Computability problems on the continuum

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1 Introduction

It is important and mathematically significant to review some theories of mathematics from an algorithmic standpoint.

In studies of algorithm in analysis, one puts the basis of considerations on the computability of real numbers and the computability of continuous functions.

Here a real number x is said to be computable if there is a sequence of rational numbers (fractions) $\{r_n\}$ which approximates x and satisfies the following two conditions.

(1) The fractional sequence $\{r_n\}$ is recursive.

(2) There is a recursive modulus of convergence (approximation).

When the condition (2) holds, we say that x is effectively approximated by $\{r_n\}$, or $\{r_n\}$ effectively converges to x. In general, we use the expression effective when a condition similar to (2) is satisfied.

A computable sequence of real numbers can also be defined in a similar manner. One needs the computability of a sequence of real numbers when one has to refer to the limit.

The family of all computable sequences of real numbers is called the computability structure of the field of real numbers.

The computability of a continuous real function on a compact interval with computable end points can be defined in a natural manner. A real function f (on a compact interval) is computable if the following hold.

(3) f preserves sequential computability, that is, for any input of a computable sequence of real numbers, its output by f is also a computable sequence.

(4) f has a recursive modulus of uniform continuity.

Computability on an open interval can be defined in terms of an approximation of the interval by a sequence of compact intervals and a modulus of uniform continuity which is recursive relative to the approximating intervals.

These notions of computability respectively of a real number (a sequence of real numbers) and of a continuous function (a sequence of continuous functions) are generally agreed to be natural and in a sense the strongest.

According to the definition described in (3) and (4) above for a continuous function, computability means that there is a way to nicely approximate the values for computable inputs, and this notion depends on the continuity.

Very often, however, we compute values and draw a graph of a discontinuous function. We can, let Mathematica, for example, draw graphs of some discontinuous functions. We thus expect that some class of discontinuous functions can be attributed a certain kind of computability. In an attempt of computing a discontinuous function, a problem arises in the computation of the value at a jump point (a point of discontinuity). This is because it is not in general decidable if a real number is a jump point, that is, the question "$x = a$?" is not decidable even for computable x and a.

(For the subsequent discussion, let us here note the following: $=, \leq, <$ on natural numbers and fractional numbers are decidable. $a < b$ is decidable for computable real numbers a and b, while $a = b$ and $a \leq b$ are not necessarily decidable even for computable real numbers.)

One method of dissolving this problem was proposed in [10] by Pour-El and Richards. It was a functional analysis approach, that is, a function is regarded as computable if it can be effectively approximated by effectively enumerated rational coefficient polynomials with respect to the norm of a function space, such as a Banach space or a Fréchet space.

In such a case, a function is regarded as computable as a point in a space. This is sufficient in order to draw a rough graph of the function, but does not supply us with information when computing individual values.

There are many ways of characterizing computation of a discontinuous function. Here we will report some of the approaches to this problem which we have undertaken so far.[1] One is to express the value of a function at a jump point in terms of limiting recursive funcions instead of recursive functions ([16]). Another is to change the topology of the domain of a function ([12]). In fact they are equivalent ([18]).

This is a report of our joint works with V.Brattka, T.Mori, Y.Tsujii and M.Washihara. References of related works and some other approaches are listed in **References**, details of which will not be mentioned here. Pour-El theory as well as its succeeding works on computability structures for Fréchet spaces and metric spaces are also explained in [20].

2 Preliminaries

The basic definitions below are taken from [10]. A sequence of ratinal numbers $\{r_n\}$ is called *recursive* if

$$r_n = (-1)^{\beta(n)} \frac{\gamma(n)}{\delta(n)}$$

with recursive β, γ and δ.

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A real number x is called *computable* ($\mathbf{R}$-computable) if

$$\forall m \geq \alpha(p).|x - r_m| < \frac{1}{2^p}$$

for recursive α and $\{r_m\}$. We will express such a circumstance as $x \simeq \langle r_m, \alpha\rangle$.

These definitions can be extended to a *computable sequence* of real numbers.

A real (continuous) function f is *computable* ($\mathbf{R}$-computable) if the following hold.

(i) f preserves *sequential computability*, that is, for a computable $\{x_n\}$, $\{f(x_n)\}$ is computable.

(ii) f is continuous with *recursive modulus of continuity*, say β;

$$\forall p \forall n \in \mathbf{N}^+ \forall k \geq \beta(n,p) \forall x, y \in [n, n+1].$$

$$|x - y| < \frac{1}{2^k} \Rightarrow |f(x) - f(y)| < \frac{1}{2^p}.$$

This can be extended to a computable sequence of functions.

3 Computation in the limit

As a start, we will try to compute $g(x) = \frac{1}{2}[x]$, where $[x]$ is the Gaussian function, according to [16].

Let x be a computable real number with $x \simeq \langle r_m, \alpha\rangle$, and let us consider how to compute the value $g(x)$. For the sake of simplicity, we assume $x > 0$. From the information on x, one can effectively find an n such that $n < x < n+2$. Then check

$$r_{\alpha(p)} < (n+1) - 1/2^p \quad ?$$

According to the answer to this inquiery, we define a sequence of integers $\{N_p\}$ as follows. While the answer is *No*, put $N_p = n + 1$. Once the answer becomes *Yes* at p, then put $N_q = n$ for all q satisfying $q \geq p$. The sequence $\{N_p\}$ is well-defined and recursive.

Define next a recursive sequence of rational numbers $r_p = \frac{N_p}{2}$. If $N_p = n+1$ holds for all p, then the limit of the sequence $\{r_p\}$ is $\frac{n+1}{2}$; otherwise, the limit is $\frac{n}{2}$. In either case, the sequence $\{r_p\}$ approximates the value $\frac{1}{2}[x]$. For each case, there is a recursive modulus of continuity; only, we cannot decide which is the case.

This undecidability indicates that, although there is a computation algorithm for each x, it does not guarantee a master program to compute the value $\frac{1}{2}[x]$. Indeed, there is a computable sequence of real numbers $\{x_n\}$ for which the sequence of values $\{\frac{1}{2}[x]\}$ is not computable. On the other hand, if we allow a limiting recursive function for a modulus of convergence, then we can claim the following: for any computable sequence of real numbers $\{x_m\}$, there is a recursive sequence of rational numbers $\{q_{mi}\}$ which approximates $\{\frac{1}{2}[x_m]\}$ with a modulus of convergence which is limiting recursive.

The limiting recursive function is defined as follows.

Definition 3.1 (Limiting recursive function: Gold[4]) Let $r, s \geq 0$ be integers and let g and $\phi_1, \cdots, \phi_r$ be recursive functions. The partial function h defined as follows will be called *limiting recursive*:

$$h(p_1, \cdots, p_s) = \lim_n g(\tilde{\phi}_1(n), \cdots, \tilde{\phi}_r(n), p_1, \cdots, p_s, n),$$

where $\tilde{\phi}(n)$ is a code for the finite sequence

$$\langle \phi(0, p_1, \cdots, p_s), \cdots, \phi(n, p_1, \cdots, p_s) \rangle .$$

Examples

$$h(p_1, \cdots, p_s) = \lim_n \phi(n, p_1, \cdots, p_s).$$

$h(p_1, \cdots, p_s) =$the least value of $\phi(n, p_1, \cdots, p_s)$ with respect to n.

There are many examples of real functions which can be computed using the limiting recursive modulus of convergence: see examples below. They are all piecewise continuous functions, jumping at some computable points. It is hence sensible to confine ourselves to such functions as a start of studying computability problems of discontinuous functions.

Examples ([16], [21]) $h(x) = x - [x]$; $]x[= n$ if and only if $n < x \leq n+1$; $\sigma(x) = 1(x \in (0, \infty)), = \frac{1}{2}(x = 0), = 0(x \in (-\infty, 0))$; the Rademacher function system; $\tau(x) = \tan x$ if $\frac{2n+1}{2}\pi < x < \frac{2n+3}{2}\pi$ and $\tau(x) = 0$ if $x = \frac{2n+1}{2}\pi$.

4 Topological computability

In computing the values or drawing the graph of a piecewise continuous function, it is a usual practice to first compute the value or plot a dot at a jump point, and then compute values or draw a curve on the open interval where the function is continuous. Such an action corresponds to the mathematical notion of isolating the jump points. We are thus led to the uniform topology of the real line induced from the Euclidean topology by isolating the jump points.

Let X be a non-empty set.

A sequence $\{V_n\}_{n \in \mathbf{N}}$ such that $V_n : X \to P(X)$ is called a *uniformity* if it satisfies some axioms, say, Axioms $A_1 \sim A_5$ (to be stated below). In particular, A_1 and A_2 can be unified to

$$\cap_n V_n(x) = \{x\}.$$

We will state Axioms $A_3 \sim A_5$ in the form of effective uniformity. $\mathcal{T} = \langle X, \{V_n\} \rangle$ forms a uniform topological space. Subsequent definitions are due to [12].

Definition 4.1 (Effective uniformity) A uniformity $\{V_n\}$ on X is *effective* if there are recursive functions $\alpha_1, \alpha_2, \alpha_3$ which satisfy the following.

$$\forall n, m \in \mathbf{N} \forall x \in X, V_{\alpha_1(n,m)}(x) \subset V_n(x) \cap V_m(x) \quad \text{(effective } A_3\text{)};$$

$$\forall n \in \mathbf{N} \forall x, y \in X, x \in V_{\alpha_2(n)}(y) \to y \in V_n(x) \quad \text{(effective } A_4\text{)};$$

$$\forall n \in \mathbf{N} \forall x, y, z \in X, x \in V_{\alpha_3(n)}(y), y \in V_{\alpha_3(n)}(z) \to x \in V_n(z) \quad \text{(effective } A_5\text{)}.$$

Definition 4.2 (Effective convergence) $\{x_k\} \subset X$ *effectively converges* to x in X if there is a recursive function γ satisfying $\forall x \forall k \geq \gamma(n)(x_k \in V_n(x))$.

This can be extended to effective convergence of a multiple sequence.

Definition 4.3 (Computability structure) Let $\mathcal{S}$ be a family of sequences from X (multiple sequences included). $\mathcal{S}$ is called a *computability structure* if the following hold.

C1: (Non-emptiness) $\mathcal{S}$ is nonempty.

C2: (Re-enumeration) If $\{x_k\} \in \mathcal{S}$ and α is a recursive function, then $\{x_{\alpha(i)}\}_i \in \mathcal{S}$.

This can be extended to multiple sequences.

C3: (Limit) If $\{x_{lk}\}$ belongs to $\mathcal{S}$, $\{x_l\}$ is a sequence from X, and $\{x_{lk}\}$ converges to $\{x_l\}$ effectively, then $\{x_l\} \in \mathcal{S}$. ($\mathcal{S}$ is closed with respect to effective convergence.)

This can be extended to multiple sequences.

A sequence belonging to $\mathcal{S}$ is called *computable*, and x is *computable* if $\{x, x, \cdots\}$ is computable.

We will henceforth consider the space

$$\mathcal{C}_{\mathcal{T}} = \langle X, \{V_n\}, \alpha_1, \alpha_2, \alpha_3, \mathcal{S} \rangle .$$

Definition 4.4 (Effective approximation) $\{e_k\} \in \mathcal{S}$ is an *effective approximating set* of $\mathcal{S}$: $\forall \{x_l\}$ computable, there is a recursive function ν such that

$$\forall n \forall l (e_{\nu(n,l)} \in V_n(x_l)).$$

Definition 4.5 (Effective separability) Suppose$\{e_k\}$ is an effective approximating set and dense in X:

$$\forall n \forall x \exists k (e_k \in V_n(x)).$$

Then $\mathcal{C}_{\mathcal{T}}$ is *effectively separable*, and $\{e_k\}$ is called an *effective separating set*.

Note Classically, a general method to define a metric d^* from a countable uniformity is known. It is an open problem if this induced metric preserves computability. We can, however, show that, effective convergence respectively with respect to an effective uniformity and with respect to the induced metric are equivalent ([19]).

Definition 4.6 (Relative computability) (1) $f : X \to \mathbf{R}$ is *relatively computable* (with respect to $\mathcal{S}$) if:

(i) f preserves sequential computability, that is, if $\{x_m\}$ is $\mathcal{E}_{\mathcal{T}}$-computable, then $\{f(x_m)\}$ is an $\mathbf{R}$-computable sequence of real numbers.

(ii) For any $\{x_m\} \in \mathcal{S}$ there exists a recursive function $\gamma(m, p)$ such that $y \in V_{\gamma(m,p)}(x_m)$ implies $|f(y) - f(x_m)| \leq \frac{1}{2^p}$.

(2) (1) can be extended to a sequence of functions.

Definition 4.7 (Computable function) (1) $f : X \to \mathbf{R}$ is *computable* if the following hold.

(i) f preserves sequential computability.

(ii) f is relatively computable, and there exist an effective approximating set, say $\{e_k\} \in \mathcal{S}$, and a recursive function $\gamma_0(k,p)$ for which

$$y \in V_{\gamma_0(k,p)}(e_k) \text{ implies } |f(y) - f(e_k)| \leq \tfrac{1}{2^p}$$

and

$$\bigcup_{k=1}^{\infty} V_{\gamma_0(k,p)}(e_k) = X$$

for p.

(2) (1) can be extended to a sequence of functions.

Definition 4.8 (Uniform computability) f is *uniformly computable* if f preserves sequential computability and there is a recursive modulus of uniform continuity for f.

We will henceforth confine ourselves to real functions, and assume the notations below. Most of the subsequent definitions and results are taken from [18].

[**Assumption**] $\{a_k\}_{k \in \mathbf{Z}}$ will denote an $\mathbf{R}$-computable sequence satisfying the following.

$$a_k < a_{k+1}, \quad \cup_k [a_k, a_{k+1}] = \mathbf{R}, a_k \simeq \langle v_{kp}, \gamma \rangle \,,$$

where $\mathbf{R}$ is the set of real numbers and $\{v_{kp}\}$ and γ are recursive.

Definition 4.9 ($\mathcal{A}$-uniformity) $A_k = \{a_k\}$, $J_k = (a_k, a_{k+1})$, $J = \cup_k J_k$,
$A = \{a_k : k \in \mathbf{Z}\} = \cup_k A_k$, $\mathbf{A_R} = A \cup J$,
(As a set, $\mathbf{A_R} = \mathbf{R}$)
$n = 1, 2, 3, \cdots$, $x \in \mathbf{A_R}$;
$U_n(x) := \{x\} = \{a_k\}$ if $x \in A_k$;
$U_n(x) := \{y : y \in J_k, |x - y| < \frac{1}{2^n}\}$ if $x \in J_k$
$\mathcal{A} = \langle \mathbf{A_R}, \{U_n\} \rangle$

Notice that $\mathcal{A}_\mathbf{R} = \mathbf{R}$ as sets.

Corollary 1 $\{U_n\}$ is an effective uniformity on $\mathbf{A_R}$ ($\mathcal{A}$-uniformity).

Definition 4.10 ($\mathcal{A}$-computability) $\mathbf{A}_Q = J \cap Q$;
ι_k (ι-symbol) will denote a "symbolic name" for a_k.
$\mathbf{A}^*_Q = \mathbf{A}_Q \cup \cup_{k \in \mathbf{Z}} \{\iota_k\}$;
$\{q_{\mu n}\} \subset \mathbf{A}^*_Q$ is called an $\mathcal{A}$-sequence if, for each μ, $\exists k \in \mathbf{Z}$, $q_{\mu n} = \iota_k$. for all μ or $\{q_{\mu n}\} \subset J_k$

An $\mathcal{A}$-*recursive sequence* is a recursive $\mathcal{A}$-sequence.

For $\{x_m\} \subset \mathbf{A_R}$,

$$\{x_m\} \simeq_{\mathcal{A}} \langle q_{mn}, \alpha_A(m,p)\rangle \quad (*)$$

will denote the following relation:

$\{q_{mn}\} \subset J_k$ if $x_m \in J_k$, $\{q_{mn}\} = \{\iota_k\}$ if $x_m \in A_k$, and

$$\forall p \forall l \geq \alpha_A(m,p)(|x_m - q_{ml}|_A < \frac{1}{2^p}) \quad (**)$$

if $x_m \in J_k$ ($|a-b|_A = |a-b|$, $a, b \in J_k$).

$\{q_{mn}\}$ is said to effectively $\mathcal{A}$-approximate $\{x_m\}$ with modulus of convergence α_A.

Similarly, for $\{x_{im}\} \subset \mathbf{A_R}$,

$$\{x_{im}\} \simeq_{\mathcal{A}} \langle q_{imn}, \alpha_A(i,m,p)\rangle \quad (*)$$

can be defined.

$\{x_m\} \subset \mathbf{A_R}$ is $\mathcal{A}$-*computable* if it is effectively approximated by a recursive $\mathcal{A}$-sequence $\{q_{mn}\} \subset \mathcal{A}^*_Q$ with a recursive modulus of convergence $\alpha_A(m,p)$, that is, $(*)$ holds.

x is $\mathcal{A}$-computable if $\{x, x, x, \cdots\}$ is $\mathcal{A}$-computable.

$\{e_n\}$ will denote an effective enumeration of $\mathbf{A}_Q \cup \{a_k\}$.

Proposition 4.1 (**R**- and $\mathcal{A}$-computable real) For a single real number x, x is **R**-computable if and only if x is $\mathcal{A}$-computable.

Theorem 1 (Computability structure $\mathcal{S}_{\mathcal{A}}$) Let $\mathcal{S}_{\mathcal{A}}$ be the family of computable sequences as defined in Definition 4.10. Then $\langle \mathbf{A_R}, \{U_n\}, \{e_n\}, \mathcal{S}_{\mathcal{A}}\rangle$ is an effective uniform space with an effective separating set $\{e_n\}$.

Examples ([12]) Consider the space $\langle \mathbf{A_R}, \{U_n\}, \{e_n\}, \mathcal{S}_{\mathcal{A}}\rangle$ with $a_k = k$. In this space, the function $g(x) = \frac{1}{2}[x]$ is uniformly computable.

The function f defined by $f(n) = n$ for $n \in \mathbf{Z}$ (the set of integers) and $f(x) = \frac{1}{x-k}$ on $\mathbf{J}_k$ is computable but *not* uniformly computable.

Consider $\mathbf{J} = \cup_k (k, k+1)$ and let $\mathcal{S}_{\mathbf{J}}$ denote the set of alll sequences from $\mathcal{S}_{\mathcal{A}}$ which lie in $\mathbf{J}$. It is a computability structure on $\mathcal{A}_{\mathbf{R}}$. Let f be the function $f(x) = x$. Then f is *relatively computable* with respect to $\mathcal{S}_{\mathbf{J}}$ but *not computable* with respect to it.

Take the computability structure $\mathcal{S}_{\mathbf{Z}}$, that is, the subfamily of $\mathcal{S}_{\mathcal{A}}$ whose sequences lie in $\mathbf{Z}$. Define $f(x) = [x + \frac{1}{2}]$. f is relatively computable with respect to $\mathcal{S}_{\mathbf{Z}}$ but *not* continuous.

Similarly to $\mathcal{A}$, we can change the topology of a subset of **R**. Using this idea applied to the interval $[0, 1]$, we can further supply some examples and counter-examples.

Example Let $X = [0, 1]$, and let X_A be the set X regarded as the disjoint sum of $\{0\}$ and $(0, 1]$. Define $V_l(0) = \{0\}$ and $V_l(x) = B(x, \frac{1}{2^l}) \cap (0, 1]$ for $x \in (0, 1]$. X_A with $\{V_l\}$ is an effective uniform topological space.

Consider the function $f(x) = \frac{1}{x}$ on $(0, 1]$ and $f(0) = 0$. f is computable on X_A, but not uniformly computable.

5 Equivalence

As for a sequence of real numbers, **R**-computability and $\mathcal{A}$-computability can be related in a certain way. As for a real function, the sequential computability with respect to the Euclidean topology and the sequential computability with respect to the uniform topology are equivalent (without any assumption). To state these facts, we will first introduce a limiting recursive sequence (of natural numbers). All the subsequent definitions and results are due to [18]. [Assumption] in the previous section is still assumed.

Definition 5.1 (Judging sequence) Let A and B be recursive relations defined as follows.

$$A(m,p): \quad |v_{k_m+1}\gamma(k_m+1,p) - r_{m\alpha(m,p)}| \leq \frac{4}{2^p},$$

$$B(m,p): \quad |v_{k_m+1\gamma(k_m+1,p)} - r_{m\alpha(m,p)}| > \frac{4}{2^p}.$$

(Notice that either $\exists pA(m,p)$ or $\forall pB(m,p)$ holds.) Define next a recursive sequence of natural numbers $\{N_{mp}\}$ and a limiting recursive sequence $\{l_m\}$.

$$N_{mp} = 0 \quad \text{if} \quad A(m,p),$$
$$N_{mp} = 1 \quad \text{if} \quad B(m,p),$$
$$l_m = \lim_p N_{mp}.$$

($l_m = 0$ if and only if $\exists pA(m,p)$; $l_m = 1$ if and only if $\forall pB(m,p)$.)

$\{l_m\}$ is a limiting recursive sequence of natural numbers, which will be called a *judging sequence* (judging which of the two cases holds).

Theorem 2 (Two computabilities of real numbers) (1) Suppose $x_m \simeq \langle r_{mn}, \alpha\rangle$ is **R**-computable. Then we can construct an $\mathcal{A}$-computable double sequence of real numbers, say $\{z_{mp}\}$, which converges to $\{x_m\}$ with a modulus of convergence ν which is "recursive in $\{l_m\}$."

(2) Suppose $\{x_m\}$ is an $\mathcal{A}$-computable sequence of real numbers with $x_m \simeq_A \langle q_{mn}, \alpha_A\rangle$. Then $\{x_m\}$ is **R**-computable.

Definition 5.2 (Sequential computability) (1) f is *$\mathcal{L}$-sequentially computable* if, for any **R**-computable $\{x_m\}$ ($x_m \simeq \langle r_{mn}, \alpha\rangle$), one can construct a recursive sequence of rational numbers $\{s_{mp}\}$ and a function δ which is "recursive in $\{l_m\}$" such that $\{s_{mp}\}$ approximates $\{f(x_m)\}$ with δ as a modulus of convergence.

(2) f is $\mathcal{A}$-sequentially computable if, for any $\mathcal{A}$-computable sequence of real numbers $\{x_m\}$, we can construct a recursive sequence of rational numbers $\{s_{mp}\}$ and a recursive function β such that $f(x_m) \simeq \langle s_{mp}, \beta\rangle$.

Theorem 3 (Equivalence) As for a real function f, two notions of sequential computability as defined in Definition 5.2 are equivalent (without any assumption).

(1) (From $\mathcal{L}$ to $\mathcal{A}$) An $\mathcal{L}$-sequentially computable function f is $\mathcal{A}$-sequentially computable.

(2) (From $\mathcal{A}$ to $\mathcal{L}$) An $\mathcal{A}$-sequentially computable function f is $\mathcal{L}$-sequentially computable.

We now turn to piecewise continuous functions, and define a notion of computability which we regard natural and productive.

Definition 5.3 (Piecewise computable function) $f : \mathbf{R} \to \mathbf{R}$ is piecewise computable if the following hold.

(i) For each $\mathbf{R}$-computable real number x, $f(x)$ is $\mathbf{R}$-computable.

(ii) There is a recursive function κ with which, for any x, y such that $a_k < x, y < a_{k+1}$ and $|x - y| < \frac{1}{2^{\kappa(k,p)}}$, $|f(x) - f(y)| < \frac{1}{2^p}$.

Definition 5.4 (Para-computability) A real function f is *para-computable* if f is $\mathcal{L}$- (hence $\mathcal{A}$-) sequentially computable and is piecewise computable (cf. Definition 5.2, Theorem 3 and Definition 5.3).

Note (ii) of Definition 5.3 requires that f be effectively uniformly continuous on each interval (a_k, a_{k+1}). In fact, we can lift this condition in order to cover a wider range of piecewise computable functions, details of which will be omitted here.

The examples in Sections 3 and 4 are para-computable in the extended sense of Note above.

6 Appendix: Computability as a locally integrable function

At the end, let us briefly explain a function space approach to the computability of some piecewise continuous functions.

The family of real functions which are integrable on each compact set $[-k, k]$ for every integer k forms a Fréchet space with the sequence of semi-norms

$$p_k(f) = \int_{[-k,k]} |f| dx.$$

Let us denote this space with $\left\langle L^1_{loc}(\mathbf{R}), \{p_k\} \right\rangle$, or $\mathcal{LOC}$for short.

As a generating set of the space $\mathcal{LOC}$, take for example the sequence of monomials $1, x, x^2, x^3, \cdots, x^n, \cdots$. That is, every function in $\mathcal{LOC}$ can be approximated by a sequence of rational coefficient polynomials, or the linear span with respect to rational coefficients is dense in $\mathcal{LOC}$.

A function in $\mathcal{LOC}$ can be defined to be computable if it is effectively approximated by a recursive enumeration of rational coefficient polynomials with respect to the semi-norms $\{p_k\}$. The sequence of monomials can therefore be regarded as an effectively generating set in $\mathcal{LOC}$.

The para-computable functions in the sense of Definition 5.4 are computable in this sense.

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