Admissible inferences

Mariko Yasugi　Kyoto Sangyo University

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京都産業大学・理学部「論理とプログラム」資料

Definition 1 When the following hold for formulas A and B, we say that B is provable from A (in the system of the propositional calculus **P**).

There is a finite sequence $\mathcal{S}$ of formulas such that any formula in $\mathcal{S}$, say C, is either A, an axiom or a conclusion of a modus ponense whose premises are in $\mathcal{S}$ earlier than C.

In such a case, we write

$$\frac{B}{A}$$

and the transition from A to B will be called an *admissible* rule of inference.

There is another system of the propositional calculus, called the system of natural deduction. Let us denote it by NK_{prop}. It starts with any formula, called an *assumption*, and rules of inference have one of the following forms.

$$\frac{A}{B}\ (\gamma\iota),\quad \frac{A_1 \quad A_2}{B}\ (\gamma\iota),\quad \frac{A_1 \quad A_2 \quad A_3}{B}\ (\gamma\iota)$$

where γ denotes one of the logical connectives and ι denotes either I or E, read as *Introduction* and *Elimination* respectively.

We will henceforth show the following.

Theorem 1 The rules of inference of NK_{prop} is *admissible* in **P**.

For later uses, we will first list two lemmas. Their proofs will be presented later.

Lemma 1 $A \vee B$ is provable in **P** from $B \vee A$ (and some axioms).

Lemma 2 If B is proved from A (and some axioms) in **P**, then $A \Rightarrow B$ is proved from $A \Rightarrow A$ (and some axioms).

Note $A \Rightarrow A$ has been shown to be provable in **P** in the textbook.

In NK_{prop} In **P**

$$\frac{\begin{array}{cc}\vdots & \vdots \\ A & B\end{array}}{A \wedge B}\ (\wedge I) \qquad \frac{\begin{array}{c}\vdots \\ B\end{array} \quad \dfrac{\begin{array}{cc}\vdots & \\ A & (Ax3)A \Rightarrow (B \Rightarrow A \wedge B)\end{array}}{B \Rightarrow A \wedge B}}{A \wedge B}$$

$$\frac{\begin{array}{c}\vdots \\ A \wedge B\end{array}}{A}\ (\wedge E-1) \qquad \frac{\begin{array}{cc}\vdots & \\ A \wedge B & (Ax4)A \wedge B \Rightarrow A\end{array}}{A}$$

Similarly with the following.

$$\frac{\begin{array}{c}\vdots \\ A \wedge B\end{array}}{B}\ (\wedge E-2)$$

$$\frac{\begin{array}{c}\vdots \\ A\end{array}}{A \vee B}\ (\vee I-1) \qquad \frac{\begin{array}{cc}\vdots & \\ A & (Ax5)A \Rightarrow A \vee B\end{array}}{A \vee B}$$

$$\frac{\begin{array}{cc}\vdots & \\ B & (Ax5)B \Rightarrow B \vee A\end{array}}{B \vee A}$$

$$\frac{\begin{array}{c}\vdots \\ B\end{array}}{A \vee B}\ (\vee I-2) \qquad \begin{array}{c}\vdots \\ A \vee B\end{array}$$

$A \vee B$ is deduced from $B \vee A$ by virtue of Lemma 1.

$$\frac{\begin{array}{c}[\not A] \\ \vdots \\ B\end{array}}{A \Rightarrow B}\ (\Rightarrow I) \qquad \begin{array}{c}A \Rightarrow A \\ \vdots \\ A \Rightarrow B\end{array}$$

The admissibility of $(\Rightarrow I)$ is guaranteed by Lemma 2.

$$\frac{\begin{array}{cc}\vdots & \vdots \\ A & A \Rightarrow B\end{array}}{B}\ (\Rightarrow E)$$

This is exactly the syllogism.

$$\frac{\begin{array}{cc}\vdots & \vdots \\ A & \neg A\end{array}}{\bot}\ (\neg E) \qquad \frac{\begin{array}{cc}\vdots & \vdots \\ A & \neg A\end{array}}{A \wedge \neg A}\ (\wedge I)$$

Notice that $\wedge I$ has already been shown to be admissible.

$$\frac{\begin{array}{c}[\not A] \\ \vdots \\ \bot\end{array}}{\neg A}\ (\neg I)$$

$\bot$ is an abbreviation of any contradiction $B \wedge \neg B$. By $\wedge E$ and $\Rightarrow I$, which are already admissible, we have

$$\dfrac{\dfrac{\begin{array}{c}[\not A] \\ \vdots \\ B \wedge \neg B\end{array}}{B}\ (\wedge E)}{A \Rightarrow B}\ (\Rightarrow I)$$

and

$$\dfrac{\dfrac{\begin{array}{c}[\not A] \\ \vdots \\ B \wedge \neg B\end{array}}{\neg B}\ (\wedge E)}{A \Rightarrow \neg B}\ (\Rightarrow I)$$

Call these figures respectively $\mathbf{P}_1$ and $\mathbf{P}_2$.

$$\dfrac{\mathbf{P}_2 \qquad \dfrac{\mathbf{P}_1 \quad (Ax7)(A \Rightarrow B) \Rightarrow (A \Rightarrow \neg B) \Rightarrow \neg A}{(A \Rightarrow \neg B) \Rightarrow \neg A}}{\neg A}$$

$$\dfrac{\begin{array}{ccc} & [\not A] & [\not B] \\ \vdots & \vdots & \vdots \\ A \vee B & C & C\end{array}}{C}\ (\vee E)$$

By $\begin{array}{c}[\not A] \\ \vdots \\ C\end{array}$, $A \Rightarrow C$ is provable. Similarly $B \Rightarrow C$ is provable. With these we have the following.

Let P_1 be the following.

$$\dfrac{B \Rightarrow C \qquad \dfrac{A \Rightarrow C \quad (Ax6)(A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \vee B \Rightarrow C))}{(B \Rightarrow C) \Rightarrow (A \vee B \Rightarrow C)}}{A \vee B \Rightarrow C}$$

$$\dfrac{A \vee B \quad A \vee B \Rightarrow C}{C}$$

Proof of Lemma 1 We will show how to deduce $A \vee B$ from $B \vee A$ in $\mathbf{P}$. Notice that $(\vee I)_i$, where i is l or r was proven admissible without using Lemmas 1 and 2.

$$\dfrac{A \vee B \quad \dfrac{[\not A]}{B \vee A}\ (\vee I)_l \quad \dfrac{[\not B]}{B \vee A}\ (\vee I)_r}{B \vee A}\ (\vee E)$$

Proof of Lemma 2 Suppose that B is deduced from A in **P** with a succession of formulas and inferences (modus ponense), which we call Π. We will show that $A \Rightarrow B$ is provable by induction on the number i of inferences in Π.

$i = 0$ 1) Π consists of A alone. $A \Rightarrow A$ has been proved.

2) A in fact does not occur, and Π consists of B. Then B must be an axiom. So it is provable.

$$\frac{B \quad (Ax1)B \Rightarrow (A \Rightarrow B)}{A \Rightarrow B}\ (mp)$$

$i > 0$ Assume the lemma for $j < 0$. B must be obtained by a modus ponense:

$$\frac{X \quad X \Rightarrow B}{B}\ (mp)$$

By the induction hypotheses, $A \Rightarrow X$ and $A \Rightarrow (X \Rightarrow B)$ are provable.

$$\frac{A \Rightarrow (X \Rightarrow B) \qquad \dfrac{A \Rightarrow \quad (Ax2)(A \Rightarrow X) \Rightarrow ((A \Rightarrow (X \Rightarrow B)) \Rightarrow (A \Rightarrow B))}{(A \Rightarrow (X \Rightarrow B)) \Rightarrow (A \Rightarrow B)}\ (mp)}{A \Rightarrow B}\ (mp)$$

This completes the proof.

Proposition 1 $\neg A \vee A$ is provable in **P**. We will take such a formula as an axiom in NK_{prop}.