Inference within Knowledge

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Abstract

This paper applies the the technique of Game Logic GL_ω, which has recently been formulated by Kaneko and Nagashima, to Mccarthy's *Three Wise Men Puzzle*. The analysis makes it clear what one really knows when he or she knows it impossible to know something as well as how one can enrich his or her knowledge by knowing that someone else knows it impossible to know something. The point is that human being is capable of jumping out from the logical system within which he or she makes inferences; this may well have significant implications for the analysis of human behaviour.

1 Introduction

There have been interests in formal treatment of games which involve the notion of *knowledge*. A first comprehensive treatment of logical systems with knowledge operator is seen in [12]. Another epochmaking achievement is seen in [6] and [7], in which the authors formulate the notion of *common knowledge* as an infinite conjunction of all possible states of knowledge of finite number of players, and prove some metatheorems of the system in concern. In this paper we will present a logical system KL to formulate and solve McCarthy's *Three Wise Men Puzzle*. We shall see that the notion of *not knowing* — as well as its full use for acquiring new knowledge — is the core of such a game.

This paper is organised in the following way.

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Chapter 2 is a preliminary section which outlines the *Three Wise Girls Puzzle*. It is essentially the same as McCarthy's *Three Wise Men Puzzle* except that the three players are asked in turn rather than simultaneously. We shall discuss what one really knows when he or she knows he or she does not know something, as well as what one can see if he or she observes someone else being aware of not knowing something. The point is that human being is not imprisoned in the system where she makes inferences: he or she *jumps out from the system* [5] and revises it whenever he or she thinks it necessary.

In subsequent sections, we assume two players, instead of three, in order to make the discussion simple.

In Section 3, the logical system we are going to work with is presented. It is based on classical propositional calculus augmented with two knowledge operators. For technical reasons, we employ a sequential calculus. For the reader who is not familiar with a sequential calculus, we list two references [2] and [4]. Our system, named KL , is a small subsystem of the Game Logic GL_ω in [7] .

The *Fundamental Theorem*, which claims the eliminability of the inference rule named *cut*, is stated in Section 4. Two of its consequences are also stated. They are quoted from [7] and will be essential in proving various facts in succeeding sections. Although there is a standard proof procedure of the Fundamental Theorem, and there are proofs of its consequences in [7], we plan to give a detailed treatment of these in a later work for the reader who is not familiar with logical systems.

Coming to the expansion of knowledge, we propose a meta-treatment of adding new knowledge as a consequence of the lemmas in the preceding section. A knowledge set is a set of statements expressing the circumstances of the game. For the soundness of the game, the statements in this set have to be mutually consistent. In order to infer the correct answer from a given knowledge set, one may have to have some additional conditions which do not contradict the given circumstances. This is all discussed in Section 5.

Another problem in *Two Girls Puzzle* is the notion of *not know*. We present two possible candidates as *non-knowledge operator* in Section 6, a subjective one and an objective one. It is easily seen that the subjective one implies the objective one. It will turn out that the only property required of the non-knowledge operator for our game is the following. Player 2 knows the following: If Player 1 knows a fact (say A) and at the same time she does not know fact A, then a contradiction is yielded. Both candidates satisfy this condition.

Sections 7 and 8 are devoted to solving the *Two Girls Puzzle* in the system KL , by explicitly setting the knowledge set and its expansions. In Section 7, some unprovability results are given, so that expansions of knowledge set can be consistently obtained. They are all consequences of the lemmas in Section 4.

Finally in Section 8, a formal deduction of Player 2 to conclude the right answer is presented. The demonstration is straightforward, and it should be more or less the same in any appropriate system. The reader is invited to compare our deduction with that of Sato in [12]. As was mentioned above, the crucial fact is the contradiction caused by the assumed knowledge of Player 2 that Player 1 knows a fact and at the same time she does not know it.

We have thus proposed a formal treatment of the knowledge set, its expansion, and the non-knowledge operator for a simple finite persons' game which does not involve the time factor, the order of mutual knowledge or the common knowledge. We plan to extend our treatment to more complicated cases.

2 the outline of analysis

Let us examine the following puzzle, which is essentially the same as Mccarthy's Three Wise Men Puzzle:

> Anna, Bianca, Carla and Enrico are in a theater; the three girls are sitting while Enrico is on the platform. Every girl wears a white or red hat, but none of them can look at her own hat on her head. However, a person sitting in a higher seat can see those who are sitting in lower seats: Anna can see Bianca and Carla wearing a white hat while Bianca can see Carla having a white hat on her head. In the circumstances Enrico asks them in turn.
>
> Enrico:''Anna, you, Bianca and Carla each have a white or red hat on their head. You can see whether Bianca and Carla each wear a white hat or a red one, can't you?"
>
> Anna:''Yes, I can."
>
> Enrico:''You may assume at least one of you, Bianca and Carla wears a white hat. Can you tell whether you have a white hat or a red one on your head?"
>
> Anna:''No, I can't."
>
> Enrico:''Bianca, you know whether Carla wears a white hat or a red one. You know also what Anna knows and what she doesn't: she knows whether you and Carla each wear a white hat or a red one; she knows at least one of you, herself, and Carla wears a white hat; she does not know whether she has a white hat or red one on her head. This is all you know, isn't it?"
>
> Bianca:"Yes."
>
> Enrico:''You may assume at least one of you, Anna and Carla wears a white hat. Can you tell whether you have a white hat or a red one for sure?"
>
> Bianca: ''No, I can't."
>
> Enrico:''Carla, can you say whether you has a white hat or red one on your head?"
>
> Carla:"Yes, I can. I must have a white hat on my head."

We can readily guess how Anna, Bianca and Carla have inferred in this puzzle.

Anna:"I see Bianca and Carla wearing a white hat. Whether I wear a white hat or not, at least two of us wear a white hat so that Enrico's statement holds true. I can't conclude my hat is white or red."

Bianca:"Carla wears a white hat. Yet I can't conclude anything about the colour of my hat from this, because I know only at least one of Anna, Carla and me wears a white hat. Now let me take account of Anna's answer. If her answer depends on whether my hat is white or red, I can tell the colour of my hat. Yet, whether my hat is red or white, Anna sees at least Carla wearing a white hat so that she can't tell whether she wears a white hat or a red one; indeed she said so. I can't see whether my hat is white or red."

Carla:"Let me suppose that Bianca and I both wear a red hat. Then Bianca can reckon:

> "At least one of Bianca, Carla and me wears a white hat; since neither Bianca nor Carla wears a white hat, I must have a white hat on my head."

Yet Anna said she couldn't tell whether she wore a white hat or a red one. So at least Bianca or I wear a white hat.

Now let me suppose Bianca's hat is white and my hat is red. Then Anna can't see whether her is white or red; it's OK, but Bianca can make the following reasoning:

> "Carla wears a red hat. If my hat is red too, Anna sees both Carla and me wearing a red hat. Then Anna, knowing at least one person wears a white hat, can notice she wears a white hat. Yet she said she couldn't. So my hat must be white."

Yet Bianca said she did not know whether she wore a white hat or a red one. So she cannot wear a white hat. What it amounts to this: that I have a white hat on my head."

We can reconsider this series of reasoning more formally. Let us designate Anna, Bianca and Carla as Player 1, Player 2 and Player 3 respectively and define Proposition iW and Operator K_i ($i = 1$, 2 or 3) in the following way:

$$\begin{array}{rl} iW: & \text{Player } i \text{ wears a white hat.} \\ K_iX: & \text{Player } i \text{ knows Proposition } X. \end{array} \tag{1}$$

In addition let us use the following logical connectives: $\neg$ (*not*), $\wedge$ (*and*) and $\vee$ (*or*). As an example $\neg iW$ expresses that Player i does not wear a white hat.

Let us first reconsider Anna's reasoning. What she knows when she is asked whether she wears a white hat by Enrico consists of her direct observation: that Bianca and Carla both wear a white hat, and Enrico's hint: that at least one of

Anna, Bianca or Carla wears a white hat. In terms of the presently introduced notation,

$$K_1(2W \wedge 3W \wedge (1W \vee 2W \vee 3W)). \tag{2}$$

Anna can conclude logically neither that she wears a white hat nor that she wears a red one, because — as we shall see in the following sections — it can be proved that

$$\text{neither } K_1(1W) \text{ nor } K_1(\neg 1W) \text{ is provable from (2).} \tag{3}$$

This is not a theorem obtainable *within* a system of knowledge. It is a statement *about* the system. Even if Anna can make every inference in the system, she can conclude neither $K_1(1W)$ nor $K_1(\neg 1W)$ with the player's (insider's) view alone. One can realise he or she cannot infer something within a system only when he or she has jump out from the system to observe it from the outsider's standpoint.

Having said that she can prove neither $K_1(1W)$ nor $K_1(\neg 1W)$, Anna has actually done it. Without this ability Bianca could not have said whether she has a white hat on her head or not.[1]

We can describe Bianca's reasoning in a similar way. What Bianca directly observes is only that Carla wears a white hat: $3W$. In addition, from the conversation between Anna and Enrico, Bianca can see that Anna knows whether Bianca and Carla each wear a white hat:

$$V_1 = K_1(2W \wedge 3W) \vee K_1(2W \wedge \neg 3W) \vee K_1(\neg 2W \wedge 3W) \vee K_1(2W \wedge \neg 3W), \tag{4}$$

that Anna knows that at least one of Anna, Bianca and Carla wears a white hat:

$$V_2 = K_1(1W \vee 2W \vee 3W), \tag{5}$$

and that Anna does not know whether she wears a white hat or not:

$$V_3 = \neg K_1(1W) \wedge \neg K_1(\neg 1W). \tag{6}$$

Taking account that Bianca herself knows Enrico's hint: $1W \vee 2W \vee 3W$, we can express what Bianca knows when she is asked by Enrico as

$$K_2(V_1 \wedge V_2 \wedge V_3 \wedge 3W \wedge (1W \vee 2W \vee 3W)). \tag{7}$$

However, this is not sufficient for concluding whether Bianca wears a white hat or not. Actually the following can be proved:

$$\text{Neither } K_2(1W) \text{ nor } K_2(\neg 1W) \text{ is provable from (7).} \tag{8}$$

[1] As we shall see in the following sections, proving (3) requires such reasoning that is familiar only to logicians. From this viewpoint it seems much more probable that Anna has concluded it only by checking that neither $1W$ nor $\neg 1W$ is inconsistent with her knowledge L_1. However, though being necessary for (3), this is not apparent whether it is also sufficient. Though it might be unrealistic in the real world, we maintain that all players never fail to make such inference. Why one can intuitively arrive at correct conclusions without making rigorous logical inference is another question which we shall not discuss in this paper.

Bianca can say she cannot tell whether her hat is white or red if she can jump out from the system to prove this.

Last let us examine Carla's reasoning. She cannot observe anyone directly. Like Bianca, however, she knows $V_1 \wedge V_2 \wedge V_3$ from the conversation between Anna and Enrico. In addition Carla can see the following from the conversation between Bianca and Enrico: that Bianca knows $V_1 \wedge V_2 \wedge V_3$

$$V_4 = K_2(V_1 \wedge V_2 \wedge V_3), \tag{9}$$

that Bianca knows whether Carla wears a white hat

$$V_5 = K_2(3W) \vee K_2(\neg 3W), \tag{10}$$

that Bianca knows at least one of Anna, Bianca and Carla wears a white hat:

$$V_6 = K_2(1W \vee 2W \vee 3W), \tag{11}$$

that Bianca does not know whether she has a white hat on her head:

$$V_7 = \neg K_2(2W) \wedge \neg K_2(\neg 2W). \tag{12}$$

That is to say,

$$K_3(V_1 \wedge V_2 \wedge V_3 \wedge V_4 \wedge V_5 \wedge V_6 \wedge V_7), \tag{13}$$

from which it is possible to derive $K_3(W3)$. Since this is a theorem within the system, Anna can obtain it logically without jumping out from the system. In fact, it is a mere exercise to prove (13) $\longrightarrow K_3(W3)$. In the subsequent sections we shall concentrate our attention on (3) and (8), which will deepen our understanding about when and how one can jump out from the system as well as what one actually knows when he or she knows that he or she does not know something.

3 Logical System

Succeeding the preceding section, we will subsequently speculate on the formal treatment of the Three Wise Girls Puzzle.

Our argument will be carried out in a formal logical system, which we will call *KL* (meaning Knowledge Logic) . The system is the classical propositional calculus augmented by an inference which characterizes the *knowledge operator*. For technical reasons, we will employ the *LK* type sequent calculus. (For the system *LK* , we refer the reader, for example, to [2], and for the Japanese reader to [4] also.)

In order to make the discussion simple, we will formulate a system of two knowledge operators, that is, we assume only two players.

We will list the conditions of the puzzle below.

Conditions There are two players, Player 1 and Player 2.

Both players wear white hats.

Player 1 can see the hat of Player 2.

At least one of the players wears white, and the players know this fact.

The discussion below can be easily extended to the general case where there are arbitrary finite many players. We do not assume that Player 2 can see the hat of Player 1. It is irrelevant in solving the puzzle.

Definition 3.1 (Language $\mathcal{L}$)

1) Symbols of $\mathcal{L}$

The language $\mathcal{L}$ of our system consists of the following *symbols*.

Propositional Symbols: $1W$(Player 1 wears a white hat), $2W$(Player 2 wears a white hat)

Knowledge Operators: K_1(Player 1 knows), K_2(Player 2 knows)

Logical Connectives: $\neg(not), \wedge(and), \vee(or), \Rightarrow (implies)$

2) Formulas of $\mathcal{L}$

$\mathcal{L}$-formulas are defined as follows.

2.1) Propositional symbols $1W$ and $2W$ are (atomic) $\mathcal{L}$-formulas.

2.2) If A is an $\mathcal{L}$-formula, then so are $(\neg A)$ and $K_i(A), i = 1, 2$.

2.3) If A and B are $\mathcal{L}$-formulas, then so are $(A \wedge B), (A \vee B)$ and $(A \Rightarrow B)$.

Note. 1) For simplicity, we will call an $\mathcal{L}$-formula simply a formula.

2) We may omit parentheses when no confusion is anticipated. It is a common practice to assume that the logical connectives have stronger ties to their scopes in the order $\neg, \wedge, \vee, \Rightarrow$. So, for example, $((K_1(\neg A) \wedge (B \vee C)) \Rightarrow (C \vee D))$ can be abbreviated to $K_1(\neg A) \wedge (B \vee C) \Rightarrow C \vee D$.

3) iR will denote $\neg iW$ for $i = 1, 2$.

Definition 3.2 (System KL)

1) The basic logical system is LK_{prop}, the propositional part of LK . Precise formulation will be presented below.

2) The system KL is LK_{prop} augmented by the K-inference, which, in its essence, is to infer $K_i(A) \wedge K_i(B) \Rightarrow K_i(C)$ from $A \wedge K_i(B) \Rightarrow C$.

Presentation of System *KL*

Sequent

A *sequent* of the language $\mathcal{L}$ is an expression of the form

$$\Gamma \rightarrow \Theta,$$

where Γ and Θ are respectively finite sets of formulas, possibly empty. An expression like $A_1, A_2, \cdots, A_k$ will denote the set $\{A_1, A_2, \cdots, A_k\}$ and an expression like Γ, Δ will represent the union set $\Gamma \cup \Delta$. On the surface it may appear that, in $A_1, A_2, \cdots, A_k$, $\Gamma \cup \Delta$, there be an order in the arrangement of formulas, but it should be understood to be a set. Γ is called the *antecedent* of the sequent, and Θ is called the *succedent* of the sequent.

The intuitive idea of a sequent $A_1, A_2, \cdots, A_k \rightarrow B_1, B_2, \cdots, B_l$ is $A_1 \wedge A_2 \wedge \cdots \wedge A_k \Rightarrow B_1 \vee B_2 \vee \cdots \vee B_l$. In case $l = 0$, the sequent $A_1, A_2, \cdots, A_k \rightarrow$

represents the fact that the set $A_1, A_2, \cdots, A_k$ implies a contradiction; in other words, $A_1, A_2, \cdots, A_k$ is inconsistent as a set. If $k = 0$, then the sequent is reduced to $B_1 \vee B_2 \vee \cdots \vee B_l$. If $k = l = 0$, then the sequent will become $\rightarrow$, the empty sequent, which represents a contradiction.

Initial Sequent
A sequent of the form $A \rightarrow A$ for any formula A is called an *initial sequent*. Such a sequent represents a tautology.

Logical Inferences
The logical inferences can be classified into three categories. The first one is called *thin*, which infers a *thinned* sequent from a given sequent. The second one is called *cut*, which cuts out a formula common to the succedent of a sequent and the antecedent of another. The third one consists of *propositional* inferences, which introduce propositional connectives in antecedents or succedents.

$$\frac{\Gamma \rightarrow \Theta}{\Delta, \Gamma \rightarrow \Theta, \Lambda}\ (thin) \qquad \frac{\Delta \rightarrow \Theta, A \quad A, \Gamma \rightarrow \Lambda}{\Delta, \Gamma \rightarrow \Theta, \Lambda}\ (cut)$$

$$\frac{A, \Gamma \rightarrow \Theta}{A \wedge B, \Gamma \rightarrow \Theta}\ (\wedge \rightarrow)_l, \quad \frac{B, \Gamma \rightarrow \Theta}{A \wedge B, \Gamma \rightarrow \Theta}\ (\wedge \rightarrow)_r$$

$$\frac{\Gamma \rightarrow \Theta, A \quad \Gamma \rightarrow \Theta, B}{\Gamma \rightarrow \Theta, A \wedge B}\ (\rightarrow \wedge)$$

$$\frac{A, \Gamma \rightarrow \Theta \quad B, \Gamma \rightarrow \Theta}{A \vee B, \Gamma \rightarrow \Theta}\ (\vee \rightarrow)$$

$$\frac{\Gamma \rightarrow \Theta, A}{\Gamma \rightarrow \Theta, A \vee B}\ (\rightarrow \vee)_l, \quad \frac{\Gamma \rightarrow \Theta, B}{\Gamma \rightarrow \Theta, A \vee B}\ (\rightarrow \vee)_r$$

$$\frac{\Gamma \rightarrow \Theta, A \quad B, \Gamma \rightarrow \Theta}{ARightarrowB, \Gamma \rightarrow \Theta}\ (Rightarrow \rightarrow) \qquad \frac{A, \Gamma \rightarrow \Theta, B}{\Gamma \rightarrow \Theta, ARightarrowB}\ (\rightarrow Rightarr$$

$$\frac{\Gamma \rightarrow \Theta, A}{\neg A, \Gamma \rightarrow \Theta}\ (\neg \rightarrow) \qquad \frac{A, \Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta, \neg A}\ (\rightarrow \neg)$$

Knowledge Inference
The knowledge operator is introduced by one type of rule, *K-rule*, as below.

$$\frac{\Gamma, K_i(\Delta) \rightarrow \Theta}{K_i(\Gamma, \Delta) \rightarrow K_i(\Theta)}\ (K \rightarrow K)$$

Here $i = 1, 2$, and $|\Theta| \leq 1$, that is, there is at most one formula in Θ. Also, $K_i(A_1, A_2, \cdots, A_k)$ stands for $K_i(A_1), K_i(A_2), \cdots, K_i(A_k)$. If $k = 0$, then $K_i(A_1, \cdots, A_k)$ is empty.

Upper Sequent and Lower Sequent A sequent above the line of an inference is called an (the) *upper sequent* of the inference, and one below the line is called

the *lower sequent* of the inference. An upper sequent is an assumption and a lower sequent is the conclusion of an inference.

Proof Figures
A proof (of KL) is a tree-form figure, called a *proof figure*, consisting of finitely many sequents, where the top sequents are initial sequents, and every sequent except for a top one is the lower sequent of an inference. (We will simply speak of proofs, instead of proof figures.)

Exposition of K-rule In order to understand the K-rule, it will be convenient to first assume that Δ is empty, $\Gamma \equiv A$ and $\Theta \equiv B$, that is,

$$\frac{A \to B}{K_i(A) \to K_i(B)}$$

This can be interpreted as: If "A implies B" is proved to be true, then it is proved that $K_i(A)$ implies $K_i(B)$, or the fact that Player i knows of A implies that Player i knows of B.

A more special case is

$$\frac{\to B}{\to K_i(B)}$$

This represents that if B is provable, then one can infer that Player i knows B holds.

What does Δ do in the K-rule? Consider the case where Γ is empty, Δ is C and Θ is B.

$$\frac{K_i(C) \to B}{K_i(C) \to K_i(B)}$$

This can be interpreted as follows. Suppose it is proved that, if Player i knows that C holds, then B holds. Then it can be deduced that, if Player i knows that C holds, then Player i knows that B holds.

As a special case with $K_i(\Delta) \equiv K_i(C) \equiv B$, we have

$$\frac{K_i(C) \to K_iK_i(C)}{K_i(C) \to K_i(C)}$$

That is, if Player i knows C, then Player i knows that she knows C. The opposite direction is not guaranteed. See also the remark below.

Remark 3.1 1) In any inference *except* a cut, an upper sequent consists of subformulas of the formulas in the lower sequent. (A formula B is a subformula of A if B is a part of A.)

2) A sequent of the form $K_i(A) \to A$ is not necessarily provable. We will demonstrate this fact in the next section. That means, a statement "If Player i is aware of A, then A is true" is not formally assured.

A sequent of the form $A \to K_i(A)$ is not generally provable either. That means, "Player i knows A whenever A is true" is not formally assured.

3) On the other hand, A is provable if and only if $K_i(A)$ is provable. The "only if" part can be claimed by an application of K-rule, while the "if" part can be claimed by a theorem in the next section. This equivalence signifies that the logical ability of Player i is exactly the logical ability of the system KL .

4) The difference between 2) and 3) above is important. Since the notion of *to know* is not expected to be expressible in classical propositional calculus

or even in predicate calculus, provability of the equivalence $A \equiv K_i(A)$ would be awkward. It would annihilate the knowledge operator.

5) $K_i(A \wedge B)$ and $K_i(A) \wedge K_i(B)$ are equivalent, but $K_i(A \vee B) \rightarrow K_i(A) \vee K_i(B)$ is not necessarily provable.

6) Our system is a tiny portion of GL_ω in Section 2 of [7]. GL_ω is basically the system LK , a sequential predicate calculus applied to certain infinitary formulas augmented by K-inference. In our case, formulas are finitary and the logic is propositional calculus.

4 Fundamental Theorem on the System

For a sequential calculus, it is always questioned if the *cut elimination theorem*, known as the *Fundamental Theorem*, holds. We will here state the fundamental theorem for KL as well as two of its useful consequences, taken from [7].

Theorem 1 (The Fundamental Theorem for KL)
There is an algorithm to eliminate all the cuts from any KL -proof. That is, given a proof P of a sequent S, the claimed algorithm transforms P into a cut-free proof of S.

Corollary 1 1) The subformula property holds for proofs of KL , that is, if a sequent S is provable in KL , then there is a proof of S which consists of only the subformulas of formulas occurring in S. (A subformula of a formula A is a formula which is a part of A.)

2) As a consequence of 1), the syntactic consistency of KL follows. That is, the empty sequent $\rightarrow$ cannot be proved in KL . One can rephrase this as the unprovability of a contradiction $X \wedge \neg X$ for any formula X.

Note The proof technique of the fundamental theorem is essentially the same with any sequential system. As for KL , one can follow the proof for GL_ω, but it is by far easier with KL .

A proof of the fundamental theorem as well as those of some propositions in this section is planned to be presented, for the reader's convenience, later in a more comprehensive paper.

Subsequently, we will quote two useful lemmas from [7], modified to our system.

Let Γ be a set of formulas. According to [7], $\epsilon\Gamma$ will denote the set of formulas obtained from Γ by eliminating all the knowledge operators as well as superfluous parentheses. For example, if Γ is $X \wedge K_1(K_2(X \vee Y))$, then $\epsilon\Gamma$ stands for $X \wedge (X \vee Y)$. $\epsilon\Gamma$ is a set of formulas of LK_{prop} .

Proposition 4.1 (Elimination of Knowledge Operator: (2.2) of Section 2.1 in [7])
A sequent $\Gamma \rightarrow \Theta$ is provable in KL if and only if $\epsilon\Gamma \rightarrow \epsilon\Theta$ is provable in LK_{prop} .

Following [7], we introduce the notion of separation of a sequent.

Definition 4.1 (Decomposition of Sequent) A formula A is said to be *indecomposable* if A is either atomic or of the form $K_i(B)$ for an $i = 1, 2$.

A formula A is called a *K_i-formula* if the outermost symbol of every maximal indecomposable subformula of A is K_i. A is called a *$-K_i$-formula* (K_{-i}-formula in [7]) if K_i occurs in A only in the scope of a K_j for $j \neq i$.

For example, if A is of the form $K_1(K_2(B)) \wedge K_1(B \Rightarrow K_1(C))$, then A is a K_1-formula, where $K_1(K_2(B))$ and $K_1(B \Rightarrow K_1(C))$ are maximal indecomposable subformulas of A. $K_2(B \vee K_1(C)) \Rightarrow 1W$ is a $-K_1$-formula. $K_1(B) \wedge K_2(C)$ is neither a K_1-formula nor a $-K_1$-formula.

Proposition 4.2 (Separation Theorem: Theorem 3.3 of [7])
Let $\Gamma, \Delta \rightarrow \Theta, \Lambda$ be a sequent, where Γ, Θ consists of K_i-formulas and Δ, Λ consists of $-K_i$-formulas.

If $\Gamma, \Delta \rightarrow \Theta, \Lambda$ is provable in KL , then either $\Gamma \rightarrow \Theta$ or $\Delta \rightarrow \Lambda$ is provable in KL .

As an application of the propositions above, we show a counter example for 2) in Remark 3.1.

Counter-example
For an atomic X, $K_i(X) \rightarrow X$ is *not* provable. For, suppose it were provable; then we can separate the sequent into the K_i part and $-K_i$ part, and by virtue of the Separation Theorem above, either $K_i(X) \rightarrow$ or $\rightarrow X$ is provable. By the Fundamental Theorem, the latter is immediately seen impossible. As for the former, by virtue of Proposition 4.1, $X \rightarrow$ must be provable, which is again impossible.

Unprovability of $X \rightarrow K_i(X)$ can be demonstrated similarly.

The "if part" of 3) in **Remark** 3.1, "if $K_i(A)$ is provable, then so is A" can be obtained by the Fundamental Theorem.

5 Expansion of Knowledge Set

In formulating puzzles in a logical system, we must place the *set of circumstances* as an *axiom set*.

In this article, we will deal with the cases where the circumstances can be expressed with finitely many formulas. Generally, we call such a set of formulas an *axiom* set, but here we will call it a *knowledge set*. In deducing a sequence of inferences from an axiom set in a logical system, we must first know that the axiom set is consistent with the logical system (no false statement be deducible), for otherwise any formula would be deducible from such an axiom set.

Supposing that a given knowledge set is consistent, the next problem is to expand it if necessary. As has been explained in Section 2, there are circumstances when new statements of knowledge be added to solve a puzzle. To

secure the soundness of the puzzle, one needs to show that the knowledge set thus expanded is again consistent. To avoid redundancy, that is, to add only *new* knowledge statements, one needs to show that they are not deducible from the original knowledge set.

In this section, we will give an account of the problems described above.

Definition 5.1 (Knowledge Set and its Expansion) 1) A finite set of $\mathcal{L}$-formulas, say Π, will be called a *knowledge set.*

2) A knowledge set Π is said to be *consistent* (in KL), if $\Pi \rightarrow$ is not provable in KL .

3) A knowledge set Π is said to be *independent* if no formula in Π is deducible from the rest, that is, if A is a formula in Π, then the sequent $\Pi \setminus \{A\} \rightarrow A$ is not provable in KL .

4) Let Π and Ξ be knowledge sets. Ξ is called a (consistent and independent) *expansion* of Π if Π is a subset of Ξ and Ξ is consistent and independent.

From the definition, we notice that the unprovability of a sequent is an important issue for expansion of a knowledge set.

Two Girls Puzzle gives an example of such a treatment of knowledge sets. The general procedure to secure consistency and independency can be outlined as follows. Let Π be a knowledge set, which consists of K_1-formulas and $-K_1$-formulas (which is the case with two girls puzzle). Suppose $\Pi \rightarrow$ were provable in KL . By Proposition 4.2, either a sequent of K_1-formulas or a sequent of $-K_1$-formulas is provable. Then by Proposition 4.2, a corresponding sequent without knowledge operators would be provable in LK_{prop} . By any known method of judging validity of propositional formulas, one can check whether it be possible or not. If Π is indeed consistent, then we will know that the last step is impossible. (See Section 6 below for a concrete treatment.)

6 Non-knowledge Operator

As was explained in Section 2, the notion of *not know* plays an essential role for two girls puzzle. Let us speculate on this matter.

We will first restate the puzzle for two persons. An observer tells tow players that each wears a red hat or a white hat, but not both of them wear red hats. A player can see the opponent's hat, but cannot see her own hat. The observer asks Player 1 if she can tell the color of her hat. She answers "No, I do not know of the color of my hat." Player 2 can hear this answer. Then the observer asks Player 2 the same question. She replies "Yes, I wear a white hat."

Now let us analyze this puzzle. With this particular case, Player 2 can make inferences only under the following circumstances:

Player 2 knows that Player 1 does *not* know that she (Player 1) wears a white hat.

The problem is what this *not* refers to. If we express the statement that "Player 1 does not know X" symbolically by $N_1(X)$, then there can be several candidates for $N_1(X)$.

With the example outlined in Section 2 (for three persons puzzle), it was intrinsic that $N_1(X)$ be interpreted as $\neg K_1(X)$.

In a general setting, there may be other interpretations. Here we will take up two candidates which one would naturally come up with.

For the sake of notational simplicity, we will omit the subscript 1 in N_1, thus denoting the *non-knowledge* operator for Player 1 simply by N.

The first candidate N^1:

$$N^1(X) \equiv \neg K_1(X)$$

This is an objective interpretation of $N(X)$, that is, an observer will know that Player 1 does not know X, but Player 1 herself may not be aware of it.

The second candidate

$$N^2(X) \equiv K_1(\neg K_1(X))$$

This is a subjective interpretation of $N(X)$, that is, Player 1 is aware that she does not know X.

The two candidates are related in the following manner.

Proposition 6.1 (Non-Knowledge Operators)

$$N^2(X) \to N^1(X)$$

is provable in KL , and hence $N^2(X) \longleftrightarrow N^1(X) \wedge N^2(X)$, where $A \longleftrightarrow B$ abbreviates "$A \to B$ and $B \to A$."

On the other hand, the converse

$$N^1(X) \to N^2(X)$$

is not provable for an atomic X.

Proof

$$K_1(X), \neg K_1(X) \to$$

is provable, from which by an application of K-rule on K_1, we obtain

$$K_1(X), K_1(\neg K_1(X)) \to$$

By $(\to \neg)$, we then obtain $K_1(\neg K_1(X)) \to \neg K_1(X)$. This is $N^2(X) \to N^1(X)$.

For unprovability, suppose $N^1(X) \to N^2(X)$ were provable for an atomic X. Then there is a cut-free proof of it. In a cut-free proof, there can be four possibilities of the upper sequent of the last inference: $\to$(thin), $\neg K_1(X) \to$ (thin), $\to K_1(\neg K_1(X))$ (thin) or $\to K_1(\neg K_1(X)), K_1(X)$ $(\neg \to)$. The first three are obviously unprovable. In order that the last one be provable, either $\to K_1(\neg K_1(X))$ or $\to K_1(X)$ must be provable, since K-rule can be applied to a sequent with at most one formula in the succedent. None of these is possible,and hence $N^1(X) \to N^2(X)$ is unprovable.

Note Contrary to unprovability of $N^1(A) \longleftrightarrow N^2(A)$, provability of two non-knowledge operators is equivalent, that is, $N^1(A)$ is provable if and only if $N^2(A)$ is.

It should be noted that both N^1 and N^2 satisfy that $K(A) \wedge N(A) \rightarrow$ is provable for a knowledge operator K and the corresponding non-knowledge operator N for any formula A. It is only this property that we will need in order to play the two girls puzzle as formulated in this article. We will hence simply write N for the non-knowledge operator corresponding to K_1. We will state this fact clearly for later use.

Proposition 6.2 (Inconsistency)
For either N^1 or N^2 as defined above (denoted collectively by N),

$$K_1(A) \wedge N(A) \rightarrow$$

for any formula A.

In fact, a more important fact is $\rightarrow K_2(K_1(A) \wedge N(A) \Rightarrow \bot$

7 Some Unprovability Results and Expansion

Now we are going to formulate our puzzle.

First we will list fundamental lemmas concerning the modus ponense with the knowledge operator.

Lemma 7.1 *1.* $K(A \Rightarrow B), K(A) \longrightarrow K(B)$

2. $K(A \Rightarrow B), K(B \Rightarrow C) \longrightarrow K(A \Rightarrow C)$

3. $K(A \Rightarrow B), K(C \wedge B \Rightarrow D) \rightarrow K(A \wedge C \Rightarrow D)$

Proof
From

$$A \Rightarrow B, A \longrightarrow B$$

one obtains the sequents in the lemma by applying the K-rule.

Let Π denote the following.

$$(1W \wedge 2W) \vee (1W \wedge 2R) \vee (1R \wedge 2W)$$

(Recall that we write iR for $\neg iW$.) Then, the set up circumstances for two girls puzzle are the following.

$\Pi, K_1(\Pi), K_2(\Pi), K_1(K_2(\Pi)), K_2(K_1(\Pi)),$
$K_2(2R \Rightarrow K_1(2R)), K_2(2W \Rightarrow K_1(2W))$

Let us call this set of formulas $\mathcal{A}$. This is the starting knowledge set. (For the inferences of Player 2, only a subset of this is needed.)

Remark 7.1 $\mathcal{A}$ can be decomposed into two groups of formulas: $\mathcal{A}_1$, which consists of all the K_1-formulas, and $-\mathrm{A}_1$, which consists of all the $-K_1$-formulas:
$\mathcal{A}_1 = \{K_1(\Pi), K_1(2W)$
$-\mathcal{A}_1 = \{\Pi, K_2(\Pi), K_2(K_1(\Pi)), K_2(2R \Rightarrow K_1(2R)), K_2(2W \Rightarrow K_1(2W))\}$

We have to first assure that $\mathcal{A}$ is consistent. We will then have to make sure that some propositions are consistent with $\mathcal{A}$so that we can add them to $\mathcal{A}$.

Lemma 7.2 $\mathcal{A}$ is *consistent* in KL .

Proof Suppose $\mathcal{A}$ were inconsistent, that is $\mathcal{A} \rightarrow$ were provable in KL . Then, by Proposition 4.1,

$$\epsilon\mathcal{A} \rightarrow$$

would be provable in LK_{prop} , that is,

$$\Pi, 2W, P \Rightarrow P \Rightarrow P \rightarrow$$

would be provable, where P stands for $2R$ and $2W$. This is equivalent to

$$\Pi, 2W \rightarrow$$

It is obvious that this cannot be proved in LK_{prop} .

Lemma 7.3 $\mathcal{A} \rightarrow \neg K_1(1W)$ is *not* KL -provable.

Note This lemma claims that, $\mathcal{A}$ and $\neg K_1(1W)$ are consistent over KL .

Proof Suppose otherwise: $\mathcal{A} \rightarrow \neg K_1(1W)$ is provable in KL . Then by Proposition 4.2, the sequent obtained from this by eliminating all the K's is LK_{prop} -provable:

$$\Pi, 2W, P \Rightarrow P \rightarrow \neg 1W$$

or

$$\Pi, 2W, 1W \rightarrow,$$

which is in fact not provable.

Lemma 7.4 $\mathcal{A} \rightarrow K_1(2W \wedge \Pi \Rightarrow 1W)$ is *not* KL -provable.

Note This lemma claims that, under the circumstances $\mathcal{A}$, Player 1 cannot reason from the expanded knowledge set $\Pi \cup \{2W\}$ that her hat is white.

Proof Suppose otherwise, that is

$$\mathcal{A} \rightarrow K_1(2W \wedge \Pi \Rightarrow 1W)$$

is provable. Then by Proposition 4.2 either one of the following would be provable.
(a) $\mathcal{A}_1 \rightarrow K_1(2W \wedge \Pi \Rightarrow 1W)$

(b) $-\mathcal{A}_1 \rightarrow$

Case (a) is of the form

$$K_1(\Pi), K_1(2W), K_1(K_2(\Pi)), K_1(P \Rightarrow K_2(P)) \rightarrow K_1(2W \wedge \Pi \Rightarrow 1W)$$

By Proposition 4.2, then

$$\Pi, 2W \rightarrow 1W$$

must be LK_{prop} -provable, but that is impossible.

Case (b) is of the form

$$\Pi, K_2(\Pi), K_2(K_1(\Pi)), K_2(P \Rightarrow K_1(P)) \rightarrow$$

By Proposition 4.2, then

$$\Pi, P \Rightarrow P \rightarrow$$

must be LK_{prop} -provable, which is impossible.

Lemma 7.5 $\mathcal{A} \rightarrow K_1(2W \wedge \Pi \Rightarrow \neg 1W)$ is not KL -provable.

Note This lemma signifies that Player 1 cannot reason from the knowledge set $\Pi \cup \{2W\}$ that her hat is red.

Proof Suppose otherwise, that is

$$\mathcal{A} \rightarrow K_1(2W \wedge \Pi \Rightarrow \neg 1W)$$

is provable. Then by Proposition 4.2, either one of the following would be provable.

(a) $\mathcal{A}_1 \rightarrow K_1(2W \wedge \Pi \Rightarrow \neg 1W)$

(b) $-\mathcal{A}_1 \rightarrow$

Case (a) is

$$K_1(\Pi), K_1(2W), K_1(K_2(\Pi)), K_1(P \Rightarrow K_2(P)) \rightarrow K_1(2W \wedge \Pi \Rightarrow \neg 1W)$$

where P is one of iW or $\neg iW$. By Proposition 4.1, then

$$\Pi, 2W \rightarrow \neg 1W$$

must be LK_{prop} -provable,or $\Pi, 2W, 1W \rightarrow$ must be provable, which is impossible.

Case (b) is the same as (b) of Lemma 7.4, and hence could not be provable.

Lemma 7.6 $\mathcal{A} \rightarrow K_1(1W)$ is not provable.

Note This lemma means that Player 1 cannot recognize, under the original knowledge set, that her hat is white.

Proof Suppose otherwise, that is $\mathcal{A} \to K_1(1W)$ is provable. Then, by Proposition 4.2, either
(a) $\mathcal{A}_1 \to K_1(1W)$
or
(b) $-\mathcal{A}_1 \to$
is provable. In case of (a), by Proposition 4.2,

$$\Pi, 2W \to 1W$$

must be LK_{prop} -provable, which is impossible.
Case (b) is impossible as shown above.

Subsequent lemmas can be proved similarly to the preceding one.

Lemma 7.7

$$\mathcal{A} \to K_1(\neg K_1(1W))$$

is not provable.

Note This lemma means that it is *not* a consequence of the original knowledge set that Player 1 is aware that she does not know her hat is white.

Lemma 7.8 $\mathcal{A} \to K_2(\neg K_1(1W))$ is not provable.

Note This lemma means that it is not a consequence of the circumstances that Player 2 knows of Player 1's ignorance of her hat being white.

Lemma 7.9

$$\mathcal{A}, K_2(\neg K_1(1W)) \to$$

is not provable.

Note This lemma means that $\mathcal{A}$ and the fact that Player 2 knows of Player 1's ignorance of her hat being white are consistent.

Similarly to Lemmas 7.8 and 7.9, we can state and prove the following.

Lemma 7.10

$$\mathcal{A} \to K_2(K_1(\neg K_1(1W)))$$

is not provable.

$$\mathcal{A}, K_2K_1(\neg K_1(1W))) \to$$

is not provable.

The last four lemmas can be unified by using the non-knowledge operator N for both interpretations, that is,

Proposition 7.1

$$\mathcal{A} \rightarrow K_2(N(1W))$$

is not provable, and

$$\mathcal{A}, K_2(N(1W)) \rightarrow$$

is not provable.

Due to Proposition 7.1, we can, and must if necessary, add $K_2N(1W)$ to $\mathcal{A}$, whichever interpretation of N we take. The set $\mathcal{A} \cup \{K_2N(1W)\}$ is the expanded knowledge set that will be necessary in solving our puzzle.

8 Inference of Player 2

As previously, let Π denote the following.

$$(1W \wedge 2W) \vee (1W \wedge 2R) \vee (1R \wedge 2W)$$

Then, let Γ denote the following.

$$K_2K_1(\Pi), K_2(\Pi), K_2N(1W), K_2(2R \Rightarrow K_1(2R))$$

Of these, $K_2K_1(\Pi), K_2(\Pi), K_2(2R \Rightarrow K_1(2R))$ is a subset of $\mathcal{A}$. Γ is obtained from this set by adding $K_2N(1W)$.

Note 1) The subsequent argument will go through either with $N \equiv \neg K_1$ or with $N \equiv K_1 \neg K_1$.

2) Γ is consistent, due to Proposition 7.1 in the preceding section.

3) In the following, if not stated otherwise, the sequents are claimed to be proved in KL .

We wish to derive

(*) $\Gamma \longrightarrow K_2(2W)$

in the system KL .

(*) can be obtained as follows. We do not write Γ except when it is explicitly stated which of the formulas in Γ are used. $\perp$ will denote any form of contradiction such as $X \wedge \neg X$.

Proof

1 through 8 below are proved in KL successively from Γ.

1. $K_2(2R \Rightarrow K_1(2R))$
2. $K_2(K_1(2R) \Rightarrow K_1(1W))$
3. $K_2(K_1(1W) \wedge N(1W) \Rightarrow \perp)$
4. $K_2(2R \wedge N(1W) \Rightarrow \perp)$
5. $K_2(N(1W) \Rightarrow \neg 2R)$

6. $K_2(\neg 2R \Rightarrow 2W)$

7. $K_2(N(1W) \Rightarrow 2W)$

8. $K_2(2W)$

1 Assumption
2 The following are provable successively, by applications of K-rules.

$$\Pi, 2R \rightarrow 1W$$

$$K_1(\Pi), K_1(2R) \rightarrow K_1(1W)$$

$$K_1(\Pi) \rightarrow (K_1(2R) \Rightarrow K_1(1W))$$

$$K_2K_1(\Pi) \rightarrow K_2(K_1(2R) \Rightarrow K_1(1W))$$

3
Case 1 $N \equiv \neg K_1$

$$\rightarrow K_1(1W) \wedge \neg K_1(1W) \Rightarrow \bot$$

$$\rightarrow K_2(K_1(1W) \wedge \neg K_1(1W) \Rightarrow \bot)$$

by a K-inference.
Case 2 $N \equiv K_1\neg K_1$

$$\rightarrow K_1(1W) \wedge \neg K_1(1W) \Rightarrow \bot$$

is equivalent to

$$K_1(1W), \neg K_1(1W) \rightarrow$$

from which, by K-rule follows

$$K_1(1W), K_1\neg K_1(1W) \rightarrow$$

which in turn implies

$$\rightarrow K_1(1W) \wedge K_1\neg K_1(1W) \Rightarrow \bot$$

and hence, by K-rule, follows

$$\rightarrow K_2(K_1(1W) \wedge K_1\neg K_1(1W) \Rightarrow \bot)$$

Notice that 3 can be proved without assumptions.

4

$$1, 2 \rightarrow K_2(2R \Rightarrow K_1(1W))$$

by 2 of Lemma 7.1.

$$K_2(2R \Rightarrow K_1(1W)), 3 \rightarrow 4$$

by 3 of Lemma 7.1.

5

$$2R \wedge N(1W) \Rightarrow \bot \rightarrow N(1W) \Rightarrow \neg 2R$$

$$K_2(2R \wedge N(1W) \Rightarrow \bot) \rightarrow K_2(N(1W) \Rightarrow \neg 2R)$$

by K-rule. From this and 4 by a cut, we obtain 5.

6

Since $2R \equiv \neg 2W$, $\neg 2R \Rightarrow 2W$ is in fact $\neg\neg 2W \Rightarrow 2W$, or $2W \Rightarrow 2W$. From this by a K-inference, we obtain 6.

7

$$5, 6 \rightarrow 7$$

by 2 of Lemma 7.1.

8

$$K_2 N(1W), 7 \rightarrow 8$$

by 1 of Lemma 1.

We have thus completed the proof of (*), that is, the inference of Player 2 within her knowledge.

Remark 8.1 1) The assumptions used in the proof above are the following: $K_2(2R \Rightarrow K_1(2R))$ is 1, $K_2K_1(\Pi)$ is in the antecedent of 2, and $K_2N(1W)$ is in the antecedent of 8.

2) Since we write iR for $\neg iW$, the derivation of 6 depends on the fact $\neg\neg iW \Rightarrow iW$. That is, iW is decidable. This comes from the fact that the system KL is classical except for the knowledge operators.

3) Except for the decidability of $\neg iW$, all the proof procedure is constructive.

4) The key-point for the success of Player 2 is the definition of notion of *non-knowledge*, denoted by N, so that

$$K_2(K_1(P) \wedge N(1W) \Rightarrow \bot)$$

is provable. We have presented two candidates for N.

Note Even if we had lifted the condition that Player 2 cannot see the hat of Player 1, there is no extra information to be added to the knowledge of Player 2, since in the circumstances Player 1 wearing white does not determine the colour of the hat of Player 2.

9 Concluding Remarks

The *Three Wise Girls Puzzle* itself is not difficult to solve. Even a pupil of a primary school often gives an apt explanation of why only the last person can see she has a white hat on her head. Undoubtedly it does not seem plausible that the pupil has made such rigorous inference as in the previous sections, but he or she has certainly inferred the right answer from the hint that someone knows

someone else does not know. As is repeatedly stressed, this is not a deduction within a system but a theorem about it. Only by observing from the outside, one can see what can and must be added to the current system. In fact but for this ability one may continue to search the given system for a non-provable theorem in vain. We hope that the analysis of this paper has made it clear that such a puzzle can be solved only if the players can jump out from the system.

Though it would be presumptuous of us to referring to human nature only with our very limited study, jumping out from the system seems to be an innate ability of human being, which may play an important role in various circumstances. In fact, it might be whether agents have this ability or not that distinguishes the society of human being from the society of insects: a worker ant could repeat the same task without asking why, but it is the last thing that human being can do: people cannot but consider what they are doing, which may make them initiate new methods or make them slow down or even sabotage their work. The ability of jumping out from the system should deserve more careful attention in economic theories, experiments and simulations. Further efforts will be required for explaining how one jumps out from the system and revises it.[2]

In relation to this, returning to more concrete argument, we may probably have to reconsider the meaning of "not knowing". This concept seems to be related and relevant to reasoning about the system rather than to inferences within the system. Actually we have considered in this paper that a girl who knows she does not know something knows it is not provable within her current system of inference. We believe this is a natural interpretation, but it might not be the unique one. We hope that we could discuss it in the next paper as well as more complete treatment of the *Three Wise Girls Puzzle*.

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[2] Hofstdter discusses the ability of jumping out from the system from various viewpoints in his best-seller [7] (actually *jumping out from the system* is his terminology). Keynes refers to this ability in Chapter 6 of the *General Theory*, mentioning his well-known example: "professional investment may be likened to those newspaper competitions in which the competitors have to pick out the six prettiest faces from a hundred photographs, the price being awarded to the competitor whose choice most nearly corresponds to the average preferences of the competitors as a whole." [8](pp.156).

Incidentally we should like to mention our motivation for the present study. One of the authors, who are recently engaged in computer simulation of agent-based models, is under the following impression: [3], [9], [10] and [11]. Such agent-based models as [1] certainly generate such dynamic and complex phenomena that are (even if they are recognized) cannot but be laid aside in purely mathematical analysis, but it is doubtful whether they are really novel to those who are familiar to various complex systems in other fields such as physics and biology. The existing models, where agents lack the ability of jumping out from the system, cannot describe such economic phenomena as Keynes mentions. From this viewpoint, he has started present study with the other author, who has come to the same opinion in her study of logics (see also [13]).

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