Computability of a function with jumps –Effective uniformity and limiting recursion–

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Abstract

Given a strictly increasing computable sequence (called a base sequence) of real numbers (with respect to the Euclidean topology), one can induce an *effective uniformity* for the real line, where the elements in the base sequence are regarded as isolated. The relation between two extended notions of computability of real sequences, one with respect to the Euclidean space with a *limiting recursive* modulus of convergence and one with respect to the induced uniform space, is discussed. As a consequence, we prove the equivalence of two extended notions of *sequential computability* (called $\mathcal{L}$- and $\mathcal{A}$- sequential computability) of a real function. This indicates that the two extended notions of sequential computability provide the computational mechanisms of the same power. We will then characterize a piecewise continuous function to be computable (called para-computable here) as being $\mathcal{L}$- (hence $\mathcal{A}$-) sequentially computable and *piecewise effectively continuous*.

Key words Uniform space, limiting recursive function, sequential computability of real numbers, effectively continuous function, para-computable function

1 Introduction

The universe of our discourse will be the real line and the real functions. Our interest is in the computability properties of these objects.

We will first give an account of some preliminaries with respect to "recursiveness" and "computability" of mathematical objects for the convenience of working mathematicians.

By a number theoretic function we mean a function from natural numbers to natural numbers. It can also be regarded as a sequence of natural numbers.

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A number theoretic function is called recursive if there is an algorithm (program) to compute the values of the function. Recursiveness can be extended to a sequence of integers as well as to a sequence of rational numbers.

A real number x is called computable (with respect to the Euclidean topology) if x satisfies the following two conditions.

(i) There exists a recursive sequence of rational numbers (fractions) $\{r_n\}$ which approximates x.

(ii) There is a recursive modulus of convergence of $\{r_n\}$ to x. (We call such a case effective convergence.)

The computability of a (sequence of) real number(s) with respect to the Euclidean topology will be called **R**-computability.

A real function f (from real numbers to real numbers) is called computable if f satisfies the following conditions.

(iii) f preserves computability of sequences (sequential computability).

(iv) f is continuous and there is a recursive modulus of continuity (effective continuity).

The importance of the computability problem of the real numbers and of the real functions is growing due to wide use of computers. As is seen above, there are standard notions of computability of real numbers and of real continuous functions. As for the discontinuous functions, forming the notion of computability is still in progress, and this is our major concern.

Let us consider as an example a piecewise continuous function f which jumps at a computable point c. We cannot expect a uniform algorithm in an attempt to compute the value $f(x)$, since $x = c$? is not decidable. We do, however, easily compute $f(c)$, since we are capable of distinguishing the point c on the real line. Such a human activity can be mathematically described in terms of the method of the uniform space by forcing the point c to be an isolated element, that is the only neighborhood of c is the singleton $\{c\}$.

The same activity of computing $f(x)$ can be described in terms of the limit operator over recursive functions. The result of taking the limit of a recursive function is called limiting recursive. By admitting a limiting recursive modulus of convergence in approximating $f(c)$ by a rational sequence, we can attribute a certain notion of computation in evaluating the values $f(x)$.

In [10], the effective uniform space was defined and was applied to attribute some computational meaning to certain real functions which are discontinuous with respect to the Euclidean topology. Examples of such real functions are seen in Section 6 of [10]. In [11], on the other hand, computation in the limit of a recursive function was utilized to give a computational property to such functions. Examples of such functions are seen in Section 4 of [11].

The uniform space approach effects a change of the topology of the domain of a real function by isolating the discontinuity points, so that the function will become continuous with regards to the induced uniform topology. As a result, the computability problem of a certain discontinuous function can be reduced to the one of a continuous function in the uniform space.

The method by the limiting recursive function is to admit a computation in the limit, that is, a number theoretic function obtained from a recursive function by taking its limit (if it exists) is admitted as having some kind of computational meaning. If we admit the limiting recursive function as a modulus of convergence, then some discontinuous functions can be regarded as computable.

These two approaches have been initiated and studied by the authors together with other collaborators (cf. [10], [11], [14]). Each provides us with a description of human intellectual activity when trying to compute or draw graphs of discontinuous functions. However, they were independently developed and appeared on the surface mutually far apart as ways of dealing with computation of discontinuous functions. On the other hand, our experience indicated that they are equivalent or are mutually interpretable in a certain sense. We will here confirm that it is indeed the case.

We will fix an **R**-computable sequence of real numbers $\{a_k\}$ (called a base sequence), where $a_k < a_{k+1}$ for all $k \in \mathbf{Z}$, $\mathbf{Z}$ being the set of integers. By taking the interval topology on the open interval (a_k, a_{k+1}) and making each singleton $\{a_k\}$ an isolated point, we can induce an effective uniform topology. We can then define the notion of computability with respect to this topology, which will be called $\mathcal{A}$-computability. There is a sequence of real numbers which is **R**-computable but is not $\mathcal{A}$-computable, while a single real number is **R**-computable if and only if it is $\mathcal{A}$-computable.

We can relate, however, those two kinds of computable sequences of real numbers in a certain way (Theorem 1: Section 3).

Some basic notations as well as the definitions of the effective uniform space relative to $\{a_k\}$ and of the limiting recursive functions are presented in Section 2. Since we do not need the entire theory of the effective uniform space, we will confine ourselves only to what are necessary for our purpose.

If, for any **R**-computable sequence of real numbers $\{x_m\}$, $\{f(x_m)\}$ is approximated by a recursive sequence of rational numbers with a modulus of convergence which is limiting recursive, then we will call f "$\mathcal{L}$-sequentially computable." With $\mathcal{L}$-sequential computability, the input data and the output data are of different sorts. The input data are recursive, but the output data are limiting recursive.

If, for any $\mathcal{A}$-computable real sequence $\{x_m\}$, $\{f(x_m)\}$ is approximated by a recursive sequence of rational numbers with a recursive modulus of convergence, then we call f "$\mathcal{A}$-sequentially computable." It will be shown that f is $\mathcal{L}$-sequentially computable if and only if f is $\mathcal{A}$-sequentially computable (Theorems 2 and 3: Section 4).

An $\mathcal{L}$- (hence $\mathcal{A}$-) sequentially computable function which is effectively continuous on all open intervals (a_k, a_{k+1}) with a recursive modulus of continuity (depending on k) will be called para-computable. Some examples of para-computable functions will be presented at the end (Section 5).

For basics of the computability structure on the real line, see, for example, [8] or [15] for a quick reference.

For the classical theory of the uniform space, we have referred to [4] and [6]. The theory of the effective uniform space has been presented in [10], and the equivqlence of the two kinds of effective convergence, one with respect to an effective uniform space and one with respect to its metrization, is discussed in

[14]. The computability properties of some discontinuous functions are discussed in [11] and [10].

Formal studies of computation in terms of limiting recursive functions in relation to "proof animation" have been made by Hayashi and Nakata in [7] and [3]. Akama has studied in [1] computation in terms of limiting recursive functions relative to combinatory algebras. The uniform space treatment has a close relationship with works by Tsuiki [9].

This work has emerged from [13], with much revision and supplement.

2 Preliminaries

Let us first review significance and basics of the study of computability aspects of analysis (mainly quoted from [15]), and then definitions of the effective uniformity and some related notions (mainly quoted from [10]).

It is important and mathematically interesting to review various theories in mathematics from the viewpoint of algorithm (computability). Its significance is now recognized due to the development of computers and their uses in mathematics, although it is not totally new to review mathematics from the algorithmic viewpoints. Computable functions over natural numbers (called recursive functions) were first studied, and then the study went on to computabilities of real numbers and of continuous functions. At present, computability of mathematical objects in the abstract setting is also investigated.

The study of computability problems of discontinuous functions began to prosper in the 1980's, and there have been various approaches to deal with these problems. Among them, we have followed the method of computability structures developed by Pour-El and Richards (cf. [8]).

Roughly speaking, a number theoretic function $\chi : \mathbf{N} \to \mathbf{N}$ is called recursive if there is a program to compute $\chi(n)$ for all $n \in \mathbf{N}$, where $\mathbf{N}$ denotes the set of natural numbers (nonnegative integers). There is a recursive function $a : \mathbf{N} \to \mathbf{N}$ which is injective and whose range $a(\mathbf{N})$ is not recursive. The function a is used in order to construct counter-examples.

Investigation of algorithms in analysis is based on computable reals, whose definition is given subsequently.

Definition 2.1 A sequence of rational numbers (fractions) $\{r_n\}$ is said to be *recursive* if there is an algorithm to compute r_n for each $n \in \mathbf{N}$; namely, there exist recursive functions α, β and σ such that

$$r_n = (-1)^{\sigma(n)} \frac{\alpha(n)}{\beta(n)},$$

where $\beta(n) \neq 0$ is assumed.

The definition can be extended to multiple sequences.

Definition 2.2 A sequence of rational numbers $\{r_n\}$ is said to *effectively converge* to a real number x (or, *effectively approximates* x) if there exists a recursive function α such that, for each natural number p,

$$n \geq \alpha(p) \quad \text{implies} \quad |r_n - x| < \frac{1}{2^p}.$$

In such a case, we call α a *recursive modulus of convergence* of this sequence (to x).

Similarly, a double sequence of rational numbers $\{r_{mn}\}$ is said to effectively converges (effectively approximates) a sequence of real numbers $\{x_m\}$ if there exists a recursive function α such that, for each m and p,

$$n \geq \alpha(m,p) \quad \text{implies} \quad |r_{mn} - x_m| < \frac{1}{2^p}.$$

We will denote this fact as $x_m \simeq \langle r_{mn}, \alpha(m,p)\rangle$, or $x_m \simeq \langle r_{mn}, \alpha\rangle$ for short.

The definition can be extended to any multiple sequences.

Definition 2.3 A real number x is said to be *computable* if it is effectively approximated by a recursive sequence of rational numbers.

Similarly a sequence of real numbers $\{x_m\}$ is called *computable* if it is effectively approximated by a recursive double sequence of rational numbers.

The definition can be extended to any multiple sequences.

In order to distinguish this computability from another one to be introduced below, we call it **R**-*computable.* It is the computability with respect to the Euclidean topology.

The relations $=, <, \leq$ etc. for "integers" and "rational numbers" can be decided effectively in the sense that the characteristic functions representing these relations are recursive. However, there is an important fact with regards to comparison of two "real numbers" (cf. Chapter 0 of [8]).

Proposition 2.1 Let x be a computable real.

1) If $x > 0$ (or $x < 0$), then there is an effective procedure which demonstrates this fact.

2) In case $x = 0$, there is no general procedure which demonstrates this fact. A counter-example can be given by using the recursive injection a mentioned earlier.

Let ξ and η be computable real numbers such that $\xi < \eta$, and let $J = [\xi, \eta]$ be the closed interval with the end points ξ and η.

Definition 2.4 1) A function $f : J \to \mathbf{R}$ is called *computable* if it satisfies the following conditions (i) and (ii):

(i) It is sequentially computable, that is, if $\{x_n\}$ is **R**-computable, then $\{f(x_n)\}$ is also **R**-computable.

(ii) $f(x)$ is effectively uniformly continuous, that is, there exists a recursive function α such that, for each $x, y \in J$,

$$|x - y| < \frac{1}{2^{\alpha(p)}} \quad \text{implies} \quad |f(x) - f(y)| < \frac{1}{2^p}.$$

The function α is called a *recursive modulus of uniform continuity* of f.

2) A function defined on the real line $\mathbf{R}$ is computable if (i) above and (ii′) below hold.

(ii′) There exists a recursive function α such that, for every positive integer M,

$$x, y \in I_M, |x-y| < \frac{1}{2^{\alpha(M,p)}} \quad \text{implies} \quad |f(x)-f(y)| < \frac{1}{2^p},$$

where I_M denotes the closed interval $[-M, M]$.

Computability of a real continuous function can easily be extended to a sequence of continuous functions.

Next we present definitions of the effective uniform space and related notions according to Definitions 3.1∼3.3 of [10]. We have taken the axioms $A_1 - A_5$ of the uniformity from [4]. Let $\mathbf{N}^+$ denote the set of positive integers.

Definition 2.5 (cf. Definition 3.1 of [10]) Let X be a non-empty set, and let $\{V_n\}_{n\in\mathbf{N}^+}$ be a sequence of maps from X to the power set of X, that is,

$$V_n : X \longrightarrow \mathcal{P}(X)$$

for all positive integers n.

$\{V_n\}$ on X is called an *effective uniformity* on X if the following five axioms $A_1 \sim A_5$ hold. Effectivity is required only for $A_3 \sim A_5$.

A_1 : $x \in V_n(x)$.

A_2 : $\cap_n V_n(x) = \{x\}$.

In the following, there are recursive functions $\alpha_1, \alpha_2, \alpha_3$ which satisfy respectively A_3, A_4, A_5.

A_3 : For every $n, m \in \mathbf{N}^+$ and every $x \in X$, $V_{\alpha_1(n,m)}(x) \subset V_n(x) \cap V_m(x)$.

A_4 : For every $n \in \mathbf{N}^+$ and every $x, y \in X$, $x \in V_{\alpha_2(n)}(y)$ implies $y \in V_n(x)$.

A_5 : For every $n \in \mathbf{N}^+$ and every $x, y, z \in X$,

$$x \in V_{\alpha_3(n)}(y) \text{ and } y \in V_{\alpha_3(n)}(z) \text{ imply that } x \in V_n(z).$$

The triple $\langle X, \{V_n\}, \langle\alpha_1, \alpha_2, \alpha_3\rangle\rangle$ is called an *effective uniform space*, which will be denoted by $\mathcal{T}$. $\mathcal{T}$ is a uniform topological space with a certain effectivity.

Effective convergence and a computability structure for $\mathcal{T}$ can be defined similarly to the counterparts of metric spaces (cf. [5] and [12]).

Definition 2.6 (cf. Definition 3.2 in [10]) A sequence $\{x_m\}$ from X is said to *effectively converge* to a point x in X if there is a recursive function γ satisfying that, for every n and $m \geq \gamma(n)$, $x_m \in V_n(x)$.

A multiple sequence $\{x_{\nu m}\}$ from X is said to *effectively converge* to a (multiple) sequence $\{x_\nu\}$ if there is a recursive function β satisfying that, for every ν, n, l and $m \geq \beta(\nu, n)$, $x_{\nu m} \in V_n(x_\nu)$.

Definition 2.7 (cf. Definition 3.3 in [10]) Let $\mathcal{S}$ be a family of sequences from X. (As usual, we include multiple sequences, such as double sequences and triple sequences, when we talk about sequences.)

$\mathcal{S}$ is called a *computability structure* if it satisfies the following.

C1: (Non-emptiness) $\mathcal{S}$ is nonempty.

C2: (Re-enumeration) If $\{x_m\} \in \mathcal{S}$ and α is a recursive function, then $\{x_{\alpha(i)}\}_i \in \mathcal{S}$.

More generally, if $\{x_{\nu_1,\nu_2,\cdots,\nu_q}\} \in \mathcal{S}$, and if $\alpha_1, \alpha_2, \cdots, \alpha_q$ are recursive functions, then $\{x_{\alpha_1(\nu),\alpha_2(\nu),\cdots,\alpha_q(\nu)}\} \in \mathcal{S}$, where ν stands for $\nu_1, \nu_2, \cdots, \nu_q$.

C3: (Limit) If $\{x_{ml}\}$ belongs to $\mathcal{S}$, $\{x_m\}$ is a sequence from X, and $\{x_{ml}\}$ converges to $\{x_m\}$ effectively, then $\{x_m\} \in \mathcal{S}$. That is, $\mathcal{S}$ is closed with respect to effective convergence.

Similarly to C2 above, we can also read m as a finite sequence of indices ν.

A sequence belonging to $\mathcal{S}$ is called a *computable sequence*. An element x of X is called *computable* if the sequence $\{x, x, \cdots\}$ is computable.

For an effective uniform topological space with a computability structure $\mathcal{S}$, we will denote $\mathcal{C}_{\mathcal{T}} = \langle X, \{V_n\}, \alpha_1, \alpha_2, \alpha_3, \mathcal{S}\rangle$.

We will subsequently assume the following [Assumption] throughout this article. We will also use $\mathbf{R}$, $\mathbf{Z}$, $\mathbf{N}$ and Q to denote respectively the set of real numbers, of integers, of natural numbers and of rational numbers.

[**Assumption**] Let $\{a_k\}_{k\in\mathbf{Z}}$ be an $\mathbf{R}$-computable sequence of real numbers satisfying $a_k < a_{k+1}$ and $\cup_k[a_k, a_{k+1}] = \mathbf{R}$.

A sequence $\{a_k\}$ as above will be called a *base sequence* and each a_k will be called a *base point* in the subsequent discussion.

Assume also that $\{a_k\}$ is represented by $\langle\{v_{kn}\}, \gamma(k,n)\rangle$, where $\{v_{kn}\}$ is a recursive double sequence of rational numbers and γ is a recursive function. This fact is denoted by $a_k \simeq \langle v_{kn}, \gamma(k,n)\rangle$, or for short, $a_k \simeq \langle v_{kn}, \gamma\rangle$.

We will next define a uniformity $\{U_n\}$ on the set of real numbers $\mathbf{R}$ relative to the given base sequence $\{a_k\}$ (cf. [Assumption]), similarly to Definition 6.1 in [10]. (in [10], a special case of $a_k = k$ was treated.)

In what follows, it will be assumed $k \in \mathbf{Z}$.

Definition 2.8 Put $A_k = \{a_k\}$ (the singleton consisting of a_k alone), $I_k = (a_k, a_{k+1})$ (the open interval between a_k and a_{k+1}), $I = \cup_k I_k$, $A = \{a_k : k \in \mathbf{Z}\} = \cup_k A_k$ and $\mathbf{A_R} = A \cup I$. (Notice that, as a set, $\mathbf{A_R} = \mathbf{R}$, and hence we will simply talk about the real numbers or rational numbers when the topology is not involved.)

For $n = 1, 2, 3, \cdots$ and $x \in \mathbf{A_R}$, put $U_n(x) = \{x\} = \{a_k\}$ if $x \in A_k$; $U_n(x) = \{y : y \in I_k, |x-y| < \frac{1}{2^n}\}$ if $x \in I_k$, where $|x-y|$ is the Euclidean distance in $\mathbf{R}$. Put $\mathcal{A} = \langle\mathbf{A_R}, \{U_n\}\rangle$.

It is obvious that $\{U_n\}$ forms a uniformity on $\mathbf{A_R}$. Furthermore, it is effective as below.

Proposition 2.2 $\{U_n\}$ above is an *effective uniformity* on $\mathbf{A_R}$. So, $\mathcal{A} = \langle \mathbf{A_R}, \{U_n\}\rangle$ is an *effective uniform space.* (See Definition 2.5.)

The proof of the proposition can be carried out similarly to that of (ii) of Proposition 6.1 in [10]. Namely, recursive funtions $\alpha_1, \alpha_2, \alpha_3$ can be given as follows.

$$\alpha_1(m, n) = \max(m, n); \alpha_2(n) = n; \alpha_3(n) = n + 1.$$

Note Let $\{x_{mn}\}$ and $\{x_m\}$ be sequences from $\mathbf{A_R}$. $\{x_{mn}\}$ will be said to *effectively $\mathcal{A}$-converge* to $\{x_m\}$ if there is a recursive function β satisfying that, for every p, m and $n \geq \beta(m, p)$, $x_{mn} \in U_p(x_m)$. Namely, $x_{mn} = x_m$ if $x_m \in A_k$; $x_{mn} \in I_k$ and $\mid x_{mn} - x_m \mid < \frac{1}{2^p}$ if $x_m \in I_k$.

In [2], Gold defined the notion of *limiting recursion*, that is, if, for a recursive function $g(p, n)$, the value eventually becomes stationary with respect to n, then the computation of that stationary value for p is called the *limiting recursion*. Here we will define a limiting recursive function that is just sufficient for our purpose.

Note We say that a number theoretic function χ is called "recursive in ψ" for a number theoretic function ψ if χ can be computed by assuming the values of ψ. For example, a function χ defined by $\chi(n) = n + \psi(n)$ is recursive in ψ, since the value $\chi(n)$ can be computed by taking the sum of n and $\psi(n)$ once the value $\psi(n)$ is given. (The function $+$ is recursive.)

Since a sequence of natural numbers, say $\{l_m\}$, can be regareded as a number theoretic function l such that $l(m) = l_m$, we can define a function being recursive in the sequence $\{l_m\}$.

Definition 2.9 For any number theoretic function h, $\lim_n h(p, n) = \pi_p$ represents the fact that, for some n, for all $m \geq n$, $h(p, m) = \pi_p$.

Let g denote a recursive function, and let $\lambda(\mathbf{p})$ be a partial function defined by

$$\lambda(\mathbf{p}) = \lim_n g(\mathbf{p}, n),$$

where $\mathbf{p}$ denotes a sequence of arguments (possibly empty). When $\lambda(\mathbf{p})$ is defined for all $\mathbf{p}$, $\lambda(\mathbf{p})$ will be called *limiting recursive.*

In [11], Yasugi, Brattka and Washihara took the advantage of the limiting recursive functions in order to study the computation process of some discontinuous functions.

3 R-computability and $\mathcal{A}$-computability

We will define the notions of $\mathcal{A}$-computable real numbers and of $\mathcal{A}$-computable sequences of real numbers relative to the base sequence. (Recall that the base sequence is the **R**-computable sequence of real numbers $\{a_k\}$ assumed in [Assumption] in Section 2.) It is a generalized notion of $\mathcal{A}$-computability seen in Section 6 of [10].

Definition 3.1 1) Let Q be the set of all rational numbers, and put $\mathbf{A}_Q = I \cap Q$ (cf. Definition 2.8 for I). For each $k \in \mathbf{Z}$, we introduce a symbol ι_k. ι_k is meant to be a "symbolic name" for a_k, and will be called an *ι-symbol*.

$\mathbf{A}_Q \cup \cup_{k\in\mathbf{Z}}\{\iota_k\}$ will be denoted by $\mathbf{A}_Q^\iota$. An element of $\mathbf{A}_Q^\iota$ will be called an $\mathcal{A}$-rational element.

2) A (multiple) sequence $\{q_{\mu n}\}$ from $\mathbf{A}_Q^\iota$ is called an $\mathcal{A}$-rational sequence if, for each μ, either there is a k so that either $q_{\mu n} = \iota_k$ for all n, or there is a k so that $\{q_{\mu n}\}_n \subset I_k$. Here μ is a finite sequence of natural numbers (possibly empty).

3) A (multiple) sequence from $\mathbf{A}_Q^\iota$ is called $\mathcal{A}$-recursive if it is an $\mathcal{A}$-rational sequence that is recursive. (It is a general practice to extend the notion of recursive sequences to those with ι-symbols by way of coding.)

4) Let $\{x_m\}$ be a real sequence from $\mathbf{A_R}$.

$$x_m \simeq_{\mathcal{A}} \langle q_{mn}, \alpha_A(m,p)\rangle \quad \text{or} \quad x_m \simeq_{\mathcal{A}} \langle q_{mn}, \alpha_A\rangle \tag{1}$$

will denote the following, where α_A is a recursive function and $\{q_{mn}\}$ is an $\mathcal{A}$-recursive sequence:

$\{q_{mn}\}_n \subset I_k$ if $x_m \in I_k$, and $q_{mn} = \iota_k$ for all n if $x_m \in A_k$. Furthermore, *in case* $x_m \in I_k$,

$$\text{for every} \quad p \quad \text{and} \quad l \geq \alpha_A(m,p), q_{ml} \in U_p(x_m) \tag{2}$$

(cf. Definition 2.8 for U_p).

If (1) holds, then $\{q_{mn}\}$ is said to *effectively $\mathcal{A}$-approximate* or *effectively $\mathcal{A}$-represent* $\{x_m\}$ with a modulus of convergence α_A.

5) A sequence of real numbers $\{x_m\} \subset \mathbf{A_R}$ is called *$\mathcal{A}$-computable* if it is effectively $\mathcal{A}$-approximated by an $\mathcal{A}$-recursive sequence $\{q_{mn}\} \subset \mathcal{A}_Q^\iota$ and a recursive function $\alpha_A(m,p)$.

We can easily extend the definition of $\mathcal{A}$-computability to any multiple sequence. In particular, a double sequence $\{x_{im}\} \subset \mathbf{A_R}$ is called $\mathcal{A}$-computable if it is effectively $\mathcal{A}$-approximated by an $\mathcal{A}$-recursive sequence $\{q_{imn}\} \subset \mathbf{A}_Q^\iota$ and a recursive function $\alpha_A(i,m,n)$.

A real number x is called $\mathcal{A}$-computable if the sequence $\{x, x, x, \cdots\}$ is $\mathcal{A}$-computable.

Note 1) (2) is required only for a case where $x_m \in I_k$.

2) ι_k is *not* an element of $\mathbf{A_R}$, but is only a symbolic name. For ι-symbols, we do *not* set any kind of arithmetic. If $x_m \in A_k$, then x_m is a_k and $q_{mn} = \iota_k$ for all n. In such a case, we do not need to require a condition like (2). $\alpha_A(m,p)$ does not play any role in such a case.

3) When 1 holds, the following facts follow: $x_m \in A$ if q_{m1} is an ι-symbol, and $x_m \in I$ if $q_{m1} \in I$.

We can easily extend this definition to any multiple sequence. In particular, for a double sequence $\{x_{im}\} \subset \mathbf{A_R}$,

$$x_{im} \simeq_{\mathcal{A}} \langle q_{imn}, \alpha_A(i,m,p)\rangle \tag{3}$$

will denote the following property, where α_A is a recursive function.

Case 1: $x_{im} \in I_k$. Then $\{q_{imn}\}_n \subset I_k$ and

$$\text{for every } p \text{ and } l \geq \alpha_A(i,m,p), q_{iml} \in U_p(x_{im}) \tag{4}$$

Case 2: $x_{im} \in A_k$. Then $q_{imn} = \iota_k$ for all n.

If (3) holds, then $\{q_{imn}\}$ is said to *effectively $\mathcal{A}$-approximate* $\{x_{im}\}$ with modulus of convergence α_A.

Definition 3.2 Define $\mathcal{S}_A$ to be the set of all $\mathcal{A}$-computable sequences from $\mathbf{A_R}$ (cf. 5) above).

Proposition 3.1 $\mathcal{S}_A$ is a computability structure for $\mathcal{A}$. (See Definition 2.7 for the computability structure for a uniform space.) We can thus claim that $\mathcal{A}$ is an effective uniform space with a computability structure.

Proof C1: For any rational number $r \in I_k$, $\{r, r, \cdots\} \in \mathcal{S}_A$, and hence $\mathcal{S}_A \neq \phi$.

C2: Let ε be a recursive function, and suppose $x_l \simeq_A \langle r_{ln}, \delta\rangle$. Then $x_{\varepsilon(i)} \simeq_A \left\langle r_{\varepsilon(i)n}, \delta(\varepsilon(i), n)\right\rangle$.

C3: Suppose $x_{ml} \simeq_{\mathcal{A}} \langle r_{mln}, \delta\rangle$ and $\{x_{ml}\}_l$ converges to $\{x_m\}$ $\mathcal{A}$-effectively (cf. Definition 2.6). We will show $\{x_m\} \in \mathcal{S}_A$.

We will show that there are an $\mathcal{A}$-recursive sequence, say $\{s_{mp}\}$, and a recursive function δ_0, with which holds

$$x_m \simeq_{\mathcal{A}} \langle s_{mp}, \delta_0\rangle$$

which means $\{x_m\} \in \mathcal{S}_A$.

We have at our disposal the following premises.

[Premise 1] $x_{ml} \simeq_{\mathcal{A}} \langle r_{mln}, \delta\rangle$ for an $\mathcal{A}$-recursive sequence $\{r_{mln}\}$ and a recursive δ. This implies the following.

If $x_{ml} \in I_k$ for some k, then $\{r_{mln}\}_n \subset I_k$, and also $r_{mln} \in U_p(x_{ml})$ for all $n \geq \delta(m,l,p)$.

If $x_{ml} \in A_k$ for some k, then $r_{mln} = \iota_k$ for all n.

[Premise 2] $\lim_l x_{ml} = x_m$ with a recursive modulus of convergence β, that is, for every m, p and $l \geq \beta(m,p)$, $x_{ml} \in U_p(x_m)$.

Using the $\mathcal{A}$-recursive $\{r_{mln}\}$ and the recursive δ and β, we define s_{mp} and δ_0 as follows.

$$s_{mp} := r_{m\beta(m,p+1)\delta(m,\beta(m,p+1),p+1)}.$$

$$\delta_0(m,p) = p.$$

By definition, δ_0 is recursive.

We will show that these s_{mp} and δ_0 will do.

We will first claim that $\{s_{mp}\}$ is $\mathcal{A}$-recursive. Suppose $x_m \in A_k$ for some k. Then $x_{ml} = x_m = a_k$ for all $l \geq \beta(m, p+1)$ from [Premise 2], hence $r_{mlp} = \iota_k$ for all $l \geq \beta(m, p+1)$ and for all p. Suppose next $x_m \in I_k$ for some k. If $l \geq \beta(m, p+1)$, then $x_{ml} \in U_{p+1}(x_m)$ by [Premise 2], hence, for such l, $x_{ml} \in I_k$. So, by [Premise 1] for any n, $r_{m\beta(m,p+1)n} \in I_k$. Putting $n = \delta(m, \beta(m, p+1), p+1)$, we have $s_{mp} \in I_k$.

$\{s_{mp}\}$ is thus an $\mathcal{A}$-sequence and hence, by definition, $\mathcal{A}$-recursive.

Next, we show that s_{mp} converges to x_m effectively with modulus of convergence δ_0. It suffices to show that, when $x_m \in I_k$, $s_{mp} \in U_p(x_m)$, that is, $|x_m - s_{mp}| < \frac{1}{2^p}$ (cf. (4) above).

Now suppose $x_m \in I_k$. From [Premise 1] and [Premise 2], we have the following.

$$|x_{m\beta(m,p+1)} - r_{m\beta(m,p+1)\delta(m,\beta(m,p+1),p+1)}| < \frac{1}{2^{p+1}};$$

$$| \ x_m - x_{m\beta(m,p+1)} \ | < \frac{1}{2^{p+1}}.$$

Now we can evaluate $|x_m - s_{mp}|$ as follows.

$$|x_m - s_{mp}| \leq | \ x_m - x_{m\beta(m,p+1)} \ | + | \ x_{m\beta(m,p+1)} - s_{mp} \ |$$

$$< \frac{1}{2^{(p+1)}} + \frac{1}{2^{(p+1)}} = \frac{1}{2^p}.$$

The following proposition is a generalization of Proposition 6.4 in [10]. (In [10], $a_k = k$, while a_k can be any computable real number here.)

Proposition 3.2 1) For a single real number x, x is $\mathbf{R}$-computable if and only if x is $\mathcal{A}$-computable.

2) For the $\mathcal{A}$ with $a_k = k$ for all k, there is an $\mathbf{R}$-computable sequence of real numbers which is not $\mathcal{A}$-computable for some $\mathcal{A}$.

Proof 1) Let x be a real number. Suppose first x is $\mathbf{R}$-computable. If x is a_k, then, with $\{q_p\} = \{\iota_k, \iota_k, \cdots\}$, $a_k \simeq_A \langle q_p, \phi \rangle$, where ϕ is an arbitrary recursive function, say $\phi(p) = 1$ for all p. If $x \in I_k$ and $x \simeq \langle r_p, \alpha \rangle$, then one can compute a number p_0 so that, for all $l \geq p_0$, $a_k < r_l$ (cf. Definition 2.2 for $\simeq$). Define $q_p = r_{p_0+p}$ and $\beta(p) = \alpha(p_0 + p)$. Then $x \simeq_{\mathcal{A}} \langle q_p, \beta \rangle$.

Suppose next, x is $\mathcal{A}$-computable. If $x \in A_k$, then $x = a_k$, which is assumed to be $\mathbf{R}$-computable. If $x \in I_k$, then $x \simeq_A \langle q_p, \alpha_A \rangle$, where $\{q_p\} \subset I_k$. The condition 1 (hence 2) in Definition 3.1 implies in this case that $|x - q_l| < \frac{1}{2^p}$ for all $l \geq \alpha_A(p)$, which means $x \simeq \langle q_p, \alpha_A \rangle$.

2) Consider the uniform space $\mathcal{A}$ where $a_k = k$. (This was studied in Section 6 of [10].) The sequence $\{x_m\}$ given in Chapter 0 of [8] serves as our counter-example.

Let a be a recursive injection whose range is not recursive. $\{x_m\}$ is defined by

$$x_m = \begin{cases} \frac{1}{2^m}, & \text{if } m = a(n) \text{ for some } n \\ 0, & \text{if there is no such } n \end{cases}$$

This sequence is proven **R**-computable in [8]. From the definition, it holds that either $x_m = 0$ (hence $x_m \in A_0$) or $x_m \in (0,1)$ (hence $x_m \in I_0$), and that $x_m = 0$ if and only if $m \in a(\mathbf{N})$. The relation $m \in a(\mathbf{N})$ is *not* recursive.

Suppose $\{x_m\}$ were $\mathcal{A}$-computable. Then there are an $\mathcal{A}$-recursive sequence $\{q_{mn}\}$ and a recursive α_A satisfying the condition 1. If $x_m \in A_0$, then $q_{mn} = \iota_0$ for all n. If $x_m \in I_0$, then $q_{mn} \in I_0$ for all n. We can thus check whether $x_m = 0$ or not just by checking q_{m1}, which is a recursive procedure. In other words, one could decide recursively if m is in the range of the function a, which contradicts the property of a. So, $\{x_m\}$ is not $\mathcal{A}$-computable.

We will study below how **R**-computable sequences and $\mathcal{A}$-computable sequnces are related.

Remark From the information on $\{x_m\}$, that is, $\langle r_{mn}, \alpha \rangle$ which represents $\{x_m\}$, one can find a recursive sequence of integers $\{k_m\}$ such that $a_{k_m} < x_m < a_{k_m+2}$. $\{k_m\}$ will henceforth be used in this context.

The definition of $\{k_m\}$ can be extended to $\{k_{\mu m}\}$ for a multiple sequence $\{x_{\mu m}\}$.

[Notation] $\{k_m\}$ will denote the sequence of integers satisfying $a_{k_m} < x_m < a_{k_m+2}$, as obtained in the Remark above. ($\{k_m\}$ is recursive.)

Definition 3.3 Recall

$$a_k \simeq \langle v_{kn}, \gamma \rangle \,,$$

(cf. [Assumption] in Section 2) and suppose $\{x_m\}$ is **R**-computable with

$$x_m \simeq \langle r_{mn}, \alpha \rangle \,.$$

1) We will subsequently use two complementary conditions, $A(m,p)$ and $N(m,p)$, defined as follows.

$$A(m,p): \quad |v_{k_m+1\gamma(k_m+1,p)} - r_{m\alpha(m,p)}| > \frac{4}{2^p}$$

$$N(m,p): \quad |v_{k_{m+1}\gamma(k_m+1,p)} - r_{m\alpha(m,p)}| \leq \frac{4}{2^p}$$

$A(m,p)$ will be called the *apartness* condition, while $N(m,p)$ will be called the *nearness* condition.

2) Define a sequence of natural numbers, $\{N_{mp}\}$, by

$$N_{mp} = 1 \quad \text{if} \quad A(m,p),$$

$$N_{mp} = 0 \quad \text{if} \quad N(m,p).$$

That is to say, N_{mp} expresses that $v_{k_m+1\gamma(k_m+1,p)}$ and $r_{m\alpha(m,p)}$ are apart when $N_{mp} = 1$, and they are near when $N_{mp} = 0$.

3) Define $l_m := \lim_p\{N_{mp}\}_p$ (cf. Definition 2.9 for lim).

$\{l_m\}$ expresses that $v_{k_m+1\gamma(k_m+1,p)}$ and $r_{m\alpha(m,p)}$ are eventually apart (with respect to p) if $l_m = 1$, and are eventually near if $l_m = 0$. This sequence will be called the *criterion*-sequence for the apartness and nearness conditions $A(m,p)$ and $N(m,p)$.

General Fact Suppose, in gerenal, $x \simeq \langle r_l, \alpha\rangle$, $y \simeq \langle s_l, \beta\rangle$ and $|r_{\alpha(p)} - s_{\beta(p)}| > \frac{4}{2^p}$. Then, for all $q \geq p+2$,

$$|r_{\alpha(q)} - s_{\beta(q)}| > \frac{4}{2^q}.$$

For, assume $q \geq p+2$. Then

$$|r_{\alpha(p)} - r_{\alpha(q)}| \leq |x - r_{\alpha(p)}| + |x - r_{\alpha(q)}| \leq \frac{1}{2^p} + \frac{1}{2^q} \leq \frac{1}{2^p} + \frac{1}{2^{p+2}}.$$

Similarly

$$|s_{\beta(p)} - s_{\beta(q)}| \leq \frac{1}{2^p} + \frac{1}{2^{p+2}}.$$

Using these evaluations as well as the assumption $|r_{\alpha(p)} - s_{\beta(p)}| > \frac{4}{2^p}$, we can prove the following in succession.

$$\frac{4}{2^p} < |r_{\alpha(p)} - s_{\beta(p)}| \leq |r_{\alpha(p)} - r_{\alpha(q)}| + |r_{\alpha(q)} - s_{\beta(q)}| + |s_{\beta(q)} - s_{\beta(p)}|$$

$$\leq \frac{1}{2^p} + \frac{1}{2^{p+2}} + |r_{\alpha(q)} - s_{\beta(q)}| + \frac{1}{2^p} + \frac{1}{2^{p+2}}.$$

Therefore

$$|r_{\alpha(q)} - s_{\beta(q)}| > \frac{4}{2^p} - 2(\frac{1}{2^p} + \frac{1}{2^{p+2}}) = \frac{6}{2^{p+2}};$$

$$|r_{\alpha(q)} - s_{\beta(q)}| > \frac{6}{2^{p+2}} > \frac{4}{2^{p+2}} \geq \frac{4}{2^q}.$$

Note The condition $q \geq p+2$ is necessary, as there is a counter example for $q = p+1$ as follows.

Let p be a fixed positive integer, and suppose $x = 0$ and $y = \frac{2}{2^p} + \frac{2}{2^{p+3}}$. Suppose x is approximated in a manner that

$$r_{\alpha(p)} = -\frac{1}{2^p} + \frac{1}{2^{p+4}}, r_{\alpha(p+1)} = \frac{1}{2^{p+1}} - \frac{1}{2^{p+4}},$$

$$s_{\beta(p)} = \frac{2}{2^p} + \frac{2}{2^{p+3}} + \frac{1}{2^p} - \frac{1}{2^{p+4}}, s_{\beta(p+1)} = \frac{2}{2^p} + \frac{2}{2^{p+3}} - \frac{1}{2^{p+1}} + \frac{1}{2^{p+4}}.$$

Then

$$| s_{\beta(p)} - r_{\alpha(p)} |= \frac{4}{2^p} + \frac{2}{2^{p+4}} > \frac{4}{2^p},$$

$$| s_{\beta(p+1)} - r_{\alpha(p+1)} |= \frac{2}{2^{p+1}} + \frac{3}{2^{p+3}} < \frac{4}{2^{p+1}}.$$

Lemma 1 1) (1.1) $A(m,p)$ and $N(m,p)$ are recursive and, for every $p \in \mathbf{N}$, exactly one of $A(m,p)$ and $N(m,p)$ holds.

(1.2) If $A(m,p)$, then, for all $q \geq (p+2)$, $A(m,q)$ holds.

(1.3) Either $A(m,p)$ holds for some p, or $N(m,p)$ holds for all p.

2) $\{N_{mp}\}$ is recursive and $l_m = \lim_o \{N_{mp}\}_p$ exists for every m, and either $l_m = 1$ or $l_m = 0$.

3) $A(m,p)$ holds for some p if and only if $l_m = 1$; $N(m,p)$ holds for all p if and only if $l_m = 0$.

4) (4.1) If $A(m,p)$ holds, then $|x_m - a_{k_m+1}| > \frac{2}{2^p}$.

(4.2) If $N(m,p)$ holds, then $|x_m - a_{k_m+1}| < \frac{6}{2^p}$.

5) If $A(m,p)$ holds for some p, then $x_m \neq a_{k_m+1}$; if $N(m,p)$ holds for all p, then $x_m = a_{k_m+1}$.

Proof (1.1) The relation $A(m,p)$ is composed of recursive sequences of rationals, $v_{k_m+1\gamma(k_m+1,p)}$, $r_{m\alpha(m,p)}$ and $\frac{4}{2^p}$, as well as the function $-$, the absolute value function and the relation $>$ over the rational numbers. These are all recursive functions and relations. (It is for this reason why we take rational approximations of real numbers.)

(1.2) is a consequence of General Fact above.

(1.3) is immediate.

2) and 3) are consequences of 1).

Recall that the assumptions $a_k \simeq \langle v_{kp}, \gamma \rangle$ and $x_m \simeq \langle r_{mn}, \alpha \rangle$ imply

$$|a_{k_m+1} - v_{k_m+1\gamma(k_m+1,p)}| < \frac{1}{2^p} \quad \text{and} \quad |x_m - r_{m\alpha(m,p)}| < \frac{1}{2^p}.$$

(4.1) Assume $A(m,p)$ holds. Then by definition

$$\frac{4}{2^p} < |v_{k_m+1\gamma(k_m+1,p)} - r_{m\alpha(m,p)}|$$

$$\leq |v_{k_m+1\gamma(k_m+1,p)} - a_{k_m+1}| + |a_{k_m+1} - x_m| + |x_m - r_{m\alpha(m,p)}|$$

$$< \frac{1}{2^p} + |a_{k_m+1} - x_m| + \frac{1}{2^p} = |a_{k_m+1} - x_m| + \frac{2}{2^p}.$$

Therefore

$$\frac{2}{2^p} < |a_{k_m+1} - x_m|.$$

(4.2) Assume $N(m,p)$ holds. Then

$$|x_m - a_{k_m+1}| \leq |x_m - r_{m\alpha(m,p)}| + |r_{m\alpha(m,p)} - v_{(k_m+1)\gamma(k_m+1,p)}|$$

$$+|v_{k_m+1\gamma(k_m+1,p)} - a_{a_{k_m+1}}| < \frac{1}{2^p} + \frac{4}{2^p} + \frac{1}{2^p} = \frac{6}{2^p}.$$

5) The first property follows from (4.1) and the second property follows from (4.2).

We show that $\mathbf{R}$-computable sequences of real numbers and $\mathcal{A}$-computable sequences of real numbers are related via the criterion sequence $\{l_m\}$. (See

3) of Definition 3.3 for $\{l_m\}$.) Namely, an $\mathbf{R}$-computable sequence which may not be $\mathcal{A}$-computable can still be approximated (in the Euclidean topology) by a double $\mathcal{A}$-computable sequence recursively in $\{l_m\}$. On the other hand, an $\mathcal{A}$-computable sequence is $\mathbf{R}$-computable.

Theorem 1 ($\mathbf{R}$-computability and $\mathcal{A}$-computability of real sequences) 1) Let $\{x_m\}$ be an $\mathbf{R}$-computable sequence of real numbers with $x_m \simeq \langle r_{mn}, \alpha\rangle$ (cf. Definition 2.2). Then there is an $\mathcal{A}$-computable double sequence of real numbers, say $\{z_{mp}\}$, which converges to $\{x_m\}$ with a modulus of convergence ν which is "recursive in $\{l_m\}$." Here the convergence is meant to be the Euclidean convergence, that is, for every p and $l \geq \nu(m,p)$, $|x_m - z_{ml}| < \frac{1}{2^p}$. (See Note before Definition 2.9 for "recursive in.")

Furthermore $\{z_{mp}\}$ has the following properties: $z_{mp} = x_m$ for all large p if $l_m = 1$, that is, if $r_{m\alpha(m,p_m)}$ and $v_{k_m+1\gamma(k_m+1,p_m)}$ are apart for some p_m; $z_{mp} = a_{k_m+1}$ for all p if $l_m = 0$, that is, if $r_{m\alpha(m,p)}$ and $v_{k_m+1\gamma(k_m+1,p)}$ are near for all p. ($\{l_m\}$ is the criterion sequence:cf. Definition 3.3.)

2) Let $\{x_m\}$ be an $\mathcal{A}$-computable sequence of real numbers with $x_m \simeq_A \langle q_{mn}, \alpha_A\rangle$. Then $\{x_m\}$ is $\mathbf{R}$-computable, that is, there are a recursive sequence of rational numbers $\{h_{mn}\}$ and a recursive function λ such that $x_m \simeq \langle h_{mn}, \lambda\rangle$.

Note 1) We should note that $\{z_{mp}\}$ is a sequence from $\mathbf{A_R}$, and hence is a sequence of real numbers.

2) See 5) of Definition 3.1 for $\mathcal{A}$-computability of $\{z_{mp}\}$. Namely, there is an $\mathcal{A}$-rational recursive triple sequence from $\mathbf{A}^{\iota}_{Q}$, say t_{mpl}, and a recursive function $\delta_0(m,p,l)$ such that $z_{mp} \simeq_{\mathcal{A}} \langle t_{mpl}, \delta_0\rangle$.

3) One consequence of Theorem 1 can be stated as follows. As for the set $\mathbf{R}$, the computability structure $\mathcal{S}_{\mathcal{A}}$ of the uniform topology is contained in the computability structure of the Euclidean topology; the converse does not hold, but the latter can be approximated by the former recursive in a limiting recursive function.

Proof of Theorem 1 1) Define $\{z_{mp}\}$ as follows.

$$z_{mp} := x_m \text{ if } A(m,p); := a_{k_m+1} \text{ if } N(m,p).$$

$\{z_{mp}\}$ is a sequence from $\mathbf{A_R}$. We will show the following.

(a) $\{z_{mp}\}$ is $\mathcal{A}$-computable;

(b) $\lim_p z_{mp} = x_m$;

(c) There is a modulus of convergence for (b), say ν, which is recursive in $\{l_m\}$.

(a) Define an $\mathcal{A}$-recursive triple sequence from $\mathbf{A}^{\iota}_{Q}$, say $\{t_{mpl}\}$, as follows.

$$t_{mpl} := r_{m\alpha(m,l)} \text{ if } A(m,p); := \iota_{k_m+1} \text{ (for all } l) \text{ if } N(m,p).$$

Put $\delta_0(m,p,l) := l$ for all m,p,l. Then δ_0 is recursive and it can be shown that

$$z_{mp} \simeq_A \langle t_{mpl}, \delta_0\rangle\,,$$

and hence $\{z_{mp}\}$ is $\mathcal{A}$-computable; for

$$|z_{mp} - t_{mpl}| = |x_m - r_{m\alpha(m,l)}| < \frac{1}{2^l}$$

if $A(m,p)$, and hence for $k \geq l = \delta_0(m,p,l)$

$$|z_{mp} - t_{mpk}| < \frac{1}{2^k} \leq \frac{1}{2^l}.$$

If $N(m,p)$, then $z_{mp} = a_{k_m+1}$ and $t_{mpl} = \iota_{k_m+1}$.

(b) By 5), (1.2) and (1.3) of Lemma 1, $\{z_{mp}\}$ (classically) converges to $\{x_m\}$ (in the Euclidean topology); for, suppose $A(m,p)$ holds for some p. Then, by (1.2) of Lemma 1, $A(m,q)$ holds for $q \geq p+2$ and by 5) $x_m \neq a_{k_m+1}$. By the definition of z_{mp}, in this case, $z_{mq} = x_m$ for $q \geq p+2$. So, $\lim_p z_{mp} = x_m$. Suppose next, $N(m,p)$ holds for all p. Then, by 5), $x_m = a_{k_m+1}$, and $z_{mp} = a_{k_m+1}$ for all p. So, $\lim_p z_{mp} = \lim_p a_{k_m+1} = x_m$. Since, by (1.3), either case holds, $\lim_p z_{mp} = x_m$.

(c) As for a modulus of convergence ν, define ν as follows.

$\nu(m,p) := \max(p_m, p+2)$ if $l_m = 1$, where $p_m = \min_p A(m,p) + 2$;

$\nu(m,p) := 1$ if $l_m = 0$.

Notice that p_m is defined under the assumption $l_m = 1$. By 3) of Lemma 1, it is obvious that ν is "recursive in $\{l_m\}$." If $l_m = 1$, then for $l \geq \nu(m,p)$, $A(m,l)$ holds, hence $|x_m - z_{ml}| = 0$. If $l_m = 0$, then $|x_m - z_{ml}| = 0$ for all $l \geq 1 = \nu(m,p)$.

2) Suppose $\{x_m\}$ is $\mathcal{A}$-computable with

$$x_m \simeq_A \langle q_{mn}, \alpha_A \rangle .$$

Then define

$h_{mn} := q_{mn}$ if $q_{mn} \neq \iota_{k_m+1}$; $:= v_{(k_m+1)n}$ if $q_{mn} = \iota_{k_m+1}$.

$\{h_{mn}\}$ is a recursive sequence of rational numbers. (We can assume that "$q_{mn} = \iota_{k_m+1}$?" is decidable.) In both cases, $\lim_n h_{mn} = x_m$.

Define next

$\lambda(m,p) := \alpha_A(m,p)$ if $q_{mp} \neq \iota_{k_m+1}$; $:= \gamma(k_m+1,p)$ if $q_{mp} = \iota_{k_m+1}$.

λ is a recursive function and it holds

$$x_m \simeq \langle h_{mn}, \lambda \rangle ,$$

and hence $\{x_m\}$ is $\mathbf{R}$-computable.

4 Sequential computability of a function

Let us recall that we are assuming a base sequence (an $\mathbf{R}$-computable sequence of real numbers) $\{a_k\}$ (cf. [Assumption] in Section 2) and a recursive sequence of integers $\{k_m\}$ (cf. [Notation] in Section 3).

We will henceforth consider a real function, that is, a function $f : \mathbf{R} \to \mathbf{R}$.

We first give definitions of sequential computability of a function respectively with respect to the Euclidean topology and with respect to the $\mathcal{A}$-topology.

Definition 4.1 1) A function f will be called $\mathcal{L}$-*sequentially computable* if, for any $\mathbf{R}$-computable sequence of real numbers $\{x_m\}$ with $x_m \simeq \langle r_{mn}, \alpha \rangle$, there is a recursive sequence of rational numbers $\{s_{mn}\}$ and a function δ which is "recursive in $\{l_m\}$" such that $f(x_m) \simeq \langle s_{mn}, \delta \rangle$. ($\{l_m\}$ is the criterion sequence: cf. Definition 3.3.)

2) A function f will be called $\mathcal{A}$-*sequentially computable* if, for any $\mathcal{A}$-computable sequence of real numbers (cf. Definition 3.1), say $\{x_m\}$, $\{f(x_m)\}$ is $\mathbf{R}$-computable, that is, there is a recursive sequence of rational numbers $\{s_{mn}\}$ and a recursive function β such that $f(x_m) \simeq \langle s_{mn}, \beta \rangle$.

Notice that the convergence of $\{s_{mn}\}$ to $f(x_m)$ is considered in the Euclidean topology in both cases.

Sequential computabilities above can be extended to multiple sequences. Although the argument below will be carried out for single sequences of real numbers, it can be immediately extended to multiple sequences.

Lemma 2 For an $\mathcal{A}$-computable sequence $\{x_m\}$, the criterion sequence $\{l_m\}$ in 2) of Lemma 1 is in fact recursive. We can in fact compute the value l_m by just examining q_{m1}.

Proof Let $\{x_m\}$ be $\mathcal{A}$-computable, that is, $x_m \simeq \langle q_{mn}, \alpha_A \rangle$. We can put $l_m = 0$ if $q_{m1} = \iota_{k_m}$ for a k_m, and $l_m = 1$ if q_{m1} is not an ι-symbol. This definition is recursive, and hence the sequence $\{l_m\}$ thus defined is recursive. That this $\{l_m\}$ will do can be shown as follows. If $x_m \in A$, then there is a k_m so that $q_{mn} = \iota_{k_m}$ for all n. If $x_m \in I$, then there is a k_m so that $a_{k_m} < x_m < a_{k_{m+1}}$ or $a_{k_m+1} < x_m < a_{k_m+2}$, and so $q_{mn} \in I_{k_m}$ or $q_{mn} \in I_{k_m+1}$ for all n. So, by Lemma 1, the definition of l_m above is correct.

We will now establish the equivalence of two notions of computability.

Theorem 2 (From $\mathcal{L}$ to $\mathcal{A}$) For any function f, if f is $\mathcal{L}$-sequentially computable, then f is $\mathcal{A}$-sequentially computable.

Proof Assume f is $\mathcal{L}$-sequentially computable.

Let $\{x_m\} \simeq_A \langle q_{mn}, \alpha_A \rangle$ be an $\mathcal{A}$-computable sequence from $\mathbf{A_R}$. By 2) of Theorem 1, $\{x_m\}$ is $\mathbf{R}$-computable.

We will prove that $\{f(x_m)\}$ is $\mathbf{R}$-computable, that is, there is a recursive pair $\langle s_{mn}, \delta \rangle$ so that $f(x_m) \simeq \langle s_{mn}, \delta \rangle$. By virtue of $\mathcal{L}$-sequential computability of f as defined in 1) of Definition 4.1, it suffices to show that $\{l_m\}$ is in fact recursive. But since $\{x_m\}$ is $\mathcal{A}$-computable, Lemma 2 guarantees that $\{l_m\}$ is recursive.

Theorem 3 (From $\mathcal{A}$ to $\mathcal{L}$) For any function f, if f is $\mathcal{A}$-sequentially computable, then f is $\mathcal{L}$-sequentially computable.

Proof Assume f is $\mathcal{A}$-sequentially computable. Consider any $\mathbf{R}$-computable sequence of real numbers $\{x_m\}$ and let $\{k_m\}$ be the recursive sequence of integers as in [Notation] of Section 3, that is, $a_{k_m} < x_m < a_{k_m+2}$.

For $\{x_m\}$, let $\{z_{mp}\}$ be the $\mathcal{A}$-computable sequence of real numbers as defined in 1) of Theorem 1. Recall that z_{mp} satisfies that $z_{mp} = x_m$ for all $p \geq p_m$ if $l_m = 1$, where $p_m = \min_p A(m,p) + 2$ (cf. the proof of 1) of Theorem 1), and $z_{mp} = a_{k_m+1}$ for all p if $l_m = 0$.

Then, by the assumption that f be $\mathcal{A}$-sequentially computable, $\{f(z_{mp})\}$ is an **R**-computable sequence of real numbers. So, there is a recursive (triple) sequence of rational numbers $\{u_{mpl}\}$ and a recursive function $\eta(m,p,l)$ such that $f(z_{mp}) \simeq \langle u_{mpl}, \eta \rangle$, so that

$$|f(z_{mp}) - u_{mp\eta(m,p,l)}| < \frac{1}{2^l} \tag{5}$$

holds for all m, p, l.

Our goal is to find a recursive sequence of rational numbers $\{s_{mp}\}$ and a function δ which is recursive in $\{l_m\}$ so that

$$f(x_m) \simeq \langle s_{mp}, \delta \rangle \, .$$

Define

$$s_{mp} := u_{mp\eta(m,p,p)}.$$

We will first show that

$$f(x_m) = \lim_p s_{mp}.$$

The proof of 1) of Theorem 1 as well as 3) of Lemma 1 in Section 3 and the property of $\{z_{mp}\}$ as recalled above will be utilized.

If $l_m = 0$, then $x_m = a_{k_m+1} = z_{mp}$ for all p, and hence

$$f(x_m) = f(a_{k_m+1}) = f(z_{mp}) \simeq \langle u_{mpl}, \eta \rangle$$

for all p. From this and (5) follows that

$$|f(x_m) - s_{mp}| = |f(z_{mp}) - u_{mp\eta(m,p,p)}| < \frac{1}{2^p}$$

for all p, that is, $f(x_m) = \lim_p s_{mp}$.

If $l_m = 1$, then $x_m \neq a_{k_m+1}$. Either $a_{k_m} < x_m < a_{k_m+1}$ or $a_{k_m+1} < x_m < a_{k_m+2}$. In either case, for every $p \geq p_m$, where $p_m = \min_p A(m,p) + 2$, $z_{mp} = x_m$. Then, similarly to the case $l_m = 0$, we can show that, for $p \geq p_m$,

$$|f(x_m) - s_{mp}| = |f(z_{mp}) - u_{mp\eta(m,p,p)}| < \frac{1}{2^p},$$

hence $\lim_p s_{mp} = f(x_m)$.

Finally, the modulus of convergence δ for the limit is obtained as follows.

Put $\delta(m,p) := p$ if $l_m = 0$; $:= \max(p_m, p)$ if $l_m = 1$. δ is thus recursive in $\{l_m\}$. (Note that p_m can be defined only under the condition of $l_m = 1$.) This δ will do.

Remark Theorems 2 and 3 assert that, as for a real function, sequential computability with respect to the limiting recursion and the one with respect to the effective uniform topology are equivalent (without any assumptions). Notice that the notion of $\mathcal{L}$-sequential computability is an extension of "sequential computability" in the traditional sense, allowing a modulus of convergence which is limiting recursive, while an approximating sequence of the function values is still required to be a recursive sequence of rational numbers.

Considering that the two extended notions of pointwise computation of a real function appear quite different on the surface, their equivalence may appear rather striking, but the two theorems indicate that they both get to the core of the matter.

5 Computability of a piecewise continuous function

In order to set the notion of computability of a piecewise continuous function, we need, besides sequential computability, some kind of effective continuity.

Definition 5.1 Suppose $\{b_{ki}\}$, $k, i \in \mathbf{Z}$ (the set of all integers), is a recursive sequence of rational numbers doubly indexed with integers, such that $a_k < b_{ki} < b_{k(i+1)} < a_{k+1}$ and

$$\cup_{i \in \mathbf{Z}} [b_{ki}, b_{k(i+1)}] = (a_k, a_{k+1}).$$

Let f be a real function satisfying the following: there is a recursive function κ with which, for any x, y such that $x, y \in [b_{ki}, b_{k(i+1)}]$ and $|x - y| < \frac{1}{2^{\kappa(k,i,l)}}$, $|f(x) - f(y)| < \frac{1}{2^l}$. Then f will be called *piecewise effectively continuous.*

Definition 5.2 We call a function f *para-computable* if it is $\mathcal{L}$- (hence $\mathcal{A}$-) sequentially computable and is piecewise effectively continuous.

We can now sum up the results of Theorems 2 and 3 as follows.

Conclusion Suppose a real function f is piecewise continuous which may jump only at the base points $\{a_k\}$. Then we can regard f as having attribute of computability if f is para-computable.

One can compare para-computability with computability (as defined for a continuous function): $\mathcal{L}$- (hence $\mathcal{A}$-) sequential computability versus sequential computability, and piecewise effective continuity versus effective continuity.

Note In [10], the theory of the uniform space with $a_k = k$ has been developed, and a sufficient condition for computability of a real function is given in Proposition 6.8. A para-computable function satisfies this condition.

Examples There are examples of para-computable functions which are not continuous in Section 6 of [10] and Section 4 of [11]. These functions are claimed to be $\mathcal{A}$-sequentially computable in the former and to be $\mathcal{L}$-sequentially computable in the latter. Let us quote some of the examples here. The integer part function $[x]$ is shown to be $\mathcal{L}$- sequentially computable by means of the "least value functional" in Theorem 2 of [11]. On the other hand, the first part of the proof of Proposition 6.7 in [10] demonstrates that it is $\mathcal{A}$-sequentially computable. Some other examples are the functions, $]x[$ where $]x[= n$ if and only if $n < x \leq n+1$, $x - [x]$, $\sigma(x)$ where $\sigma(x) = 1(x \in (0, \infty)), = \frac{1}{2}(x = 0), = 0(x \in (-\infty, 0))$ (cf. Examples 3∼6 in [11]). These examples are in fact "effectively uniformly continuous" on each interval between two adjacent base points. A para-computable function which does not fall into this category is the trigonometric function tan, where the values at $\frac{2m+1}{2}\pi$, $m = 0, \pm 1, \pm 2, \cdots$, are artificially defined; for example these values are all 0 in Example 11 in [10].

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