(1. 1) If A does not contain $\exists$-quantifier, then $MR(A) = A$.

In (1. 2)$\sim$(1. 5) below, A is assumed to contain an $\exists$-quantifier. We assume the induction hypotheses $MR(B) = \exists Y^{\beta}\mathcal{B}[Y^{\beta}]$ and $MR(C) = \exists Z^{\gamma}\mathcal{C}[Z^{\gamma}]$, where $\mathcal{B}$ and $\mathcal{C}$ do not contain $\exists$-quantifiers.
(1. 2) If $A = \exists X^{\alpha}B$, then $MR(\exists X^{\alpha}B) = \exists W^{\alpha\times\beta}\mathcal{B}[\pi_1(W^{\alpha\times\beta})/X^{\alpha}, \pi_2(W^{\alpha\times\beta})/Y^{\beta}]$. If $MR(B)$ is an $\exists$-free formula $\mathcal{B}$, then $MR(A) = \exists X^{\alpha}\mathcal{B}$.
(1. 3) If $A = B \wedge C$, then $MR(B \wedge C) = \exists W^{\beta\times\gamma}(\mathcal{B}[\pi_1(W^{\beta\times\gamma})/Y^{\beta}] \wedge \mathcal{C}[\pi_2(W^{\beta\times\gamma})/Z^{\gamma}])$.
In the case where $MR(B)$ is $\exists$-free, $MR(A) = \exists Z^{\gamma}(\mathcal{B} \wedge \mathcal{C}[Z^{\gamma}])$. If $MR(C)$ is $\exists$-free, then $MR(A) = \exists Y^{\beta}(\mathcal{B}[Y^{\beta}] \wedge \mathcal{C})$.
(1. 4) If $A = B \supset C$, then $MR(B \supset C) = \exists W^{\beta\to\gamma}\forall Y^{\beta}(\mathcal{B}[Y^{\beta}] \supset \mathcal{C}[W^{\beta\to\gamma}Y^{\beta}/Z^{\gamma}])$. If $MR(B)$ is $\exists$-free, then $MR(A) = \exists Z^{\gamma}(\mathcal{B} \supset \mathcal{C}[Z^{\gamma}])$. If $MR(C)$ is $\exists$-free, then $MR(A) = \forall Y^{\beta}(\mathcal{B}[Y^{\beta}] \supset \mathcal{C})$.
(1. 5) If $A = \forall \mathbf{X}B$, then $MR(\forall \mathbf{X}B) = \exists W^{\delta}\forall \mathbf{X}\mathcal{B}[\mathbf{X},\ W^{\delta}\mathbf{X}/Y^{\beta}]$. Here δ is determined as follows: δ is $\{x\}\beta$ if $\mathbf{X}$ is a $\mathcal{BL}(c)$-variable x; δ is $\alpha \to \beta$ if $\mathbf{X} = X^{\alpha}$. If $MR(B)$ is $\exists$-free, then $MR(A) = \forall \mathbf{X}\mathcal{B}$.
(2) A contains $\mathcal{H}$ as its subformula, that is, A is an $\mathcal{H}$-formula.
(2. 1) We define the MR-interpretation of basic $\mathcal{H}$-formulas as follows:

$$MR(\mathcal{H}(i,\ t,\ X^{\alpha})) = \mathcal{H}(i,\ t,\ X^{\alpha}).$$

$$MR(\Sigma_{\mathcal{H}}(j,\ i,\ t,\ X^{\alpha})) = \Sigma_{\mathcal{H}}(j,\ i,\ t,\ X^{\alpha}).$$

(2. 2) For the general cases of formulas which contain $\mathcal{H}$, the MR-interpretations can be defined from (2. 1) by applying (1. 1)$\sim$(1. 5).

In [2], we investigated the following theorem.

Theorem 3.11 (**Completeness theorem for** MR**-interpretation.**)
Let A be a formula in **S**. Then

$$\mathbf{S} \vdash MR(A) \Leftrightarrow A.$$

Proof. By induction on the structure of A. We show the theorem only in the case where $A = \exists Y^{\beta}B[Y^{\beta}]$. (For other cases, see the proof of Theorem 4.3 of [2].)

If $MR(B[Y^{\beta}])$ is $\exists$-free, then the result follows trivially. So, we assume that $MR(B[Y^{\beta}]) = \exists X^{\alpha}\mathcal{B}[Y^{\beta},\ X^{\alpha}]$. By Definition 3.10, $MR(\exists Y^{\beta}B[Y^{\beta}]) = \exists Z^{\beta\times\alpha}\mathcal{B}[\pi_1(Z^{\beta\times\alpha})/Y^{\beta},\ \pi_2(Z^{\beta\times\alpha})/X^{\alpha}]$. By the induction hypothesis, $\mathbf{S} \vdash MR(B[Y^{\beta}]) \Leftrightarrow B[Y^{\beta}]$. So, $\mathbf{S} \vdash \exists Y^{\beta}\exists X^{\alpha}\mathcal{B}[Y^{\beta},\ X^{\alpha}] \Leftrightarrow \exists Y^{\beta}B[Y^{\beta}]$. On the other hands, by the product axiom (and by usual constructive logic),

$$\mathbf{S} \vdash \exists Z^{\beta\times\alpha}\mathcal{B}[\pi_1(Z^{\beta\times\alpha})/Y^{\beta},\ \pi_2(Z^{\beta\times\alpha})/X^{\alpha}] \Leftrightarrow \exists Y^{\beta}\exists X^{\alpha}\mathcal{B}[Y^{\beta},\ X^{\alpha}].$$

So, we have the result. □

Definition 3.12 (MR**-interpretation of TRDB-formulas**)
We define the MR^*-interpretation, which sends a formula of **TRDB** to a formula of **S**, by: $MR^*(A) = MR(A^*)$ for every **TRDB**-formula A (cf. Definition 3.4).

Definition 3.13 (**MR-type of TRDB-formula**)
For an **TRDB**-formula A, we define $MRTYP(A)$, the type of $MR(A)$ as follows.
(i) If $MR^*(A) = \exists X^\alpha \mathcal{A}[X^\alpha]$, $MRTYP(A)$ is the type-form α.
(ii) If $MR^*(A)$ is an $\exists$-free formula, $MRTYP(A)$ is the type-form $\mathbf{1}$.

By induction on the construction of A, one can show the following proposition. For the detail of the proof, see the proof of Theorem 4.6 of [2].

Proposition 3.14 (**A relation between** $MRTYP$ **and** $[\![\ ]\!]$)
Let $[\![\ ]\!]$ be the interpretation defined in Definition 2.9, which maps formulas of **TRDB** to type-forms. Then, for every **TRDB**-formula A, $MRTYP(A) = [\![A]\!]$.

4 Term-forms: review

In this section, we present the definition of *term-forms* (terms with parameter types) and reduction rules of them, which were introduced by Yasugi and Hayashi in [5] and [6]. We repeat the definitions in some detail for the reader's convenience.

Definition 4.1 (**Continuous (Bar recursive) functionals**)
Let $T(f,\ x)$ be any recursive relation with a function argument f and a number argument x. It is known that if $\forall f \exists x T(f,\ x)$ is true, then there is a *neighborhood function* a (recursive) satisfying the following properties:
(CF1) $\forall f \exists n(a(f\lceil n) \neq 0)$, where $f\lceil n$ represents the restriction of f up to the argument $n-1$;
(CF2) $\forall z \forall s(a(z) \neq 0 \supset a(z * s) \neq 0)$, where $z * s$ denotes the sequence $\langle z_1, z_2, \cdots, z_n, s\rangle$ presuming that z represents $\langle z_1, z_2, \cdots, z_n\rangle$;
(CF3) $\forall f \forall n(a(f\lceil n) \neq 0 \supset T(f,\ a(f\lceil n) - 1))$.
For such an a, define

$$\mathbf{b}_a(f) = \min\{n\ ;\ a(f\lceil n) \neq 0\ \}.$$

In what follows, $\mathbf{b}_a$ stands for a *continuous (bar recursive) functional*, and $\mathbf{b}$ stands for a functional $\mathbf{b}_a$ for a neighborhood function a as explained above.

Proposition 4.2 Let a satisfy (CF1)-(CF3).
(1) $\forall f \exists n \forall g(f\lceil n = g\lceil n \supset \mathbf{b}_a(f) = \mathbf{b}_a(g))$.
(2) $m \geq \mathbf{b}_a(f)$ if and only if $a(f\lceil m) \neq 0$.

Definition 4.3 (**Term-forms**: See also Definition 4.1 of [5].)
Term-forms are defined below. $\phi : \alpha$ will express that term-form ϕ is of type-form α. If $\alpha \sim \beta$ (cf Theorem 2.8), then we let $\phi : \alpha$ imply $\phi : \beta$.
(1) $*$ is an atomic term-form whose type-form is $\mathbf{1}$.
(2) For a natural number n, id_n is an atomic term-form whose type-form $\{x^n\}\tau(n)$. Here x^n is an n-ary $\mathcal{BL}(c)$-variable.
(3) For each natural number n and for each type-form β, X_n^β (the nth variable-form of type-form β) is an atomic term-form.
(4) If $\phi : \gamma$, then $\lambda x.\phi : \{x\}\gamma$. If $\phi : \gamma$, then $\lambda X_n^\beta.\phi : \beta \to \gamma$, where the condition $\Gamma^*(\phi,\ \beta)$ below is assumed.

$\Gamma^*(\phi,\ \beta)$**:** For each X_n^δ, where $\delta \sim \beta$ and X_n^δ occurs freely in ϕ, no free $\mathcal{BL}(c)$-variable x in δ is bound by λx in ϕ.

(5) If $\phi : \{x\}\gamma$, then $app(\phi;\ t) : \gamma[t/x]$. If $\phi : \beta \to \gamma$ and $\psi : \beta$, then $app(\phi;\ \psi) : \gamma$. We abbreviate these as ϕt and $\phi\psi$ respectively.
(6) If $\phi : \beta$ and $\psi : \gamma$, then $cond[A;\ \phi,\ \psi] : cond[A;\ \beta,\ \gamma]$.
(7) If $\phi : \mathcal{R}[j,\ s]$, then $\sigma[j <_I i;\ \phi] : \rho[j <_I i;\ \mathcal{R}[j,\ s]]$.

(8) A functional constant μ of type $\{f\}N$, where f is a unary function $\mathcal{BL}(c)$-variable. (μ represents a modulus of finiteness functional of the order $<_I$. See the beginning of Preliminaries.)
(9) Bar recursion. Let $\mathbf{b}$ denote a functional evaluating a *barred tree*. If $\phi : \{z\}\gamma$ and $\psi : \{z\}(\{s\}\gamma[z * s/z] \to \gamma)$, then $\mathcal{B}[\mathbf{b};\ \phi,\ \psi;\ m,\ \ f] : \gamma[f\lceil m/z]$, where m and f are $\mathcal{BL}$-terms of arities respectively 0 and 1, and $f\lceil m$ represents the restriction of f to the domain m.

Notice that the terms of $\mathbf{S}$ defined in Definition 3.2 form a subset of these term-forms. The language of $\mathbf{S}$ is *not* extended to include these term-forms.

Definition 4.4 (**Conversion of term-forms**: See Definition 4.3 of [5].)
(1) Conversion rules of term-forms are the following.
(1. 1) $(\lambda x.\phi)t \Rightarrow \phi[t/x]$.
(1. 2) $(\lambda X^{\beta}_{n}.\phi)\psi \Rightarrow \phi[\psi/X^{\delta}_{n}]$, where $\beta \sim \delta$ and X^{δ}_{n} is free in ϕ. (ψ is substituted for X^{δ}_{n} for any such X^{δ}_{n}.)
(1. 3) $cond[A; \phi, \psi] \Rightarrow \chi$, where χ is ϕ if A is a true sentence and χ is ψ if A is a false sentence.
(1. 4) $\sigma[j <_I i; \phi] \Rightarrow \chi$, where χ is ϕ if $j <_I i$ is a true sentence and χ is $*$ if $j <_I i$ is a false sentence.
(1. 5) The conversion rule for $\mathcal{B}$ is as follows.

$$\mathcal{B}[\mathbf{b}; \phi, \psi, m, f] \Rightarrow app(\phi;\ f\lceil m),$$

if $\mathbf{b}(f) \leq m$ is a true sentence with $\mathbf{b}$ added to $\mathcal{BL}$, and

$$\mathcal{B}[\mathbf{b}; \phi, \psi, m, f] \Rightarrow app(app(\psi;\ f\lceil m);\ \lambda k.\mathcal{B}[\mathbf{b};\ \phi, \psi, m+1, (f\lceil m)\#k])$$

if $\mathbf{b}(f) > m$ is a true sentence. Here $(f\lceil m)\#k$ is a unary term being an extension of the finite sequence $f\lceil m$ of the form $\langle f(0),\ f(1), \cdots,\ f(m-1),\ k,\ 0,\ 0, \cdots\rangle$.
(1. 6) Let s and t be n-ary $\mathcal{BL}(c)$-terms. If t is the result of computation of s, then $app(id_n;\ s) \Rightarrow app(id_n;\ t)$.
(2) We can define "ϕ 1-reduces to ψ" and "ϕ is reducible to ψ" similarly to Definition 2.6, with the strategies that the conversions of σ and $\mathcal{B}$ have priorities of reduction. A term-form is said to be *normal* if it can not be reduced any longer.

Definition 4.5 (**Term system TRM**)
TRM is the system consisting of type-forms and term-forms together with their reductions.

Since the definition of type-forms depends on a central type-form η (cf. Remark 2.7), we should write **TRM** as $\mathbf{TRM}(\eta)$. However, since we fix such η, we abbreviate $\mathbf{TRM}(\eta)$ to **TRM** in what follows.

Theorem 4.6 (**Strong normalizability of TRM**; see Theorem 3 in [5])
Every term-form is strongly normalizable to a unique normal form (under the σ- and $\mathcal{B}$-strategies).

Notation. Let ϕ and ψ be term-forms. We denote $\phi \sim \psi$ (ϕ is equivalent to ψ) if ϕ and ψ have the same normal form.

5 Semantics for S-formulas

In this section, we define a truth-value ET of **S**-formulas under certain term-forms. As our main objective, we present the main theorem of this paper, which assures validity of **S** with respect to the truth-value ET. In order to define ET, we need *degree* of an **S**-formula.

Definition 5.1 (**Degree of S-formula**)
(1) For a primitive recursive order structure $\mathcal{I} = (I,\ <_I)$, which we assumed in defining **TRDB**, we define $\mathcal{I}^* = (I^*,\ <^*)$ as in [5] and [1].

$$I^{\sim} = \{i^{\sim}\ ;\ i \in I\}\ ;\ I^* = I \cup I^{\sim} \cup \{\infty\}\ ;\ i <^* i^{\sim} <^* j <^* \infty \text{ when } i <_I j.$$

Moreover, we put $\mathcal{I}_* = \omega^{\mathcal{I}^*}$, where we identify $\mathcal{I}^*$ with its order type.
(2) Let A be either $\mathcal{H}(i,\ t,\ X^{\alpha})$ or $\Sigma_{\mathcal{H}}(i,\ j,\ t,\ X^{\alpha})$. We define the *rank* $r(A)$ of A as follows.
(i) $r(\mathcal{H}(i,\ t,\ X^{\alpha})) = i^{\sim}$ if i is closed; $r(\mathcal{H}(i,\ t,\ X^{\alpha})) = \infty$ otherwise.
(ii) $r(\Sigma_{\mathcal{H}}(i,\ j,\ t,\ X^{\alpha})) = j$ if j is closed; $\Sigma_{\mathcal{H}}(i,\ j,\ t,\ X^{\alpha})) = \infty$ otherwise.
(3) Let A be a formula. Then, we define the *degree* $d(A)$ of A as follows.
(i) If A is an atomic formula except an $\mathcal{H}$-formula, then $d(A) = 1$.
(ii) $d(B \wedge C) = d(B \vee C) = d(B \supset C) = \max(d(B),\ d(C)) + 1$.
(iii) $d(\forall \mathbf{X} B[\mathbf{X}]) = d(\exists \mathbf{X} B[\mathbf{X}]) = d(B[\mathbf{X}]) + 1$, where $\mathbf{X}$ is a $\mathcal{BL}(c)$-variable or a variable-form.
(iv) $d(\mathcal{H}(i,\ t,\ X^{\alpha})) = \omega^{r(\mathcal{H}(i,\ t,\ X^{\alpha}))}$; $d(\Sigma_{\mathcal{H}}(i,\ j,\ t,\ X^{\alpha})) = \omega^{r(\Sigma_{\mathcal{H}}(i,\ j,\ t,\ X^{\alpha}))}$.

Lemma 5.2 (1) Let $G^*[i,\ t]$ be the formula which determines the axiom $TRD^*(G, \mathcal{I})$. (See Definition 3.7.) If i is closed, then $d(G^*[i,\ t]) <_* d(\mathcal{H}(i,\ t,\ X^{\alpha}))$.
(2) For every **S**-formula A, there exists a natural number n such that $d(A) = d(MR(A)) + n$.
(3) For every **S**-formula $A[x]$ and BL(c)-terms s and t, $d(A[s]) = d(A[t])$ whenever $s = t$.

Proof. (1) By (i) of (2) and (iv) of (3) in Definition 5.1, $d(\mathcal{H}(i,\ t,\ X^{\alpha})) = \omega^{i^{\sim}}$ and by (ii) of (2) and (3), $d(G^*[i,\ t]) = \omega^i \cdot m + n$ for some natural numbers m and n. Since $i <^* i^{\sim}$, it holds that $\omega^i \cdot m + n <_* \omega^{i^{\sim}}$ for all natural numbers m and n. So, we have $d(G^*[i,\ t]) <_* d(\mathcal{H}(i,\ t,\ X^{\alpha}))$.
(2) and (3) are easily proved by induction on the construction of A. □

We now define a truth-value of **S**-formulas. The following definition is essentially equivalent to that given in Section 4 of [4].

Definition 5.3 (**only-$\vec{x}$-open term-forms**)
Let $\vec{x}$ stand for a finite set of $\mathcal{BL}$-varialbes $x_1, \cdots, x_n$. A term-form ϕ is said to be only-$\vec{x}$-open if ϕ has no free $\mathcal{BL}(c)$-variable except $\vec{x}$.

Definition 5.4 (**Semantics**)
We consider **S** and **TRM** (See Definitions 3.2, 3.7, 4.3, 4.4, and 4.5).

Let $A[X_1^{\alpha_1}, \cdots,\ X_n^{\alpha_n},\ x_1, \cdots,\ x_m]$ be an $\exists$-free formula of **S**, where $X_1^{\alpha_1}, \cdots, X_n^{\alpha_n}$ are all free variable-forms of A and $x_1, \cdots,\ x_m$ are all free $\mathcal{BL}(c)$-variables of A. Note that α_i $(i = 1, \cdots, n)$ may contain x_k $(k = 1, \cdots, m)$. We abbreviate $A[X_1^{\alpha_1}, \cdots,\ X_n^{\alpha_n},\ x_1, \cdots,\ x_m]$ to $A[\vec{X}, \vec{x}]$.

Let $s_1, \cdots,\ s_m$ be closed $\mathcal{BL}(c)$-terms, where each s_k has the same arity as that of x_k, and let $\phi_1, \cdots, \phi_n$ be only-$\vec{x}$-open term-forms, where each ϕ_j has a type-form β_j with $\alpha_j[\vec{s}/\vec{x}] \sim \beta_j[\vec{s}/\vec{x}]$. We define the truth-value of $A[\vec{X}, \vec{x}]$ under environments of $\phi_1, \cdots, \phi_n$ for $X_1^{\alpha_1},\ \cdots,\ X_n^{\alpha_n}$ and $s_1, \cdots, s_m$ for $x_1,\ \cdots,\ x_m$, which is denoted by $ET(A[\vec{X}, \vec{x}];\ \vec{\phi}, \vec{s})$, by transfinite induction on the degree of $A[\vec{X}, \vec{x}][\vec{s}/\vec{x}]$. (If $n = m = 0$, we denote the truth-value of A (under the empty environment) by $ET(A)$.)
(1) $A[\vec{X}, \vec{x}]$ does not contain H.
(1. 1) $A[\vec{X}, \vec{x}]$ is an atomic formula, that is, $A[\vec{X}, \vec{x}]$ is of the form $\Phi[\vec{X}, \vec{x}] = \Psi[\vec{X}, \vec{x}]$. We define the truth-value by: $ET(A[\vec{X}, \vec{x}];\ \vec{\phi}, \vec{s}) = T\ (true)$ if $\Phi(\vec{\phi}[\vec{s}/\vec{x}],\ \vec{s})$ and $\Psi(\vec{\phi}[\vec{s}/\vec{x}], \vec{s})$ have the same normal form; $ET(A[\vec{X}, \vec{x}];\ \vec{\phi}, \vec{s}) = F\ (false)$ otherwise.
(1. 2) The connectives $\wedge$ and $\supset$ are interpreted classically.
(1. 3) $A[\vec{X}, \vec{x}]$ is of the form $\forall y B[\vec{X}, \vec{x}, y]$. If, for every closed $\mathcal{BL}(c)$-term t of the same arity as y, $ET(B[\vec{X}, \vec{x}, y];\ \vec{\phi}, \vec{s}, t) = T$, then $ET(A[\vec{X}, \vec{x}];\ \vec{\phi}, \vec{s}) = T$. $ET(A[\vec{X}, \vec{x}];\ \vec{\phi}, \vec{s}) = F$ otherwise.

(1. 3′) $A[\vec{X},\vec{x}]$ is of the form $\forall W^{\delta}B[W^{\delta},\vec{X},\vec{x}]$. If, for every only-$\vec{x}$-open term-form ψ^{γ} with $\delta[\vec{s}/\vec{x}] \sim \gamma[\vec{s}/\vec{x}]$, $ET(B[W^{\delta},\vec{X},\vec{x}];\ \psi,\vec{\phi},\vec{s}) = T$, then $ET(A[\vec{X},\vec{x}];\ \vec{\phi},\vec{s}) = T$. $ET(A[\vec{X},\vec{x}];\ \vec{\phi},\vec{s}) = F$ otherwise

(2) $A[\vec{X},\vec{x}]$ contains $\mathcal{H}$.

(2. 1) $A[\vec{X},\vec{x}]$ is of the form $\mathcal{H}(i,\ t,\ X_1^{\alpha_1})$. Let $\mathcal{G}[i,\ t,\ X^{\xi}]$ be the formula obtained from $MR(G^*[i,\ t]) = \exists X^{\xi}\mathcal{G}[i,\ t,\ X^{\xi}]$. Note that, by Proposition 3.14 and Definitions 2.9 and 2.6, it holds that $\alpha_1 \sim \xi$. We define the truth-value by

$$ET(\mathcal{H}(i,\ t\ X_1^{\alpha_1});\ \phi_1,\vec{s}) = ET(\mathcal{G}[i,\ t,\ X^{\xi}];\ \phi_1,\vec{s}).$$

(2. 2) $A[\vec{X},\vec{x}]$ is of the form $\Sigma_{\mathcal{H}}(j,\ i,\ t,\ X_1^{\alpha_1})$. By the definition of $\Sigma_{\mathcal{H}}$-formulas, $\alpha_1 \sim \rho[j <_I i;\ \mathcal{R}[j,\ t]]$, and hence, the type-form β_1 of ϕ_1 satisfies that $\beta_1[\vec{s}/\vec{x}] \sim \alpha_1[\vec{s}/\vec{x}] \sim \rho[j <_I i;\ \mathcal{R}[j,\ t]][\vec{s}/\vec{x}]$. Therefore, if $j[\vec{s}/\vec{x}] <_I i[\vec{s}/\vec{x}]$, then $\beta_1[\vec{s}/\vec{x}] \sim \mathcal{R}[j[\vec{s}/\vec{x}],\ t[\vec{s}/\vec{x}]]$ by Definition 2.6 (1. 3).

We define the truth-value as follows.

(i) If $ET(j <_I i;\ \vec{s}) = T$, that is, $j[\vec{s}/\vec{x}] <_I i[\vec{s}/\vec{x}]$, then

$$ET(\Sigma_{\mathcal{H}}(j,\ i,\ t, X_1^{\alpha_1});\ \phi_1,\vec{s}) = ET(\mathcal{H}(\bar{j},\ \bar{t},\ X_1^{\mathcal{R}[\bar{j},\bar{t}]});\ \phi_1[\vec{s}/\vec{x}]),$$

where $\bar{j} = j[\vec{s}/\vec{x}]$ and $\bar{t} = t[\vec{s}/\vec{x}]$.

(ii) If $ET(j <_I i;\ \vec{s}) = F$, then

$$ET(\Sigma_{\mathcal{H}}(j,\ i,\ t, X_1^{\alpha_1});\ \phi_1,\vec{s}) = F.$$

Note that $x <_I y$ is an $\mathcal{H}$-free formula with the intended meaning of the order of $\mathcal{I}$, which is a primitive recursive predicate. (See Definition 3.7 (2).)

(2. 3) If A is an $\mathcal{H}$-formula which is not atomic, then we follow the cases in (1).

Proposition 5.5 *ET* is well-defined.

Proof. By transfinite induction on the degree of $A[\vec{X},\vec{s}]$ in the above definition. It suffices to show this lemma in the case where $A[\vec{X},\vec{x}] = \mathcal{H}(i,\ t,\ X^{\alpha})$.

Let $\mathcal{G}[i,\ t,\ X^{\xi}]$ be the formula defined in Definition 5.4 (2. 1) and let $\vec{s}$ be closed $\mathcal{BL}(c)$-terms in which each s_k has the same arity as that of x_k. By Lemma 5.2, we obtain

$$d(\mathcal{G}[i[\vec{s}/\vec{x}],\ t[\vec{s}/\vec{x}],\ X^{\xi[\vec{s}/\vec{x}]}]) <_* d(\mathcal{H}(i[\vec{s}/\vec{x}],\ t[\vec{s}/\vec{x}],\ X^{\alpha[\vec{s}/\vec{x}]})).$$

Therefore, $ET(\mathcal{G}[i,\ t,\ X^{\xi}];\ \phi,\ \vec{s})$ is well-defined for every only-$\vec{x}$-open term-form ϕ^{β} with $\beta[\vec{s}/\vec{x}] \sim \alpha[\vec{s}/\vec{x}]$, and hence, so is $ET(\mathcal{H}(i,\ t,\ X^{\alpha});\ \phi,\ \vec{s})$. □

Lemma 5.6 (1) Given $\vec{X}$, $\vec{x}$, $\vec{\phi}$ and $\vec{s}$ defined in Definition 5.4, we consider an $\exists$-free formula $A[X^{\alpha},\vec{X},\vec{x}]$. Let $\Phi[Z_1^{\gamma_1},\cdots,Z_p^{\gamma_p}]$ be an **S**-term having a type-form β with $\alpha[\vec{s}/\vec{x}] \sim \beta[\vec{s}/\vec{x}]$. We also let $\psi_1,\cdots,\psi_p$ and ξ be only-$\vec{x}$-open term-forms, where ψ_j $(j = 1,\cdots,p)$ has a type-form δ_j with $\delta_j[\vec{s}/\vec{x}] \sim \gamma_j[\vec{s}/\vec{x}]$ and ξ has a type-form δ with $\delta[\vec{s}/\vec{x}] \sim \alpha[\vec{s}/\vec{x}]$. (We also abbreviate $Z_1^{\gamma_1},\cdots,Z_p^{\gamma_p}$ to $\vec{Z}$.) If $\Phi[\psi_1,\cdots,\psi_p][\vec{s}/\vec{x}]$ and $\xi[\vec{s}/\vec{x}]$ share a normal form, then

$$ET(A^*[\vec{Z},\vec{X},\vec{x}];\ \vec{\psi},\vec{\phi},\vec{s}) = ET(A[X^{\alpha},\vec{X},\vec{x}];\ \xi,\vec{\phi},\vec{s}),$$

where $A^*[\vec{Z},\vec{X},\vec{x}] = A[\Phi[\vec{Z}],\vec{X},\vec{x}]$.

(2) Assume A and $\vec{\phi},\vec{s}$ which are given in the above (1), and let ψ_1 and ψ_2 be only-$\vec{x}$-open term-forms, where ψ_j has a type-form α_j $(j = 1,2)$ with $\alpha_1[\vec{s}/\vec{x}] \sim \alpha_2[\vec{s}/\vec{x}] \sim \alpha[\vec{s}/\vec{x}]$. If $\psi_1[\vec{s}/\vec{x}]$ and $\psi_2[\vec{s}/\vec{x}]$ share a normal form, then

$$ET(A[X^{\alpha},\vec{X},\vec{x}];\ \psi_1,\vec{\phi},\vec{s}) = ET(A[X^{\alpha},\vec{X},\vec{x}];\ \psi_2,\vec{\phi},\vec{s}).$$

Proof. (2) is a particular case of (1) where $\Phi[Z_1^{\gamma_1}, \cdots, Z_p^{\gamma_p}] = Z_1^{\gamma_1}$. (1) is easily shown by transfinite induction over the definition of ET, that is, by transfinite induction on the degree of $A[X^\alpha, \vec{X}, \vec{x}][\vec{s}/\vec{x}]$. □

Remark 5.7 Preceding the following theorem, we rewrite axioms of **S** as inference rules. The resulting system is equivalent to the original system.
(1) Definition by transfinite recursion for $\mathcal{H}$:

$$\frac{G^*[i,\ t]}{\exists X^{\mathcal{R}(i,\ t)}\mathcal{H}(i,\ t,\ X^{\mathcal{R}(i,\ t)})}\mathcal{H}\text{-}I \qquad \frac{\exists X^{\mathcal{R}(i,\ t)}\mathcal{H}(i,\ t,\ X^{\mathcal{R}(i,\ t)x})}{G^*[i,\ t]}\mathcal{H}\text{-}E.$$

(2) Implication axiom:

$$\frac{A \supset \exists Y^\beta B[Y^\beta]}{\exists Y^\beta(A \supset B[Y^\beta])}IMP,$$

where A is $\exists$-free and B is any formula.
(3) Axiom of choice:

$$\frac{\forall \mathbf{X}\exists Y^\beta A[\mathbf{X},\ Y^\beta]}{\exists Y^\gamma \forall \mathbf{X} A[\mathbf{X},\ Y^\gamma \mathbf{X}]}AC,$$

where A is an $\exists$-free formula, $\mathbf{X}$ is either a $\mathcal{BL}(c)$-variable x (in which case γ is $\{x\}\beta$) or a variable-form X^α (in which case γ is $\alpha \to \beta$).
(4) Product axiom:

$$\frac{\exists X^\alpha \exists Y^\beta A[X^\alpha,\ Y^\beta]}{\exists Z^{\alpha\times\beta} A[\pi_1(Z^{\alpha\times\beta})/X^\alpha,\ \pi_2(Z^{\alpha\times\beta})Y^\beta]}PA,$$

where A is $\exists$-free.

The following theorem can be proved in a manner similar to the proof of the Main Theorem in Section 4 of [4], adding the new cases of implication axiom, axiom of choice, and product axiom.

Theorem 5.8 (Validity of S-theorem)
Let $A\ (= A[\vec{Y}, \vec{y}])$ be a theorem of **S**, where $\vec{Y}\ (= Y_1^{\beta_1}, \cdots, Y_n^{\beta_n})$ are all free variable-forms of A and $\vec{y}\ (= y_1, \cdots, y_m)$ are all $\mathcal{BL}(c)$-variables of A. If $MR(A[\vec{Y},\ \vec{y}]) = \exists X^\alpha \mathcal{A}[X^\alpha, \vec{Y}, \vec{y}]$, then there exists a term-form $\Phi\ (= \Phi[\vec{Y}, \vec{y}])$ satisfying the following:
(i) Φ has no free variable (-form) except $\vec{Y}, \vec{y}$. (That is, Φ is only-$\vec{y}$-open.)
(ii) For all closed $\mathcal{BL}(c)$-terms $\vec{t}\ (= t_1, \cdots, t_m)$, where the arity of t_j is the same as that of y_j, and for all only-$\vec{y}$-open term-forms $\vec{\psi}\ (= \psi_1, \cdots, \psi_n)$, where ψ_i has a type-form β_i' with $\beta_i'[\vec{t}/\vec{y}] \sim \beta_i[\vec{t}/\vec{y}]$, it holds that

$$ET(\mathcal{A}[X^\alpha, \vec{Y}, \vec{y}];\ \Phi[\vec{\psi}/\vec{Y}], \vec{\psi}, \vec{t}) = T.$$

Remark 5.9 (1) In the above theorem, if $MR(A[\vec{Y}, \vec{y}])$ is $\exists$-free, then $ET(MR(A);\ \vec{\psi},\ \vec{t}) = T$ for all $\vec{t}$ and $\vec{\psi}$.
(2) If A is a closed formula, then the term-form Φ above is closed.

Proof of Theorem 5.8. We prove this theorem in an extended form as follows.

Let Π be a deduction of **S**, where A is the consequence of Π and $A_1, A_2, \cdots, A_n$ are live assumptions in Π. Note that we can let any live assumption do not have any free variable(-form) except those of A.

Put

$$\mathcal{A}_\Pi^* = MR(A_1 \supset (A_2 \supset (\cdots \supset (A_n \supset A)\cdots))) = \exists X^\alpha \mathcal{A}_\Pi[X^\alpha].$$

We prove the following proposition $(*)$: There exists a term-form $\Phi[\vec{Y}, \vec{y}]$ of **TRM** such that $ET(\mathcal{A}_\Pi[X^\alpha, \vec{Y}, \vec{y}];\ \Phi[\vec{\psi}/\vec{Y}], \vec{\psi},\ \vec{t}) = T$ for all $\vec{\psi}$, $\vec{t}$ satisfying the condition (ii) of this theorem.

In particular, this means $ET(\mathcal{A}[X^\alpha, \vec{Y}, \vec{y}];\ \Phi[\vec{\psi}/\vec{Y}], \vec{\psi}, \vec{t}) = T$ if Π has no live assumptions. This completes the proof of the theorem.

Note. (1) For term-forms $\phi_1, \cdots, \phi_n$, the list $\phi_1 \cdots \phi_n$ will stand for the expression $app(app(\cdots app(\phi_1;\ \phi_2); \cdots; \phi_{n-1});\ \phi_n)$ for short.
(2) In what follows, we abbreviate a term-form $\lambda x.cond[x = 0;\ \phi,\ cond[x = 1; \psi,\ *]]$ (*pairing*) to $\langle \phi,\ \psi \rangle$.
(3) In the proof, we omit the type-form of a variable-form whenever it is not important what the type-form is.

We show $(*)$ by induction on the structure of Π.
(1) Π consists of only an assumption or an axiom.
(1. 1) Π consist of an assumption alone. Let $MR(A) = \exists X^\alpha \mathcal{A}[X, \vec{Y}, \vec{y}]$. Then

$$MR(A \supset A) = \exists Z^{\alpha\to\alpha} \forall X^\alpha (\mathcal{A}[X, \vec{Y}, \vec{y}] \supset \mathcal{A}[(ZX)/X, \vec{Y}, \vec{y}])\ (= \exists Z^{\alpha\to\alpha} \mathcal{A}_\Pi[Z]).$$

It holds that $(\lambda X^\alpha.X^\alpha)\phi \Rightarrow \phi$ for every ϕ^β with $\alpha \sim \beta$. Therefore, by Lemma 5.6,

$$ET(\mathcal{A}[ZX, \vec{Y}, \vec{y}];\ \lambda X^\alpha.X^\alpha,\ \phi, \vec{\psi}, \vec{t}) = ET(\mathcal{A}[X];\ \phi, \vec{\psi}, \vec{t})$$

for every ϕ^β, $\vec{\psi}$ and $\vec{t}$. So, by Definition 5.4 (1. 3), we have

$$ET(\forall X^\alpha(\mathcal{A}[X, \vec{Y}, \vec{y}] \supset \mathcal{A}[ZX, \vec{Y}, \vec{y}]);\ \lambda X^\alpha.X^\alpha, \vec{\psi}, \vec{t}) = T.$$

(1. 2) Π consists of a closed axiom. An axiom of **S** is an arithmetic true sentence (cf. Definition 3.7 and (2)$\sim$(3) in Definition 2.3). By Definition 5.4 (1. 1), this case is trivial.
(2) Π has **R** as its last inference rule.
(2. 2) If **R** is either implication axiom, axiom of choice or product axiom, then $(*)$ follows immediately since there is only one premise of **R** and the premise formula is translated to the *MR*-interpretation of the consequence formula. For example, if **R** is the inference rule obtained from product axiom (cf. Remark 5.7 (4)), then for the premise formula $\exists X^\alpha \exists Y^\beta A[X,\ Y]$,

$$MR(\exists X^\alpha \exists Y^\beta A[X,\ Y]) = \exists Z^{\alpha\times\beta} A[\pi_1(Z)/X,\ \pi_2(Z)/Y]$$

since A is $\exists$-free.
(2. 3) **R** is a logical inference rule.
(2. 3. 1) **R** is a $\forall$-introduction. Let Π be of the following form

$$\frac{\begin{array}{c} A_1 \\ \Sigma \\ A[x] \end{array}}{\forall x A[x]}\ \mathbf{R}.$$

It will suffice to show the result in the case $MR(A_1)$ and $MR(A)$ have $\exists$-quantifiers. (For simplifying, we assume A_1 is the solo live assumption in Σ.) Let $MR(A_1) = \exists X_1^{\alpha_1} \mathcal{A}_1[X_1, \vec{Y}, \vec{y}]$, $MR(A[x]) = \exists X^\alpha \mathcal{A}[X, \vec{Y}, x, \vec{y}]$. Then, $\mathcal{A}^*_\Sigma = \exists Z^{\alpha_1\to\alpha} \forall X_1(\mathcal{A}_1[X_1, \vec{Y}, \vec{y}] \supset \mathcal{A}[ZX_1/X, \vec{Y}, x, \vec{y}])$. By the induction hypotheses, there exists term-form Ψ $(= \Psi[\vec{Y}, x, \vec{y}])$ such that, for any $\vec{\psi}$, $\vec{t}$ and s, $ET(\forall X_1(\mathcal{A}_1[X_1, \vec{Y}, \vec{y}] \supset \mathcal{A}[ZX_1, \vec{Y}, x, \vec{y}]);\ \Psi[\vec{\psi}/\vec{Y}], \vec{\psi}, s, \vec{t}) = T$. That is, For all $\vec{\psi}$, $\vec{t}$ and for any ϕ^γ with $\gamma[\vec{t}/\vec{y}] \sim \alpha_1[\vec{t}/\vec{y}]$, if $ET(\mathcal{A}_1[X_1, \vec{Y}, \vec{y}];\ \phi, \vec{\psi}, \vec{t}) = T$, then for every s

$$ET(\mathcal{A}[W, \vec{Y}, x, \vec{y}]);\ (\Psi[\vec{\psi}])\phi, \vec{\psi}, s, \vec{t}) = T. \qquad (2.3.1)$$

Using Ψ, we define a term-form Φ by

$$\Phi = \lambda X_1.\{x\}(\Psi X_1).$$

Since $((\Phi[\vec{\psi}])\phi)s \sim ((\Psi[\vec{\psi}])\phi)[s/x]$, it follows that $ET(\forall x\mathcal{A}[Wx, \vec{Y}, x, \vec{y}];\ (\Phi[\vec{\psi}])\phi,\ \vec{\psi}, \vec{t}) = T$ from (2.3.1). So, we have that $ET(\mathcal{A}_\Pi[Z, \vec{Y}, \vec{y}]; \Phi[\vec{\psi}], \vec{\psi}, \vec{t}) = T$ where

$$\mathcal{A}_\Pi = \forall X_1(\mathcal{A}_1[X_1, \vec{Y}, \vec{y}] \supset \forall x\mathcal{A}[ZX_1x, \vec{Y}, x, \vec{y}]).$$

Note. In order to simplify the discussion, in what follows, we let A be closed, that is, $\vec{Y}$ and $\vec{y}$ be empty. Therefore, we can let all live assumptions $A_1, \cdots, A_n$ be closed.

(2. 3. 2) **R** is a $\wedge$-introduction. Let Π be of the following form

$$\dfrac{\begin{array}{cc} A_1 & B_1 \\ \Pi_1 & \Pi_2 \\ A & B \end{array}}{A \wedge B}\ \mathbf{R}.$$

It will suffice to show the result in the case $MR(A_1)$, $MR(B_1)$, $MR(A)$ and $MR(B)$ have $\exists$-quantifiers. (For simplifying, we assume A_1 and B_1 are the solo live assumptions in Π_1 and Π_2.) Let $MR(A_1) = \exists X_1^{\alpha_1}\mathcal{A}_1[X_1]$, $MR(B_1) = \exists Y_1^{\beta_1}\mathcal{B}_1[Y_1]$, $MR(A) = \exists X^\alpha\mathcal{A}[X]$ and $MR(B) = \exists Y^\beta\mathcal{B}[Y]$. Then, $\mathcal{A}^*_{\Pi_1} = \exists Z_1^{\alpha_1\to\alpha}\forall X_1(\mathcal{A}_1[X_1] \supset \mathcal{A}[Z_1X_1/X])$ and $\mathcal{A}^*_{\Pi_2} = \exists Z_2^{\beta_1\to\beta}\forall Y_1(\mathcal{B}_1[Y_1] \supset \mathcal{B}[Z_2Y_1/Y])$. Moreover, for W of the type $\alpha_1 \to (\beta_1 \to (\alpha \times \beta))$, it holds that

$$\mathcal{A}^*_\Pi = \exists W\forall X_1\forall Y_1(\mathcal{A}_1[X_1] \supset (\mathcal{B}_1[Y_1] \supset (\mathcal{A}[\pi_1(WX_1Y_1)] \wedge \mathcal{B}[\pi_2(WX_1Y_1)]))).$$

By the induction hypotheses, there exist term-forms Φ_1 and Ψ_1 such that, for any ϕ_1 and ϕ_2 with $ET(\mathcal{A}_1[X_1];\ \phi_1) = T$ and $ET(\mathcal{B}_1[Y_1];\ \phi_2) = T$ respectively, it holds that $ET(\mathcal{A}[X];\ \Phi_1\phi_1) = T$ and $ET(\mathcal{B}[Y];\ \Phi_2\phi_2) = T$. Using Φ_1 and Φ_2 we define Φ by:

$$\Phi = \lambda X_1Y_1.\langle \Phi_1X_1,\ \Phi_2Y_1\rangle.$$

For any ϕ_1 and ϕ_2, we can have $\pi_i(\Phi\phi_1\phi_2) \sim \pi_i(\langle \Phi_1\phi_1,\ \Phi_2\phi_2\rangle) \sim \Phi_i\phi_i$. Therefore, for any ϕ_1 and ϕ_2 with $ET(\mathcal{A}_1[X_1];\ \phi_1) = T$ and $ET(\mathcal{B}_1[Y_1];\ \phi_2) = T$ respectively, it holds that

$$ET(\mathcal{A}[X] \wedge \mathcal{B}[Y];\ \pi_1(\Phi\phi_1\phi_2),\ \pi_2(\Phi\phi_1\phi_2)) = T.$$

So, $ET(\mathcal{A}_\Pi[W];\ \Phi) = T$.

(2. 3. 3) **R** is an $\supset$-elimination. Let Π be of the following form

$$\dfrac{\begin{array}{cc} A_1 & B_1 \\ \Pi_1 & \Pi_2 \\ A & A \supset B \end{array}}{B}\ \mathbf{R}.$$

It will suffice to show the results in the case

$$\mathcal{A}^*_{\Pi_1} = \exists Z_1\forall X_1(\mathcal{A}_1[X_1] \supset \mathcal{A}[Z_1X_1/X]) \text{ and}$$

$$\mathcal{A}^*_{\Pi_2} = \exists Z_2\forall Y_1\forall X(\mathcal{B}_1[Y_1] \supset (\mathcal{A}[X] \supset \mathcal{B}[Z_2Y_1X/Y])).$$

By the induction hypotheses, for $i = 1, 2$, there exists a term-form Φ_i with $ET(\mathcal{A}_{\Pi_i}[X_i];\ \Phi_i) = T$. Using Φ_1 and Φ_2, we define Φ by:

$$\Phi = \lambda X_1Y_1.(\Phi_2Y_1(\Phi_1X_1)).$$

By Definition 5.4 (1. 2), it suffices to show that, for any ϕ_1 and ψ_1 with $ET(\mathcal{A}_1[X_1];\ \phi_1) = T$ and $ET(\mathcal{B}_1[Y_1];\ \psi_1) = T$ respectively, it holds that $ET(\mathcal{B}[Y];\ \Phi\phi_1\psi_1) = T$. Given such term-forms ϕ_1 and ϕ_2, it holds that

$$ET(\mathcal{A}[X];\ \Phi_1\phi_1) = T \text{ and } \ ET(\forall X(\mathcal{A}[X] \supset \mathcal{B}[Y]);\ \Phi_2\psi_1 X)) = T. \qquad (1)$$

By Definition 5.4 (1. 2), (1) implies that $ET(\mathcal{B}[Y];\ \Phi_2\psi_1(\Phi_1\phi_1)) = T$. Therefore, since $\Phi\phi_1\psi_1$ and $\Phi_2\psi_1(\Phi_1\phi_1)$ share a normal form, we have the conclusion.
(2. 3. 4) If $\mathbf{R}$ is any logical inference rule except the above cases, then we can easily treat it similarly to (2. 3. 1) or (2. 3. 3).
(2. 4) $\mathbf{R}$ is a Bar induction. Let Π be of the following form

$$\dfrac{\begin{array}{cc} A_1 & A_2 \\ \Pi_1 & \Pi_2 \\ \forall z(R[z] \supset A[z]) & \forall z(\forall x A[z * x] \supset A[z]) \end{array}}{A[t]}\ \mathbf{BI},$$

where t is a number term. It will suffice to show the result in the case where $MR(A[z]) = \exists X^{\alpha}\mathcal{A}[z,\ X]$;
$\mathcal{A}^*_{\Pi_1} = \exists Z^{\alpha_1 \to \{z\}\alpha}\forall X_1^{\alpha_1}(\mathcal{A}_1[X_1] \supset \forall z(R[z] \supset \mathcal{A}[z,\ ZX_1z/X]))$; and
$\mathcal{A}^*_{\Pi_2} = \exists W^{\delta}\forall X_2^{\alpha_2}(\mathcal{A}_2[X_2] \supset \forall z\forall Y^{\{x\}\alpha}(\forall x\mathcal{A}[z * x,\ Yx/X] \supset \mathcal{A}[z,\ \mathbf{W}/X]))$,
where $\delta = \alpha_2 \to \{z\}(\{x\}\alpha \to \alpha)$ and $\mathbf{W} = app(app(app(W^{\delta};\ X_2^{\alpha_2});\ z);\ Y^{\{x\}\alpha}) = WX_2zY$.
By the induction hypotheses, there exist term-forms Φ_1 and Φ_2 satisfying (I) and (II) below.
(I) For any term-form χ_1 (with type α_1) satisfying $ET(\mathcal{A}_1[X_1];\ \chi_1) = T$ and for any closed number term s_1 satisfying $ET(R[z];\ s_1) = T$, it holds that $ET(\mathcal{A}[z,\ X];\ \Phi_1\chi_1 s_1,\ s_1) = T$.
(II) Let χ_2 be a term-form (with type α_2) satisfying $ET(\mathcal{A}_2[X_2];\ \chi_2) = T$, let ϕ be a term-form (with type $\{x\}\alpha$), and let s_2 be a closed number term. If ϕ and s_2 satisfy that $ET(\mathcal{A}[z,\ X];\ \phi t,\ s_2 * t) = T$ for any closed number term t, then $ET(\mathcal{A}[z,\ X];\ \Phi_2\chi_2 s_2\phi,\ s_2) = T$.

Using Φ_1 and Φ_2, we define $\Phi[m,\ f]$ by

$$\Phi[m,\ f] = \lambda V_1^{\alpha_1}V_2^{\alpha_2}.\mathcal{B}[\mathbf{b};\ \Phi_1V_1,\ \Phi_2V_2;\ m,\ f],$$

where $\mathbf{b}$ is a continuous functional corresponding to the elementary formula $R[z]$, that is, $\mathbf{b}$ is defined by: $\mathbf{b}(f) = \min\{n :\ R[f\lceil n]\}$ for each functional term f. ($f\lceil n$ denotes the f restricted to the domain n. For $\mathbf{b}$, see also Definition 4.1.) Let $\mathbf{Q}(a)$, where a ranges over natural numbers, stand for the following proposition:

$\mathbf{Q}(a)$: Let f be a closed unary term, and let m be a closed number term. We also let χ_1 and χ_2 be term-forms as in conditions (I) and (II) respectively. Then,
$ET(\mathcal{A}[z,\ X];\ \mathcal{B}[\mathbf{b}; \Phi_1\chi_1,\ \Phi_2\chi_2;\ m,\ f],\ f\lceil m) = T$ if $f\lceil m = a$.

We will prove $\forall z\mathbf{Q}(z)$. Then, we have $\mathbf{Q}(t)$, that is,

$$ET(\mathcal{A}[z,\ X] :\ \mathcal{B}[\mathbf{b}; \Phi_1\chi_1,\ \Phi_2\chi_2;\ lg(t),\ t^*],\ t) = T,$$

where $lg(t)$ stands for the length of t and t^* stands for a functional argument which is an extension of t. By the above conditions (I) and (II), and by Lemma 5.6, $\mathbf{Q}(t)$ implies that $\Phi[lg(t),\ t^*]$ is a term-form satisfying this theorem.

Let $\mathbf{P}(a)$ stand for $\forall f\forall m(a = f\lceil m \Rightarrow \mathbf{b}(f) \leq m)$. By an informal Bar induction, we can complete the proof if we can establish the following propositions:

Hyp 1 $\forall f\forall l\,(\mathbf{P}(f\lceil l) \Rightarrow \forall m > l\ \mathbf{P}(f\lceil m))$;
Hyp 2 $\forall f\exists n\mathbf{P}\,(f\lceil n)$;
Hyp 3 $\forall w\,(\mathbf{P}(w) \Rightarrow \mathbf{Q}(w))$;

Hyp 4 $\forall w\,(\forall k\mathbf{Q}(w*k) \Rightarrow \mathbf{Q}(w))$.

Hyp 1 and Hyp 2 are trivial by the properties of $\mathbf{b}$ and $R[z]$.
Hyp 3: If $f\lceil m = a$, then $\mathbf{b}(f) \leq m$. So, $\mathcal{B}[\mathbf{b};\ \Phi_1\chi_1,\ \Phi_2\chi_2;\ m,\ f]$ is converted to $\Phi_1\chi_1(f\lceil m)$, and $R[f\lceil m]$ by the definition of $\mathbf{b}$. From the condition (I), $ET(\mathcal{A}[z,\ X];\ \Phi_1\chi_1(f\lceil m),\ f\lceil m) = T$.
Hyp 4: If $\mathbf{b}(f) \leq m$, then the proof goes in the same way as that of Hyp 3. Let $\mathbf{b}(f) > m$. It suffices to show that if f^*, an extension of $f\lceil m$, satisfies that $ET(\mathcal{A}[z,\ X] :\ \mathcal{B}[\mathbf{b};\Phi_1\chi_1,\ \Phi_2\chi_2;\ m+1,\ f^*],\ f^*\lceil(m+1)) = T$, then $ET(\mathcal{A}[z,\ X] :\ \mathcal{B}[\mathbf{b};\Phi_1\chi_1,\ \Phi_2\chi_2;\ m,\ f],\ f\lceil m) = T$. Since $\mathbf{b}(f) > m$, $\mathcal{B}[\mathbf{b};\ \Phi_1\chi_1,\ \Phi_2\chi_2;\ m,\ f]$ is converted to

$$app(app(\Phi_2\chi_2;\ f\lceil m);\ \lambda k.\mathcal{B}[\mathbf{b};\Phi_1\chi_1,\ \Phi_2\chi_2;\ m+1,\ (f\lceil m)\#k]), \qquad (**)$$

where $(f\lceil m)\#k$ is a unary term defined in Definition 4.4 (1. 5). We abbreviate $\lambda k.\mathcal{B}[\mathbf{b};\Phi_1\chi_1,\ \Phi_2\chi_2;\ m+1,\ (f\lceil m)\#k]$ to $\mathcal{B}$ and the above term $(**)$ to $\Phi_2\chi_2(f\lceil m)\mathcal{B}$.

Obviously, $((f\lceil m)\#k)\lceil(m+1) = (f\lceil m)*k$. So, by the hypothesis, for every closed number term s, $ET(\mathcal{A}[z,\ X] :\ \mathcal{B}s,\ (f\lceil m)*s) = T$. Therefore, by the condition (II), $ET(\mathcal{A}[z,\ X] :\ \Phi_2\chi_2(f\lceil m)\mathcal{B},\ f\lceil m) = T$. By Lemma 5.6, we have $ET(\mathcal{A}[z,\ X] :\ \mathcal{B}[\mathbf{b};\Phi_1\chi_1,\ \Phi_2\chi_2;\ m,\ f],\ f\lceil m) = T$.
(2. 5) $\mathbf{R}$ is either of TRG^* or $\Sigma_{\mathcal{H}}$-rule. According to Definition 5.4 (2. 1), one can easily show the result for this case, since the ET-value of the formula obtained from MR(the premise formula of $\mathbf{R}$) is the same as that of the consequence of $\mathbf{R}$. □

As an immediate consequence of Theorem 5.8, we have the following result with respect to the existence of a function.

Corollary 5.10 For any Π^0_2-sentence of arithmetic, say $\forall x\exists y A[x,\ y]$, if $\forall x\exists y\ A[x,\ y]$ is a theorem of $\mathbf{S}$, then there exists a closed term-form Φ such $ET(\forall x A[x, Yx];\ \Phi) = T$. That is, all functions which are provably total in $\mathbf{S}$ can be realized in **TRM**.

Corollary 5.11 $\mathbf{S}$ is consistent.

Proof. If $0 = 1$ is provable in $\mathbf{S}$, then $ET(MR(0 = 1)) = ET(0 = 1) = T$ by Theorem 5.8. On the other hand, $ET(0 = 1) = F$ by Definition 5.4. These yield a contradiction. □

Remark 5.12 In Theorem 5.8, it turned out that given a formula $\exists X^\alpha\mathcal{A}[X^\alpha]$, the MR-interpretation of a theorem in the course of the proof $\mathbf{S}$, not only can we show the existence of a term Φ with $ET(\mathcal{A}[X^\alpha];\ \Phi) = T$, but also we can indeed construct such a term Φ from a deduction. Furthermore, let us compare the term-form obtained in Definition 8.3 of [5] with regards to **TRDB**-formulas. For **TRDB**-formulas, the construction algorithm of $\Phi = \Phi_\Pi$ in Theorem 5.8 amounts to the same as $[\![\Pi]\!]$ obtained in §4 of Part III in [4]. This fact and Proposition 8.5 in [5] then claim that the algorithm in the proof of Theorem 5.8 is a way of abstraction for computations of **TRDB**-deductions. Conversely, since $ET(\mathcal{A}[X^\alpha];\ [\![\Pi]\!]) = T$, this theorem assures that $[\![\ ]\!]$ in [4] and [5] is a natural interpretation with respect to ET-values.

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