A proof-theoretic approach to knowledge

Mariko Yasugi *and Sobei H. Oda †
Faculty of Science and Faculty of Economics
Kyoto Sangyo University
yasugi@cc.kyoto-su.ac.jp
http://www.kyoto-su.ac.jp/~yasugi
oda@cc.kyoto-su.ac.jp

Abstract We speculate on logical aspects of the so-called *three wise men puzzle* with *proof-theoretic* methods. We propose and study the following. (1) Logical ability of each player of a puzzle is equivalent to a logical system KL, which is based on classical propositional calculus augmented with inferences of knowledge operators. (2) Enriched pieces of knowledge of players are obtained as results of meta-theorems of KL. (3) Solvability of a puzzle is guaranteed by automated theorem proving. All this is attained by means of the cut elimination theorem for KL and its corollaries.

Kew words *puzzle, propositional calculus, knowledge operator, cut elimination, unprovable sequent, theorem proving*

1 Origin of the problem

The origin of our study of knowledge operators, initiated in [9], was our awareness of the limitation of modeling methods of human behaviors. (See, for example, [6], [7], [8] for such modeling.)

Let us quote from Section 1 and Section 9 of [9].

"In existing models of consumers' behaviors, consumers are assumed to be automata who act according to some rules. In such models, the consumers are deprived of their ability to speculate on these rules themselves. Such an approach to human behavior disregards an important aspect of human ability. That is, a human being can observe the rules according to which he/she acts and then can take into account of such observation."

"Hofstadter discusses the ability (of humans) of jumping out of systems, and Keynes also refers to this ability. It is this aspect of humans that we became concerned."

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For Hofstadter see [3], and for Keynes see [5].

To start with, we have taken up in [9] a concrete example of a system observation, that is, a so-called "three wise men puzzle", modified to *two wise girls puzzle*, and have defined a formal system of propositional logic with knowledge operators in which this puzzle can be solved. We then speculated on the notion of *jumping out* of the system in a formal manner.

Two girls' puzzle is a puzzle to hit the right color of a player's hat, without seeing it. It is explained in Section 2 below. A logical formulation of the puzzle involves the following(See Section 3 for details.): a proper language for formulas (statements of the puzzle), finite sets of formulas (knowledge sets), formal system with knowledge operators (players' logical ability), proof-theory of unprovability results (jumping out of a system), consistency results (expansion of knowledge set), formal proof (solving a puzzle) and automated theorem proving (solvability of the puzzle).

These will be worked out successively in subsequent sections.

[9] and this paper form a twin article. In [9], the emphasis is on analysis of human behaviors in terms of meta-theory of logical systems, while here we focus our attention on the proof-theoretic side of the problem.

For the reader's convenience, we will briefly describe the results in [9] along with new developments. For introduction to proof-theoretic tools, one can refer to [1], [2] (for a Japanese reader) and [4].

Let us just mention that there are many semantic approaches to the knowlege logic. [10] was a pioneering work on the related problems.

This article is composed of 13 sections. Here in Section 1, we have explained the origin of the problem we are to deal with.

In Section 2, we explain the puzzle and raise some questions which arise from it.

In Section 3, the problems we are to investigate are formulated in logical terms and some requisites are placed on the logical system to be used.

The logical system upon which our theory will be based, called KL, is presented in Section 4. It is classical propositional logic with knowledge operators added. For technical reasons, Gentzen style sequential formulation is adopted.

The proof-theoretic tool which is at the basis of our theory is the *cut elimination theorem* for KL. Two consequences of the cut elimination theorem, the elimination lemma and the separation lemma, which originate in [4], are stated in Section 5. Their proofs will be given in Sections 11 and 12.

A *knowledge set* is a finite set of formulas which is consistent over the system KL. It is explained in Section 6.

In dealing with knowledge, the notion of "non-knowledge" is a delicate issue. We propose two definitions of non-knowledge, an objective one and a subjective one, and show that both satisfy some desired condition. This is in Section 7.

Expansion of knowledge set is the very issue that stimulated us to work on knowledge. A player is entitled to claim she does not know a fact when its negation is not provable from the original knowledge set. Related lemmas for the *two wise girls puzzle* are stated in Section 8.

The last player's inference, which is a usual practice, is demonstrated in Section 9.

Automated theorem proving, which represents solvability of a puzzle, is described in Section 10, and is treated in detail in Section 13.

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2 Two wise girls' puzzle

Let us describe the purzzle to be considered as follows. (To make the discussion simple, we assume two players, instead of three, called Anna and Bianca.)

Two wise girls' puzzle

There are one red and two white hats. Anna and Bianca are told of this by the observer. Then two girls are put on white hats. Each can see the other's hat, but not her own.

Anna, asked if she can tell the color of her hat, answers "No!" Then Bianca concludes that her hat is white.

Now, let us analyze the process of solving this puzzle.

First, Anna speculates as follows.

"I see that Bianca wears white, and all I know is that at least one of us wears white. Bianca's hat already satisfies my knowledge, regardless of my color. Now, with all my knowledge and reasoning about the colors of our hats, I do not have a way to know my color."

The knowledge of Anna consists of Bianca's color being white and at least one of the two girls wearing white. In order that Anna can claim her inability to decide her own color, (1) she has to be able to know all the logical consequences of her knowledge, and (2) to conclude that no such consequence tells her the color of her hat.

Let us express these conditions in meta-theoretic terms as follows.
(1) Anna's reasoning ability is equivalent to that of a logical system (to be set up appropriately).
(2) Anna is capable of *jumping out* of the system and deduce the unprovability of certain statements meta-theoretically.

Next, Bianca will reason as follows.

"I now know that Anna does not know her color, and that at least one of us wears white. Now suppose I wore red. Then, by virtue of the condition that at least one of us wears white, Anna would have concluded that her hat was white. She did not. So, my hat must be white."

The knowledge of Bianca consists of Anna's ignorance of her color and at least one of the girls wearing white. (Bianca may also know that Anna wears white, but this fact does not contribute to her reasoning.)

In Bianca's case, her guess is based on Anna's "non-knowledge" (the first piece of knowledge), and a fact (the second piece of knowledge).

In order that Bianca be able to reach proper conclusion, one has to formulate an appropriate logical system with the notion of *knowledge*.

We also know that this puzzle is *solvable*. That is, Bianca is bound to reach a proper conclusion. But, what does this mean?

If she *is wise*, then she can of course reason properly. If she weren't wise, however, how can we be assured that the puzzle be solved?

We regard this *solvability* of a puzzle, whether Bianca happens to be wise or not, as the automated theorem proving.

Also, in Bianca's knowledge, Anna's ignorance of her own color can be interpreted in various ways. Sensible interpretations are an objective negation of "Anna knows," and an affirmative description that "Anna is aware of her being ignorant of her color."

In existing studies of knowledge, it is mainly Bianca's last reasoning that has been questioned. However, this part is not at all difficult for anybody who is acquainted with logic, and we have now seen that meta-theoretical aspects of the problem are so much more problematic.

Now we proceed to a formal treatment of the puzzle.

3 Problem setting and requisites for the system

We will sort out what are involved in solving the puzzle.

1 Statements Statements describing the puzzle will be expressed in terms of formulas of propositional calculus with *knowledge operators*.

2 Envioronment The envioronment is the set of conditions upon which the puzzle is asked. It will be expressed with a *set of formulas* and called a *knowledge set*, which is disclosed to players.

3 Inferences In order to act, a player makes inferences from an envioronment. We will set up a *logical* system KL and assume that a player's reasoning ability is *equivalent* with this system.

4 Jumping out A player's ability to speculate on the system, by jumping out of it, is expressed as her ability to recognize *meta-theorems* of the system, that is, provability, unprovability and consistency of formulas.

5 Expansion of knowledge set A player may gradually gain further pieces of knowledge, thus expanding a knowledge set. In order to reach a correct conclusion, the player has to know that a statement can be legitimately added to a knowledge set. This is guaranteed by its *consistency* with the old knowledge set.

6 Solving a puzzle Solving a puzzle means *constructing a proof* within the system KL from a knowledge set.

7 Solvability Even when we know that a puzzle is solvable, not everybody can do proper inferences. Solvability can then be secured by *automatic theorem-proving*.

In setting up a logical system to solve *two wise girls' puzzle*, we first introduce *knowledge operators* for Anna and Bianca, K_1 and K_2 respectively, and then formulate a *logical system KL* which characterizes these knowledge operators.

For either of these operators, K_i, it is natural to require the following.

Requisites for KL

(a) The logical ability of Player i is equivalent to KL. That is,

$$A \text{ is a theorem of } KL \text{ if and only if } K_iA \text{ is a theorem of } KL$$

where K_iA expresses that Player i "knows that A is true."

(b) $K_iA \longrightarrow K_iK_iA$ is KL-provable.

(c) The converse of (b) is not necessarily provable.

Remark 3.1 It is impoartant to notice that the equivalence in (a) is restricted to the case where both A and K_iA are theorems without assumptions.

Furthermore this equivalence does *not* ensure that $A \longleftrightarrow K_iA$ be formally provable. In fact, we do not require either $A \rightarrow K_iA$ or $K_iA \rightarrow A$ as general theorems.

(b) above is Axiom **4** for $\Box$ in modal logic. Since $K_iA \rightarrow A$ is not a theorem in the system to be considered, it is weaker than modal logic S**4**.

4 Logical system

The logical basis of the system KL is the classical propositional calculus. For technical reasons, we adopt a system of sequential calculus.

We add knowledge operators for Anna and Bianca, K_1 and K_2 respectively.

Definition 4.1 (Language and system) (1) The language: There are propositional variables $X_1, X_2, \cdots, X_k, \cdots$. The propositional constants are $iW, i = 1, 2$. iW is read as "player i wears white." ($\neg iW$ will be abbrviated to iR, read as "Player i wears red.")

(2) Formulas are defined from propositional variables and propositional constants (atomic formulas) by applications of propositional connectives and the operators K_1 and K_2. If A is a formula, then K_iA is a formula.

(3) $\Gamma \rightarrow \Delta$ will denote a sequent, that is, Γ and Δ each denotes a finite set of formulas (possibly empty). Γ is called the antecedent of the sequent, and Δ is called the succedent of the sequent.

(4) Propositional inferences are formulated as in [1]. We list a few of them.

$$\frac{\Delta \rightarrow \Theta, A \quad A, \Gamma \rightarrow \Lambda}{\Delta, \Gamma \rightarrow \Theta, \Lambda} \ (cut)$$

$$\frac{\Gamma \rightarrow \Theta}{\Delta, \Gamma \rightarrow \Theta, \Lambda} \ (thin)$$

$$\frac{A, \Gamma \to \Theta}{A \wedge B, \Gamma \to \Theta}\ (\wedge \to)_l, \quad \frac{B, \Gamma \to \Theta}{A \wedge B, \Gamma \to \Theta}\ (\wedge \to)_r$$

$$\frac{\Gamma \to \Theta, A \quad B, \Gamma \to \Theta}{A \Rightarrow B, \Gamma \to \Theta}\ (\Rightarrow \to) \qquad \frac{A, \Gamma \to \Theta, B}{\Gamma \to \Theta, A \Rightarrow B}\ (\to \Rightarrow)$$

There is only one inference rule to characterize the knowledge operators.

$$\frac{\Gamma, K_i(\Delta) \to \Theta}{K_i(\Gamma, \Delta) \to K_i(\Theta)}\ (K \to K)$$

Here $i = 1, 2$, and $|\Theta| \leq 1$, that is, there is at most one formula in Θ. Also, $K_i(A_1, A_2, \cdots, A_k)$ stands for $K_i(A_1), K_i(A_2), \cdots, K_i(A_k)$. If $k = 0$, then $K_i(A_1, \cdots, A_k)$ is empty.

(5) Proof-figures of KL can be defined as usual, starting with any initial sequent of the form $D \to D$.

See Section 3 of [9] for details. See also [1] for the sequential calculus. One can also refer to [2].

Remark 4.1 We will expound how we obtain (a), (b), and (c) in **Requisites**. Let K denote either K_1 or K_2.

(1) One direction of (a) in **Requisites** in Section 3 is immediately derived:

$$\frac{\to A}{\to KA}\ (K \to K)$$

The converse is a consequence of the cut elimination theorem and will be explained later.

(2) (b) in **Requisites**, $KA \to KKA$, is KL-provable:

$$\frac{KA \to KA}{KA \to KKA}\ (K \to K)$$

(3) (c) in **Requisites** is a consequence of the cut elimination theorem and will be explained later.

(4) The inference $\Box$ in modal logic,

$$\frac{\Gamma \to A}{\Box\Gamma \to \Box A}\ (\Box)$$

is a special case of $(K \to K)$, reading K for $\Box$.

(5) The system KL is "constructive" with respect to the operator K; that is, $K(A \vee B) \to KA \vee KB$ is not necessarily provable. Such characteristics of KL is the same as the system of game logic in [4].

(6) What are not provable in KL will be listed as consequences of the cut elimination theorem.

5 Proof-theoretic tools

The *cut elimination theorem* holds for KL. Two consequences of the cut elimination theorem, the *elimination lemma* and the *separtion lemma* are extremely useful for our treatment of the problem. They were first stated and prove in [4] for a system of *common knowledge*. (In our case, the notion of common knowledge is not necessary.) Proofs of the subsequent proof-theoretic properties for the system KL will be included in later sections.

The results in this section are necessary for **4** (jumping out of the system) and **5** (expansion of knowledge set) in Section 3.

Theorem 1 (The cut elimination theorem for KL) There is an algorithm to eliminate all the cuts from any KL-proof-figure. That is, given a proof-figure P of a sequent S, the claimed algorithm transforms P into a cut-free proof-figure of S.

It is an immediate consequence of the theorem that KL is syntactically consistent.

Let Γ be a set of formulas. According to [4], $\epsilon\Gamma$ will denote the set of formulas obtained from Γ by eliminating all the knowledge operators as well as superfluous parentheses. For example, if Γ is $X \wedge K_1(K_2(X \vee Y))$, then $\epsilon\Gamma$ stands for $X \wedge (X \vee Y)$. $\epsilon\Gamma$ is a set of formulas of the propositional part of LK.

Proposition 5.1 (Elimination lemma: (2.2) of Section 2.1 in [4]) If a sequent $\Gamma \rightarrow \Theta$ is provable in KL, then $\epsilon\Gamma \rightarrow \epsilon\Theta$ is provable in the propositional part of LK.

Following [4], we introduce the notion of separation of a sequent.

Definition 5.1 (Separation) A formula A is called *K_i-atomic* if the outermost symbol of A is K_i. A formula A is called a K_i-formula if it is constructed from K_i-atomic formulas by applications of propositional connectives. A is called a *$-K_i$-formula* (K_{-i}-formula in [4]) if K_i occurs in A only in the scope of a K_j for $j \neq i$.

For example, if A is of the form $K_1(K_2B) \wedge K_1(B \Rightarrow K_1C)$, then A is a K_1-formula, where $K_1(K_2B)$ and $K_1(B \Rightarrow K_1C)$ are K_1-atomic subformulas of A. $K_2(B \vee K_1C) \Rightarrow 1W$ is a $-K_1$-formula. $K_1B \wedge K_2C$ is neither a K_1-formula nor a $-K_1$-formula.

A sequent is called *K_i-separable* if it consists of K_i-formulas and $-K_i$-formulas.

Proposition 5.2 (Separation lemma: Theorem 3.3 of [4])
Let $\Gamma, \Delta \rightarrow \Theta, \Lambda$ be a sequent, where Γ and Θ each consists of K_i-formulas and Δ and Λ each consists of $-K_i$-formulas.

If $\Gamma, \Delta \rightarrow \Theta, \Lambda$ is provable in KL, then either $\Gamma \rightarrow \Theta$ or $\Delta \rightarrow \Lambda$ is provable in KL.

Remark 5.1 (1) As consequences of the facts above, we can show that, for an atomic X, neither $K_iX \to X$ nor $X \to K_iX$ is provable. (See Section 5 in [9].)

(2) A is provable if and only if K_iA is provable. The "only if" part has been shown. For the "if" part, suppose KA is provable. Then, by the cut elimination theorem (and syntactic consistency as its consequence), the only possibility of the last inference in a cut-free proof-figure of $\to KA$ is $(K \to K)$, and so A must already be provable.

(3) As was mentioned above, $K(X \vee Y) \to KX \vee KY$ is not provable. The excluede middle in the form of $\to KX \vee K\neg X$ is not provable (while $K(X \vee \neg X)$ is provable).

6 Knowledge set

Definition 6.1 (Knowledge Set and its Expansion) 1) A finite set of KL-formulas, say Π, will be called a *knowledge set*. The initial knowledge set in a puzzle is called the envioronment of the puzzle. (This corresponds to **2** Envioronment in Problem setting.)

2) A knowledge set Π is said to be *consistent* (in KL), if $\Pi \to$ is not provable in KL.

3) A knowledge set Π is said to be *independent* if no formula in Π is deducible from the rest, that is, if A is a formula in Π, then the sequent $\Pi \setminus \{A\} \to A$ is not provable in KL, where $\Pi \setminus \{A\}$ denotes the set Π minus the formula A.

4) Let Π and Ξ be knowledge sets. Ξ is called a (consistent and independent) *expansion* of Π if Π is a subset of Ξ and Ξ is consistent and independent. (This corresponds to **5** Expansion in Section 3.)

From the definition, we notice that unprovability of a sequent is an important issue for expansion of a knowledge set.

Two wise girls' puzzle gives an example of such a treatment of knowledge sets and their expansions. The general procedure to secure consistency and independency can be outlined as follows. Let Π be a knowledge set, which consists of K_1-formulas and $-K_1$-formulas (which is the case with two girls puzzle). Suppose $\Pi \to$ were provable in KL. By the separation lemma, either a sequent of K_1-formulas or a sequent of $-K_1$-formulas is provable. Then by the elimination lemma (Proposition 5.1), a corresponding sequent without knowledge operators would be provable in the propositional part of LK. By any known method of judging validity of propositional sequents, one can check whether it be possible or not. If Π is indeed consistent, then we will know that the last step is impossible.

7 Non-knowledge operator

In our puzzle, Bianca (the second player) can make inferences only under the following circumstances:

Bianca knows that Anna (the first player) does *not* know that Anna wears a white hat.

Let us speculate on what this *not* refers to. We can think of two possibilities. The first candidate N^1(objective one):

$$N^1 X \equiv \neg K_1 X$$

The second candidate (subjective one):

$$N^2 X \equiv K_1(\neg K_1 X)$$

Then $N^2 X \to N^1 X$ is provable in KL, but the converse

$$N^1 X \to N^2 X$$

is not provable for an atomic X. (See Section 6 of [9] for details.)

For either N^1 or N^2 as defined above (denoted collectively by N),

$$K_1 A \wedge NA \to$$

for any formula A. A more important fact is the following.

Proposition 7.1 (Contradiction) $K_2(K_1 A \wedge NA \Rightarrow \bot)$ is KL-provable for both N^1 and N^2. ($\bot$ denotes any formula of contradiction, $X \wedge \neg X$ for example.)

Proof For N^1:

$$\dfrac{\dfrac{K_1 A, \neg K_1 A \to}{\to K_1 A \wedge \neg K_1 A \Rightarrow \bot}\ (\to\Rightarrow)}{\to K_2(K_1 A \wedge \neg K_1 A \Rightarrow \bot)}\ (K_2 \to K_2)$$

For N^2:

$$\dfrac{\dfrac{\dfrac{K_1 A, \neg K_1 A \to}{K_1 A, K_1 \neg K_1 A \to}\ (K_1 \to K_1)}{\to K_1 A \wedge K_1 \neg K_1 A \Rightarrow \bot}\ (\to\Rightarrow)}{\to K_2(K_1 A \wedge K_1 \neg K_1 A \Rightarrow \bot)}\ (K_2 \to K_2)$$

Remark 7.1 It is only this fact that we need for *two girls' puzzle*, and so we will simply write N for either N^1 or N^2. (See Section 6 of [9].)

8 Expansion of knowledge set

In order to *expand* one's knowledge set (corresponding to **5**), one usually has to *jump out* of the system of one's reasoning (corresponding to **4**) and see that some statements are independent of the system. Some lemmas which makes it possible have been stated and proved in Section 7 of [9].

For Anna to be able to claim she does not know the color of her hat, she has to jump out of the system and see that neither $\mathcal{A} \rightarrow K_1(1W)$ nor $\mathcal{A} \rightarrow K_1(1R)$ be provable in KL, where $\mathcal{A}$ is the envioronment of the puzzle. (Recall that we write iR for $\neg iW$.)

Anna having told she does not know the color of her hat, Bianca can assume $K_2(N(1W))$ and $K_2(N(1R))$.

These facts can be proof-theoretically assured. We will list some of these results.

Let Π denote the following.

$$(1W \wedge 2W) \vee (1W \wedge 2R) \vee (1R \wedge 2W)$$

Then, the envioronment for *two girls' puzzle* is the following.

$\Pi, K_1(\Pi), K_2(\Pi), K_1(K_2(\Pi)), K_2(K_1(\Pi)), K_1(2W),$
$K_2(2R \Rightarrow K_1(2R)), K_2(2W \Rightarrow K_1(2W))$

Let us call this set of formulas $\mathcal{A}$. (For the inferences of Bianca, only a subset of this is needed.)

Remark 8.1 $\mathcal{A}$ can be decomposed into two groups of formulas: $\mathcal{A}_1$, which consists of all the K_1-formulas, and $-\mathcal{A}_1$, which consists of all the $-K_1$-formulas:
$\mathcal{A}_1 = \{K_1(\Pi), K_1(K_2(\Pi)), K_1(2W)\}$
$-\mathcal{A}_1 = \{\Pi, K_2(\Pi), K_2(K_1(\Pi)), K_2(2R \Rightarrow K_1(2R)), K_2(2W \Rightarrow K_1(2W))\}$

Lemma 8.1 $\mathcal{A}$ is *consistent* in KL. (The puzzle is a sound one.)

Lemma 8.2 $\mathcal{A} \rightarrow K_1(1W)$ is *not* provable. (It cannot be formally claimed that Anna knows she wears white.)

So, looking at the system from outside, Anna can claim that she cannot tell if she wears white.

Lemma 8.3 $\mathcal{A} \rightarrow \neg K_1(1W)$ is *not* KL-provable. (It cannot be formally claimed that Anna does not know she wears white.)

So, Anna cannot claim she does not know her hat's color if she stays inside the system.

Lemma 8.4

$$\mathcal{A} \rightarrow K_2(N(1W))$$

is not provable, and

$$\mathcal{A}, K_2(N(1W)) \rightarrow$$

is not provable.

These indicate the following. Bianca *can* consistently add to her knowledge set the statement which expresses the fact that she knows Anna does not know she wears white, and also she *has to* add it if she is to use it, since she cannot know it inside the system.

The set $\mathcal{A} \cup \{K_2N(1W)\}$ is a consistent expansion of $\mathcal{A}$.

9 Inference of Bianca

In the formal reasoning, the following lemma, proving cuts within knowledge operators is a useful one. We state a most general form of a cut within knowledge.

Lemma 9.1 (Cut within knowledge)

$$K(A \Rightarrow B), K(C \wedge B \Rightarrow D) \rightarrow K(A \wedge C \Rightarrow D)$$

This is an immediate consequence of

$$A \Rightarrow B, C \wedge B \Rightarrow D \rightarrow A \wedge C \Rightarrow D$$

by an application of $(K \rightarrow K)$.

Recall that Π denotes the following.

$$(1W \wedge 2W) \vee (1W \wedge 2R) \vee (1R \wedge 2W)$$

Then, let Γ denote the following.

$$K_2K_1(\Pi), K_2N(1W), K_2(2R \Rightarrow K_1(2R))$$

Of these, $K_2K_1(\Pi), K_2(2R \Rightarrow K_1(2R))$ is a subset of $\mathcal{A}$. Γ is obtained from this set by adding $K_2N(1W)$, and is consistent due to Lemma 8.4. (One also has to add this due to Lemma 8.4.)

Now, for **6**, Solving a puzzple, we wish to derive
(*) $\Gamma \longrightarrow K_2(2W)$
in the system KL.

(*) can be obtained as follows. (We do not write Γ except when we need to explicitly state which of the formulas in Γ are used.)

1 through 8 below are provable in KL successively from Γ.

1. $K_2(2R \Rightarrow K_1(2R))$
2. $K_2(K_1(2R) \Rightarrow K_1(1W))$
3. $K_2(K_1(1W) \wedge N(1W) \Rightarrow \bot)$
4. $K_2(2R \wedge N(1W) \Rightarrow \bot)$
5. $K_2(N(1W) \Rightarrow \neg 2R)$
6. $K_2(\neg 2R \Rightarrow 2W)$
7. $K_2(N(1W) \Rightarrow 2W)$
8. $K_2(2W)$

(8) is (*). Proof procedure is given in Section 8 of [9]. We will later give a cut-free proof of (*) as an example of automated theorem proving.

10 Theorem proving

As we mentioned at the start, not everybody can make a correct inference. Nonetheless, we claim that the puzzle be solvable (**7** Solvability). A meta-theoretic characterization of this claim would be automatic theorem proving. That is, $\Gamma \to K_2 2W$ is automatically proved.

This can be attained as a consequence of the cut elimination theorem and its corollaries. The propositional part of the theorem proving can be executed just as the usual Wang's algorithm (a resolution method). In resolving a sequent, only the new inference, $(K \to K)$ has to be taken care of. The resolution looks like this:

$$\frac{\Gamma_1, K_i(\Gamma_2) \to \Theta}{\Delta, K_i(\Gamma) \to K_i(\Theta), \Lambda}\ (K \to K)$$

where Δ and Λ are atomic, $\Gamma_1 \bigoplus \Gamma_2 = \Gamma$ (disjoint sum) and either Γ_1 or Θ is non-empty.

There can be finitely many upper sequents to a given sequent according to this resolution, and, in each case, the number of the operator K decreases.

We will later present a more thorough treatment.

Now we are going to proceed to establish proof-theoretic properties announced in earlier sections.

11 Outline of cut elimination proof

The cut elimination theorem is proved similarly to that of LK, following also the idea of [4].

Let $\mathbf{P}$ be a KL proof-figure such that $\mathbf{P}$ has a sole *cut* as the last inference. (We will denote this condition by $(\mathcal{C})$.) So, $\mathbf{P}$ has the form

$$\frac{\dfrac{\overset{\cdot}{\Gamma \to \Theta, M}}{} \qquad \dfrac{\overset{\cdot}{M, \Delta \to \Lambda}}{}}{\Gamma, \Delta \to \Theta, \Lambda}\ (\mathbf{C})$$

where $\mathbf{C}$ denotes the (only) *cut* in $\mathbf{P}$.

We will prescribe an algorithm to transform $\mathbf{P}$ into a cut-free proof-figure of the same sequent $\Gamma, \Delta \to \Theta, \Lambda$. Given any proof-figure of KL, one can apply the algorithm as above repeatedly so that one obtains a cut-free proof-figure of the same sequent.

We will present an algorithm as above to eliminate the cut **cut** in $\mathbf{P}$ by a triple induction. For that purpose, we will first introduce three scales for *cuts*.

Definition 11.1 (Scales) Let $\mathbf{P}$ be a proof-figure satisfying (C) above. Formulas, sequents and *cuts* mentioned below are all occurrences in $\mathbf{P}$.

1. Grade: The *grade* of a formula A, $g(A)$, is defined as follows.

   (1) $g(A) = 0$ if A is atomic.

(2) $g(\neg A) = g(KA) = g(A) + 1$.

(3) $g(A \wedge B) = g(A \Rightarrow B) = g(A \vee B) = \max(g(A), g(B)) + 1$

The grade of *cut* $\mathbf{C}$ in $\mathbf{P}$, $g(\mathbf{C})$, and the grade of $\mathbf{P}$, $g(\mathbf{P})$, is defined to be $g(M)$.

2. Left rank: The *left rank* of a sequent S in $\mathbf{P}$, $\rho_l(S)$, is defined as follows. $\rho_l(S) = 0$ if S is an initial sequent. Otherwise,

$$\rho_l(S) = \left\{ \begin{array}{r} \max\{\rho_l(S') : S' \quad \text{is an upper sequent of} S\} + 1, \\ \text{if } S \quad \text{contains } M \quad \text{in the succedent} \\ 0, \\ \text{otherwise} \end{array} \right\}$$

Then the left rank of *cut* $\mathbf{C}$ is defined to be

$\rho_l(\mathbf{P}) = \rho_l(\mathbf{C}) = \rho_l(S_0)$, where S_0 is the left upper sequent of *cut* $\mathbf{C}$.

3. Right rank: The *right rank* of a sequent S in $\mathbf{P}$ and of $\mathbf{P}$, $\rho_r(S)$ and $\rho_r(\mathbf{P})$, is defined similarly to $\rho_l(S)$, changing "the succedent" to "the antecedent" and "the left upper sequent" to "the right upper sequent."

4. The comlexity of $\mathbf{P}$ with respect to *cut* $\mathbf{C}$, $c(\mathbf{P})$, is $\langle g(\mathbf{P}), \rho_l(\mathbf{P}), \rho_r(\mathbf{P})\rangle$, lexicographically ordered.

12 Cut elimination process

The *cut* elimination process is more or less the same with Gentzen's original reduction method in [1], but modified in the way in [4]. [4] deals with a system of common knowledge, which is much more complicated than our system. So, we will conform the proof technique to our case.

Since the proof procedure is quite a usual practice, and we can follow more or less the proof in [4], we will treat only some exemplary cases. For the reader who may not be acquainted with Gentzen's method, we post a version of this article with more comprehensive treatemtn of Section 12 at:

http://www.kyoto-su.ac.jp/ỹasugi/Recent/

Assume that *cut* $\mathbf{C}$ can be eliminated from any proof-figure $\mathbf{P}'$ with $c(\mathbf{P}') < c(\mathbf{P})$, where $\mathbf{P}\ '$ has a sole *cut* as the last inference. Upon this assumtion, we will show that *cut* $\mathbf{C}$ can be eliminated from $\mathbf{P}$.

1) M occurs in the end-sequent $\Gamma, \Delta \rightarrow \Theta, \Lambda$.

1.1) M occurs in Γ. Then the end-sequent of $\mathbf{P}$, $\Gamma, \Delta \rightarrow \Theta, \Lambda$, is obtained from $M, \Delta \rightarrow \Lambda$ by some applications of *thin*.

1.2) M occurs in Λ. Then $\Gamma, \Delta \rightarrow \Theta, \Lambda$ is obtained from $\Gamma \rightarrow \Theta, M$ by some applications of *thin*.

Now we may assume that M does not occur in $\Gamma, \Delta \rightarrow \Theta, \Lambda$. Then neither of the upper sequents of *cut* $\mathbf{C}$ in $\mathbf{P}$ is an initial sequent, and hence $\mathbf{P}$ is of the

form

$$\dfrac{\dfrac{\cdot}{\Gamma \to \Theta, M}\ (I_1) \quad \dfrac{\cdot}{M, \Delta \to \Lambda}\ (I_2)}{\Gamma, \Delta \to \Theta, \Lambda}\ (M)(cut)$$

We will show how *cut* **C** can be eliminated according to the kinds of inferences I_1 and I_2, to obtain a cut-free proof-figure of the end-sequent $\Gamma, \Delta \to \Theta, \Lambda$.

Induction step 1 Put $g(\mathbf{P}) = \gamma$ and assume $\rho_l = \rho_2 = 1$.

2) I_1 or I_2 is a *thin* (which introduces M.) Simply eliminate that *thin*, and obtain the endsequent by applying *thin*.

If 2) is not the case, since $\rho_l = \rho_r = 1$, M is the principal formula of each inference of I_1 and I_2 , and that $\gamma > 0$. (Notice that the case where $\gamma = 0$ is included in 1) and 2) above.)

3) I_1 and I_2 are propositional inferences or K-inference whose principal formulas (the consequences of these inferences) are M.

If M is of the form $A \Rightarrow B$, then instead of introducing M by inferences on $\Rightarrow$, execute cuts on B and then on A successively.

Notice that $A \notin \Gamma, B \notin \Delta, B \notin \Theta$. Since $g(B) < \gamma$ and $g(A) < \gamma$, by the induction hypotheses, one can first eliminate the *cut* on B, and next eiliminate the *cut* on A, thus obtaining a *cut*-free proof-figure of $\Gamma, \Delta \to \Theta, \Lambda$.

If M is of the form KA,then I_1 is of the form

$$\dfrac{\Gamma', K\Gamma'' \to A}{K\Gamma', K\Gamma'' \to KA}\ (K \to K)$$

and I_2 is of the form

$$\dfrac{A, \Delta', K\Delta'' \to \Lambda'}{KA, K\Delta', K\Delta'' \to K\Lambda'}\ (K \to K)$$

Since $\rho_l(M) = 1$, $KA \notin K\Delta''$. Also $A \notin \Delta'$. (Recall that a sequent consists of sets of formulas.)

Transform **P** into the following.

$$\dfrac{\dfrac{\Gamma', K\Gamma'' \to A \quad A, \Delta', K\Delta'' \to \Lambda'}{\Gamma', K\Gamma'', \Delta', K\Delta'' \to \Lambda'}\ (A)(cut)}{K\Gamma', K\Gamma'', K\Delta', K\Delta'' \to K\Lambda'}\ (K \to K)$$

Since $g(A) < \gamma$, one can eliminate the *cut* on A, thus obtaining a cut-free proof-figure of the end-sequent.

Induction Step 2 $g(M) = \gamma, \rho_l = 1, \rho_r > 1$.

4) If I_2 is a *thin* (which is *not* on the *cut* formula M), then apply a *cut* on M to the upper sequent of I_2 , which leaves γ and ρ_l invariant but reduces ρ_r.

5) When I_1 is a *thin*, the *cut* formula M is introduced, and hence we can obtain the end-sequent from the upper sequent of I_1 .

Now we may assume that I_1 is not a *thin*.

6) The case where I_2 is a propositional inference.

6.1) Suppose I_2 is a $\Rightarrow\rightarrow$ with the principal formula $A \Rightarrow B$.

6.1.1) M is not $A \Rightarrow B$. Then first cut out M, and later introduce $A \Rightarrow B$ by $\Rightarrow\rightarrow$. New *cuts* can be eliminated, since with this transformation γ and ρ_l remain invariant while ρ_r decreases.

6.1.2) M is $A \Rightarrow B$. By virtue of 6.1.1) above, $A \Rightarrow B, \Gamma, \Delta \rightarrow \Theta, \Lambda$ can be obtained cut-free and ρ_r decreases, while $\rho_l = 1$ remains invariant. So, by the induction hypothesis, the end-sequent can be obtained cut-ree.

7) I_2 is a K-inference. Since $\rho_r > 1$, $\mathbf{P}$ is of the following form.

$$\dfrac{\Gamma \rightarrow \Theta, KN \quad \dfrac{KN, \Delta', K\Delta'' \rightarrow \Lambda'}{KN, K\Delta', K\Delta'' \rightarrow K\Lambda'}\ (K \rightarrow K)}{\Gamma, K\Delta', K\Delta'' \rightarrow \Theta, K\Lambda'}\ (cut)$$

Since $\rho_l = 1$, the KN in $\Gamma \rightarrow \Theta, KN$ is obtained either by a *thin* or by a K-inference. The former case has been dealt with earlier in 2.1). Consider the latter case. (In this case Θ is empty.)

$$\dfrac{\Gamma', K\Gamma'' \rightarrow N}{K\Gamma', K\Gamma'' \rightarrow KN}\ (K \rightarrow K)$$

where $K\Gamma', K\Gamma''$ is Γ.

Case 1 $N \notin \Delta'$. Using the figure above, one can transform $\mathbf{P}$ so that first KN is cut out, and then $K \rightarrow K$ is executed.

Then, ρ_r decreases, $\rho_l = 1$ and γ remains invariant. So, the induction hypothesis applies to the new *cut*.

Case 2 $N \in \Delta'$. Then $\Delta' = N, \Delta_1$ and $K\Delta' = KN, K\Delta_1$.

Consider first the following.

$$\dfrac{K\Gamma', K\Gamma'' \rightarrow KN \quad KN, N, \Delta_1, K\Delta'' \rightarrow \Lambda'}{K\Gamma', K\Gamma'', N, \Delta_1, K\Delta'' \rightarrow \Lambda'}\ (KN)(cut)$$

For this *cut*, γ and $\rho_l = 1$ remain invariant, while ρ_r decreases. So, one can eliminate this *cut*, and obtain a cut-free proof-figure of the end-sequent, which we will denote by $\mathbf{P}_0$. Now consider the following.

$$\dfrac{\dfrac{K\Gamma'' \rightarrow N \quad \mathbf{P}_0}{\Gamma', K\Gamma', K\Gamma'', \Delta_!, K\Delta'' \rightarrow \Lambda'}\ (N)(cut)}{K\Gamma', K\Gamma'', K\Delta_1, K\Delta'' \rightarrow K\Lambda'}\ (K \rightarrow K)$$

For this *cut*, $g(N) < \gamma$, and hence one can obtain a cut-free proof-figure of the end-sequent.

Induction Step 3

$$g(\mathbf{P}) = \gamma, \rho_l = l > 1, \rho_r = r(\geq 1)$$

In this case, at least one of the upper sequents of I_1 contains M.
8) I_1 is a *thin*. First cut out M and then apply I_1.

Here γ remains invariant, while ρ_l decreases, and so the induction hypothesis applies.
9) I_1 is propositional.
9.1) I_1 is $\Rightarrow\rightarrow$.

$$\dfrac{\dfrac{\Gamma' \rightarrow \Theta, M, A \quad B, \Gamma' \rightarrow \Theta, M}{A \Rightarrow B, \Gamma' \rightarrow \Theta, M}\,(I_1) \qquad M, \Delta \rightarrow \Lambda}{A \Rightarrow B, \Gamma' \rightarrow \Theta, \Lambda}\,(M)(cut)$$

There are three possibilities, that is, (a) neither A nor B is M, (b) A is M and (c) B is M. According to these cases, we transform $\mathbf{P}$ as follows.
(a) First consider

$$\dfrac{\Gamma' \rightarrow \Theta, M, A \quad M, \Delta \rightarrow \Lambda}{\Gamma', \Delta \rightarrow \Theta, A, \Lambda}\,(M)(cut)$$

Since γ remains invariant while ρ_l decreases, one can obtain a cut free proof-figure of the last sequent, which we will call $\mathbf{P}_1$.

Next consider

$$\dfrac{B, \Gamma' \rightarrow \Theta, M \quad M, \Delta \rightarrow \Lambda}{B, \Gamma', \Delta \rightarrow \Theta, \Lambda}\,(M)(cut)$$

As above, one can obtain a cut-free proof-figure of the last sequent, which we will call $\mathbf{P}_2$. Then transform $\mathbf{P}$ into the following.

$$\dfrac{\mathbf{P}_1 \quad \mathbf{P}_2}{A \Rightarrow B, \Gamma', \Delta \rightarrow \Theta, \Lambda}\,(\Rightarrow\rightarrow)$$

(b)

$$\dfrac{B, \Gamma' \rightarrow \Theta, M \quad M, \Delta \rightarrow \Lambda}{B, \Gamma', \Delta \rightarrow \Theta, \Lambda}\,(M)(cut)$$

From this one obtains a cut-free proof-figure of the last seqnent. Since M is A, the sequent $\Gamma' \rightarrow \Theta, M, A$ is in fact $\Gamma' \rightarrow \Theta, A$. From these, by an application of $\Rightarrow\rightarrow$, we obtain a cut-free proof-figure of the original end-sequent.
(c) Consider first

$$\dfrac{\Gamma' \rightarrow \Theta, M, A \quad M, \Delta \rightarrow \Lambda}{\Gamma', \Delta \rightarrow \Theta, A, \Lambda}\,(M)(cut)$$

From this one can obtain a cut-free proof-figure of the last sequent, which we will call $\mathbf{P}'$. Then we obtain a cut-free proof-figure as follows.

$$\dfrac{\dfrac{B, \Delta \rightarrow \Lambda}{B, \Gamma', \Delta \rightarrow \Theta, \Lambda}\,(thin) \qquad \mathbf{P}'}{A \Rightarrow B, \Gamma', \Delta \rightarrow \Theta, \Lambda}\,(\Rightarrow\rightarrow)$$

9.2) Similarly with other cases.
10) I_1 is a K-inference. But this case is impossible, due to the fact that $\rho_l > 1$, for then M is $K\Theta$ and the last part of the proof-figure looks as follows.

$$\frac{\Gamma', K\Gamma'' \to \Theta}{K\Gamma', K\Gamma'' \to K\Theta}\ (K \to K)$$

Here M is introduced at the last moment, contradicting $\rho_l > 1$.

This completes the cut elimination proof.

We can show some consequences of the cut elimination theorem as in the case of LK.

Corollary 1 (Syntactic consistenry) 1) The system KL satisfies syntactic consistency, that is, the empty sequent $\to$ is not provable in KL.
2) The subformula property holds in KL.
3) Propositions 5.1 and 5.2 can be proved.

Proofs of 1) and 2) are traditional ones.
Proposition 5.1 can be easily proved by just checking each sequent in a cut-free proof-figure starting from initial sequents.
For a proof of 5.2, we first need some definitions.

Definition 12.1 ($K \to K$-free part) Let $\mathbf{P}$ be a cut-free proof-figure. The $K \to K$-free part of $\mathbf{P}$, denoted by P_K is a part of $\mathbf{P}$ defined as follows.
The end-sequent belongs to P_K. If

$$\frac{S_1 \quad S_2}{S}\ (I)$$

is in $\mathbf{P}$, where there may be only one upper sequent, if I is not $K \to K$ and if S belongs to P_K, then S_1 and S_2 belong to P_K. If I is $K \to K$, then S_1 does not belong to P_K. If S does not belong to P_K, then neither S_1 nor S_2 belongs to P_K.

The following lemma can be easily obtained from the definitions of $K \to K$-free part and of separability of a sequent, by checking sequents from the end-sequent upwards.

Lemma 12.1 Let $\mathbf{P}$ be a cut-free proof-figure of the end-sequent S_0 and suppose S_0 is separable. Then every sequent in P_K is also separable.

Proposition 5.2 then follows immediately from thie lemma.

13 Resolution tree and theorem proving

In Section 10 above, we proposed that automated theorem proving represents *solvability* of a puzzle.

In the *two wise girls' puzzle*, Bianca's conclusion can be expressed by either $K_2 2W$ or $K_2 2R$ (that is, $K_2(\neg 2W)$). So she can try to prove either $S_1 : \Gamma \to K_2 2W$ or $S_2 : \Gamma \to K_2(\neg 2W)$.

The automated theorem proving for the system KL is similar to that of the classical propositional calculus. One first constructs a resolution tree of S_i and then transforms it to a proof-figure. The resolution with respect to a propositional connective is exactly as usual. The resolution with respect to the knowledge operator has a multiple choice, according as which occurrences of K in the antecedent be eliminated.

Definition 13.1 (Resolution tree) The *resolution rules* are defined according to the propositional connectives one chooses to resolve.

1) Propositional resolutions:

$$\frac{\Gamma \to \Delta, A \quad \Gamma \to \Delta, B}{\Gamma \to \Delta, A \wedge B} \ (\to \wedge)$$

$$\frac{A, B\Gamma \to \Delta}{A \wedge B, \Gamma \to \Delta} \ (\wedge \to)$$

$$\frac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A \Rightarrow B} \ (\to\Rightarrow)$$

$$\frac{\Gamma \to \Delta, A \quad B, \Gamma \to \Delta}{A \Rightarrow B, \Gamma \to \Delta} \ (\Rightarrow\to)$$

For other propositional connectives, the rules can be defined similarly.

The resolution rule $\to \wedge$, for example, is read as "The sequent $\Gamma \to \Delta, A \wedge B$ is *resolved* into $\Gamma \to \Delta, A$ and $\Gamma \to \Delta, B$."

The sequent below the line is called the lower sequent of the resolution rule, while each sequent above the line is called an upper sequent. The figures above should be read from the lower sequent to upper sequents.

2) K-resolution: Suppose all the formulas in a sequent are either atomic or K-atomic (See Definition 5.1). Suppose it is of the form

$$\Delta, K\Gamma \to K\Theta, \Lambda$$

where Δ and Λ consist of atomic formulas.

We consider all the decompositions of the formulas in Γ, say Γ_1 and Γ_2. If Θ consists of multiple formulas, then choose one, say Θ_i.

Let $\delta(\Gamma_1, \Gamma_2, \Theta_i)$ denote such a decomposition of Γ and a choice of Θ. Then the resolution rule $(K \to K)$ for $\delta(\Gamma_1, \Gamma_2, \Theta)$ is the following.

$$\frac{\Gamma_1, K\Gamma_2 \to \Theta_i}{\Delta, K\Gamma_1, K\Gamma_2 \to K\Theta, \Lambda}\ (\delta(\Gamma_1, \Gamma_2, \Theta_i))(K \to K)$$

where either Γ_1 or Θ_i is nonempty.

3) Application of propositional resolution: One can apply any propositional resolution as long as there is a formula with the corresponding connective.

4) A resolution tree of a sequent S_0, denoted by T is a collection of figures starting with S_0 and continuing the resolutions upward. In an application of $(K \to K)$ rule, there may be multiple possibility of upper sequents. In that case, one generates a multiple set of figures. Each will form a tree.

5) In a tree T, consider the following condition:
(i) there is a formula which occurs both in the antecedent and the succedent of a sequent in T,
or
(ii) all the formulas in a sequent is atomic.
Whichever comes first, one stops resolving. (If (ii) is the case, the resolution automatically halts.) A sequent satisfying (i) will be called *valid*.

6) Forest: At each $K \to K$ rule, as many trees as the number of decompositions and choices are generated. We call the collection of such trees a *forest* of S_0, and denote it by $\mathcal{F}_{S_0}$

Remark 13.1 (1) Notice that in any of the resolution rule, each upper sequent has fewer number of propositional connectives and K than its lower sequent.

(2) It is also obvious that a propositional rule of inference and a propositional resolution rule correspond to one another, and a $(K \to K)$ resolution corresponds to a $(K \to K)$ inference rule, with possibly some applications of *thin*.

(3) In 2) in the definitn, one may wonder why the sequent $\Delta \to \Lambda$ is not an alternative. The situation is as follows. If one chose $\Delta \to \Lambda$, then the resolution halts there. If this is *valid*, then the lower sequent is also valid, and hence the resolution would have halted there. Otherwise, it will not contribute to provability of the original sequent. So, we need not choose $\Delta \to \Lambda$.

Proposition 13.1 (1) For any sequent S_0, and for any of its resolution tree T, T is finite, that is, every path of sequents in T halts.

(2) A sequent S_0 is provable in KL if and only if, in its forest $\mathcal{F}_{S_0}$, there is a tree T where every topmost sequent in T is valid.

Proof of (2): If S_0 is provable, then a resolution tree of S_0 satisfying the condition can be easily constructed from its *cut-free* proof-figure.

Suppose there is a resolution tree of S_0 satisfying the condition. Then by some applications of *thin*, we obtain a (cut-free) proof-figure of S_0.

Solving the puzzle

We will now give a tree of $\Gamma \to K_2 2W$. In order to make expressions simple, we will demonstrate a tree (in a forest) which turns out to give a proof-figure of the desired sequent.

In a similar manner, one can construct a forest of $\Gamma \to K_2\neg 2W$, which shows that this sequent cannot be provable.

(1) Solvability for N^1.

As before, put Π:

$$(1W \wedge 2W) \vee (1W \wedge \neg 2W) \vee (\neg 1W \wedge 2W)$$

Then the target sequent S_0 is

$$K_2K_1\Pi, K_2(\neg K_1 1W), K_2(\neg 2W \Rightarrow K_1\neg 2W) \to K_2(2W)$$

if we take $N^1 A \equiv \neg K_1 A$ as a non-knowledge operator.

The following is a desired tree of S_0.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Pi \to 1W, 2W}{\Pi, \neg 2W \to 1W}\,(\to\neg)}{K_1\Pi, K_1\neg 2W \to 2W, K_1 1W}\,(K_1 \to K_1)}{K_1\Pi, \neg K_1 1W, K_1\neg 2W \to 2W}\,(\neg\to) \quad \dfrac{2W, K_1\Pi, \neg K_1 1W \to 2W}{K_1\Pi, \neg K_1 1W \to 2W, \neg 2W}\,(\to\neg)}{K_1\Pi, \neg K_1 1W, \neg 2W \Rightarrow K_1\neg 2W \to 2W}\,(\Rightarrow\to)}{K_2K_1\Pi, K_2\neg K_1 1W, K_2(\neg 2W \Rightarrow K_1\neg 2W) \to K_2 2W}\,(K_2 \to K_2)$$

Here $(K_2 \to K_2)$ applies to $\delta(\Gamma, \phi, 2W)$ and $(K_1 \to K_1)$ applies to $\delta(\Pi, 2W, 1W)$.

It is obvious that $\Pi \to 1W, 2W$ can be resolved into a vaild sequent.

By inserting some applications of *thin*, one can immediately obtain a (cut-free) proof-figure of S_0.

The resulting proof-figure appears quite different from that in Section 9. The proof procedure in Section 9 is a natural one, while the cut-free proof-figure constructed here is a mechanical one.

(2) Solvability for N^2

For $N^2 \equiv K_1\neg K_1$, we can choose the following resolution tree.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Pi \to 1W, 2W}{\Pi, \neg 2W \to 1W}\,(\to\neg)}{K_1\Pi, K_1\neg 2W \to 2W, K_1 1W}\,(K_1 \to K_1)}{K_1\Pi, \neg K_1 1W, K_1\neg 2W \to 2W}\,(\neg\to) \quad \dfrac{2W, K_1\Pi, K_1\neg K_1 1W \to 2W}{K_1\Pi, K_1\neg K_1 1W \to 2W, \neg 2W}\,(\to\neg)}{K_1\Pi, K_1\neg K_1 1W, \neg 2W \Rightarrow K_1\neg 2W \to 2W}\,(\Rightarrow\to)}{K_2K_1\Pi, K_2K_1\neg K_1 1W, K_2(\neg 2W \Rightarrow K_1\neg 2W) \to K_2 2W}\,(K_2 \to K_2)$$

(3) Unprovability of $S_0 : \Gamma \to K_2\neg 2W$ for N^2.

One tree of S_0 in its forest is the following.

$$\dfrac{\dfrac{K_1\Pi, K_1\neg 2W \to \neg 2W, K_1 1W \quad \dfrac{\dfrac{\Pi \to 1W}{K_1\Pi, 2W \to K_1 1W}\,(K_1 \to K_1)}{K_1\Pi \to \neg 2W, \neg 2W, K_1 1W}\,(\neg\to)}{K_1\Pi, \neg K_1 1W, \neg 2W \Rightarrow K_1\neg 2W \to \neg 2W}\,(\Rightarrow\to, \neg\to)}{\Gamma \to K_2\neg 2W}\,(K_2 \to K_2)$$

We have abbreviated the left part of the tree. It can be easily shown that the topmost sequent of the right path, $\Pi \to 1W$, is not valid. We can construct other trees in a similar manner, and can show that they are not valid.

(4) Unprovability of $X \to KX$ and $KX \to X$

$$\frac{\to X}{X \to KX}\ (K \to K)$$

$$\frac{X \to}{KX \to X}\ (K \to K)$$

The top sequent of either tree is not valid.

(5) Unprovability of $S_0 : K(X \vee Y) \to KX \vee KY$. One tree in a forest of S_0 is the following.

$$\cfrac{\cfrac{\cfrac{X \to X \quad Y \to X}{X \vee Y \to X}\ (\vee \to)}{K(X \vee Y) \to KX, KY}\ (K \to K)}{K(X \vee Y) \to KX \vee KY}\ (\to \vee)$$

A topmost sequent is not valid. Construction of other possible trees is left to the reader.

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